SE1CY15 Cybernetics and Circuits
Feedback – Part D
Prof Richard Mitchell

In the final quarter of the course the topics are

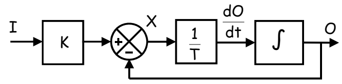
- Using the Laplace Operator instead for ∫ (or 1/jω)
- Simulation of Systems – including 'animal' systems
- Second order systems – time and frequency responses

This builds on models developed so far

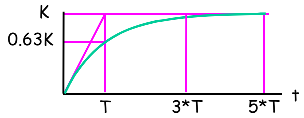
- There we form block diagrams and then transfer functions
- For simulation, though, we revert back to differential equations

16 : The Laplace Operator

We have investigated the step response of systems, for instance



$$\frac{dO}{dt} = \frac{1}{T}(K{*}I - O) \Rightarrow O = K - Ke^{-\frac{t}{T}}$$



K is steady state response

$-Ke^{-\frac{t}{T}}$ is transient response

T sets the speed of response

Rather than using Diff Eqn, easier to use Laplace operator …

Introducing s – Laplace operator

Very briefly s is introduced : it will help re block diagrams

It fits neatly with the Freq response we have already met

As an approximation, s means differentiation (1/s means integration)

Let's apply it to a differential equation we have met …

$$\frac{dO}{dt} = \frac{1}{T}(K{*}I - O) \text{ which can be written } T\frac{dO}{dt} + O = K{*}I$$

We write $\frac{dO}{dt}$ as s O, so differential equation becomes

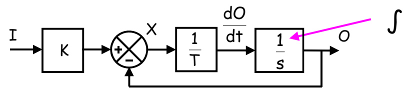
$Ts\,O + O = K{*}I$ or $(Ts + 1)\,O = K{*}I$

Hence, $\frac{O}{I} = \frac{K}{Ts+1}$

Then, if I unit step and O = 0 at t = 0: $O = K - Ke^{-t/T}$

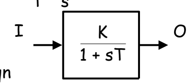
Why s helps – re block diagrams

Integration is inverse of differentiation, so ∫ = 1/s : as in diagram



$$\frac{O}{I} = \frac{\text{Forward}}{\text{1-Loop}} = K{*}\frac{\frac{1}{T}{*}\frac{1}{s}}{1-\frac{-1}{T}{*}\frac{1}{s}} = \frac{K}{1+sT}$$

So can redraw system as



Same as found from Diff Eqn

Consistent with freq response

Key Point

For frequency response, all signals are sinusoids at ang freq ω

- We replace all ∫ by 1/jω – and analyse block diagrams
- Strictly, all signals are sinusoids under steady state

For other inputs (eg step), and for finding the transient response,

- We replace all ∫ by 1/s – and analyse block diagrams

For electronic circuits with sinusoids impedances are

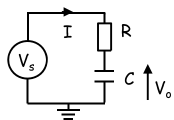
Resistor : $Z = R$ Capacitor : $Z = \frac{1}{j\omega C}$ Inductor: $Z = j\omega L$

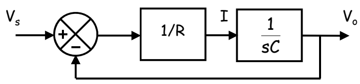
For electronic circuits transients and other signals

Resistor : $Z = R$ Capacitor : $Z = \frac{1}{sC}$ Inductor: $Z = sL$

Use with circuit theory methods (Ohm, Kirchhoff, Thevenin, etc)

RC Circuit Example

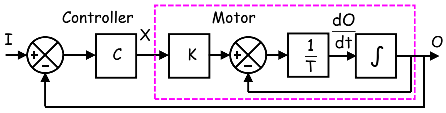




So $\frac{V_o}{V_s} = \frac{\text{Forward}}{\text{1-Loop}} = \frac{\frac{1}{R}{*}\frac{1}{sC}}{1-\frac{-1}{R}{*}\frac{1}{sC}} = \frac{1}{sCR+1}$

Then, if V_s unit step and $V_o = 0$ at t = 0: $V_o = 1 - e^{-t/RC}$

Why s helps : Motor Control Example



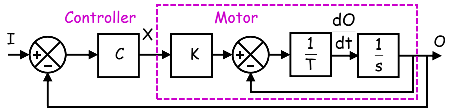
More difficult if form differential equation :

$$\frac{dO}{dt} = \frac{(I - O) * C * K - O}{T} = \frac{C * K * I - (1 + C * K) * O}{T}$$

Get in form $\frac{K' * I - O}{T'}$ by dividing by 1 + C*K

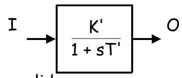
$$\frac{dO}{dt} = \frac{\frac{C * K}{1 + C * K} * I - * O}{\frac{T}{1 + C * K}} \qquad K' = \frac{C * K}{1 + C * K};\ T' = \frac{T}{1 + C * K}$$

Simpler using Laplace



$$\frac{O}{I} = \frac{\text{Forward}}{1 - \sum \text{Loops}} = \frac{\frac{CK}{Ts}}{1 - -\frac{CK}{Ts} - -\frac{1}{Ts}} = \frac{CK}{sT + CK + 1} = \frac{\frac{CK}{1 + CK}}{s\frac{T}{1 + CK} + 1}$$

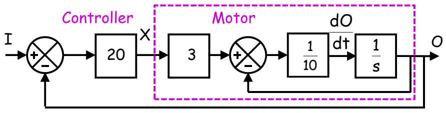
So can redraw system as



Where K' and T' are as on previous slide

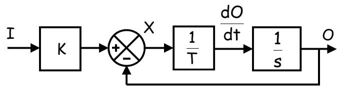
Then, if I unit step and O = 0 at t = 0: $O = K' - K'e^{-t/T'}$

Lecture 16 - In Class Exercise



Find O/I and hence final value K & time constant T if I is unit step

Finding the System Response



$$\frac{O}{I} = \frac{K}{1 + sT}$$

If I is a unit step, we know that $O = K – K\,e^{-t/T}$

But how do we get this expression for O?

Answer – we analyse the transfer function

Concept applies to more complicated systems too

Finding the Step Response

Key point, O has two components $\qquad \frac{O}{I} = \frac{K}{1 + sT}$

steady state , K and transient, $-K\,e^{-t/T}$

Can find both from TF

Steady state found by setting s to 0 (ie no change)

$O_{ss} = \frac{K}{1 + 0} * I = K$ (if I unit step)

Transient : find 'value of s' (root) so that denominator is 0:

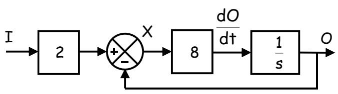
sT + 1 = 0 if s = -1/T

Then transient is const * exp(st) = c * exp(-t/T)

So O = K + c exp(-t/T);

If O = 0 at t = 0; 0 = K + c so c = -K, and hence $O = K – K\,e^{-t/T}$

Example



K = 2 T = 1/8

I is unit step

O = 0 at t=0

$$\frac{O}{I} = \frac{\frac{16}{s}}{1 - -\frac{8}{s}} = \frac{16}{s + 8} \text{ or } \frac{2}{1 + s/8}$$

Steady state is 2/1 = 2; Denominator is 0 when s = -8

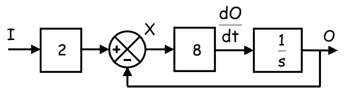
Hence $O = 2 + c\,e^{-t*8}$

At t = 0, 0 = 0, so $0 = 2 + c\,e^{0} = 2 + c$; so c = -2

Hence, complete response is

$O = 2 – 2\,e^{-t*8}$

Relating to Solving Differential Eqn



$$\frac{dO}{dt} = 8*(2*I - O)$$

$$\frac{dO}{dt} + 8O = 16I = 16$$

Mathematicians find 'particular integral', as I is a constant..

O_{PI} found by $8O_{PI} = 16$; $O_{PI} = 2$

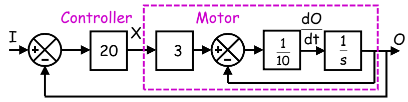
For 'complementary function', find root of auxiliary equation

$m + 8 = 0$, so $m = -8$; $O_{CF} = c\,e^{-t8}$

Hence $O = 2 + c\,e^{-t8}$

In effect doing same method when using transfer function

Control System



In exercise found $\frac{O}{I} = \frac{60/61}{1 + s\,10/61} = \frac{0.984}{1 + s0.164}$

Set s to 0, $O_{ss} = 0.984$

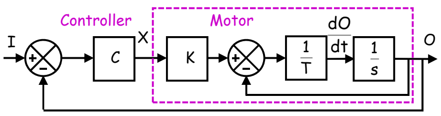
Roots of denominator is $s = -1/0.164 = -6.1$, so $O_t = c\,e^{-t*6.1}$

So $O = 0.984 + c\,e^{-t*6.1}$

If O = 0, at t = 0, 0 = 0.984 + c

So $O = 0.984 - 0.984\,e^{-t*6.1}$

P + I Control of Motor



$$\frac{O}{I} = \frac{\frac{CK}{1+CK}}{1 + s\frac{T}{1+CK}}$$

If I is unit step, $O_{ss} = \frac{CK}{1 + CK}$ which is close to 1 only if CK big

We noted last term integral or proportional plus integral control..

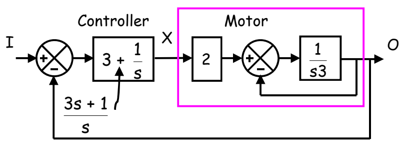
Make $C = C_1 + C_2 \int$, so O at steady state if X is constant

Integrator output constant if its input is 0 : ie O = I

Using s, $C = C_1 + C_2 \frac{1}{s} = \frac{sC_1 + C_2}{s}$

P+I + Feedback Control

Consider : numerically contrived example



$$\frac{O}{I} = \frac{\frac{3s+1}{s}*2*\frac{1}{3s}}{1 - -\frac{3s+1}{s}*2*\frac{1}{3s} - -\frac{1}{3s}} = \frac{\frac{6s+2}{3s^2}}{1 + \frac{6s+2}{3s^2} + \frac{1}{3s}} = \frac{6s+2}{3s^2 + 6s + 2 + s}$$

$$= \frac{2(3s+1)}{s(3s+1) + 2(3s+1)} = \frac{2}{s+2} = \frac{1}{s/2 + 1}$$

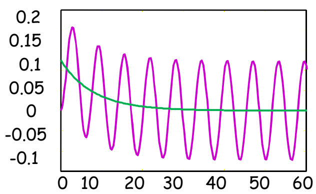
If I unit step, $O = 1 - e^{-2t}$ reaches 63% of 1 at time 1/2s

What if Input is Sinusoid?

Output = transient + steady state

Transient again found by root of denominator of transfer function

Decays to 0



Steady state : replace $s = j\omega$ in TF, find gain and phase

$$O_{ss} = \frac{K}{\sqrt{1+\omega^2T^2}}\sin(\omega t - \phi) \quad \text{where } \phi = \tan^{-1}(\omega T)$$

Much easier than finding particular integral

Example – not examinable (in Feedback)

$$\frac{O}{I} = \frac{60/61}{1+s\,10/61} = \frac{0.984}{1+s0.164}$$

Let I = sin(5t) and O = 0 at t = 0

$$\frac{O}{I} = \frac{0.984}{1+j\omega0.164} = \frac{0.984}{1+j0.82} \qquad \angle\frac{O}{I} = 0 - \tan^{-1}0.82 = -0.687 \text{ rad}$$

$$\left|\frac{O}{I}\right| = \frac{0.984}{\sqrt{1+0.82^2}} = \frac{0.984}{\sqrt{1+0.6724}} = \frac{0.984}{1.293} = 0.761$$

$O_{ss} = 0.761 \sin(5t - 0.687)$ $O_t = c\,e^{-t*6.1}$ as before

$O = 0.761 \sin(5t - 0.687) + c\,e^{-t*6.1}$

At t = 0, 0 = 0.761 sin(-0.687) + c = -0.483 + c , so c = 0.483

So $O = 0.761 \sin(5t - 0.687) + 0.483\,e^{-t*6.1}$

Summary

We have seen how the s operator can be used to analyse systems

We replace integrators by 1/s, and then use forward/1-loop(s)

From the resultant transfer function we can find step response

- Roots of denominator give transient
- Setting s to zero gives steady state if input is step
- Modulus/Argument gives steady state if input sinsuoid

The analysis has been a little informal : fuller information in Part 2.

Next week look at simulating systems –

- Initially first order systems we have met
- Then we consider second order systems

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17 : Simulating Systems

We have seen how to use s and jω to analyse first order systems

Form transfer functions/differential equations, and solve.

In the first half of this lecture, see how computer can solve

In the second half, we start to look at second order systems

On Simulation

We find the differential of a variable and numerically integrate it

Specifically $\frac{dO}{dt}$ = *function*(O and others) : find O

We will look at a simple computer simulation

Then how MATLAB has a better method

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Simulation of First Order System

Our system can be modelled by differential eqn

$$\frac{dO}{dt} = \frac{1}{T}(K*I - O)$$

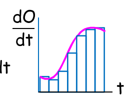
K and T are constants
I = input, O = output

To simulate, calculate values at regular times : the sampling instants

Specifically we find dO/dt then 'integrate' to get O

Simplest method (Euler's method) is as follows

initialise O
For each time step
  Calculate dO/dt (using O, I, K and T)
  new O = current O + sample time * dO/dt



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MatLab Implementation

Have matrix for signals: column 1 = time, col 2 = dO/dt, col 3 = O

Illustrate calculations for K = 2, T = 5, Ts = 0.5, O init = 0

Matrix S to 0, S[1,3]=0

0	0	0
0	0	0
0	0	0
0	0	0

First Iteration
S[2,1] = Ts*1
S[2,2] = dO/dt
 = (K*I – S[1,3])/T
S[2,3] = O + dO/dt*Ts
 = S[1,3] + S[2,2]*Ts

0	0	0
0.5	0.4	0.2
0	0	0
0	0	0

Next Iteration
S[3,1] = Ts*2
S[3,2] = dO/dt
 = (K*I – S[2,3])/T
S[3,3] = O + dO/dt*Ts
 = S[2,3] + S[3,2]*Ts

0	0	0
0.5	0.4	0.2
1.0	0.36	0.38
0	0	0

Next

0	0	0
0.5	0.4	0.2
1.0	0.3	0.38
1.5	0.324	0.542

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MATLAB Function to do this

```
function sigs = ktsim (K, T, I, Oinit, tsamp, numpts);
% sigs = KTSIM (K, T, I, TSAMP, NUMPT)
% Simulates system described by dO/dt = (K * I - O) / T
% returns sigs = matrix [t, dO/dt and O] at each calculation step
%    first column has successive t's, next has dO/dt and last has O
% Prof Richard Mitchell 29/3/11
sigs = zeros (numpts+1, 3);                % initialise mat to zero
sigs(1,3) = Oinit;                         % initialise output
for ct = 1:numpts
   sigs(ct+1, 1) = ct * tsamp;             % store time in col 1
   sigs(ct+1, 2) = (K * I – sigs(ct, 3)) / T;    % dO/dt in col 2
   sigs(ct+1, 3) = sigs(ct, 3) + sigs(ct+1, 2) * tsamp;   % output, col 3
end
```

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Using that MATLAB function

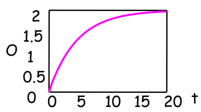
```
>> sigs = ktsim (K, T, I, Oinit, tsamp, numpts);
```

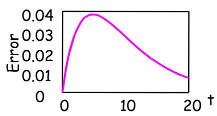
K, T, I are obvious; Oinit is value of O at t=0

numpts iterations are done, at instants tsamp apart

sigs is matrix: col 1 = sigs(:,1) has time; col 2 has dO/dt; col3 has O

```
>> sigs = ktsim(2, 5, 1, 0, 0.5, 40);
>> ythy = 2 – 2 * exp(-sigs(:,1)/5);     % ie 2 – 2 exp(-t/5)
>> plot (sigs(:,1), sigs(:,3));     plot(sigs(:,1), sigs(:,3)-ythy);
```





4% max error

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Solving ODEs in MATLAB

Model is an 'ordinary' differential equation, ODE.

The Euler numerical solution has a maximum error of 4%

Likely to be higher for more advanced systems

Better to using MATLAB's ODE45 function

(uses so called 4th order Runge Kutta to solve ODE)

For which you write an m-file which returns do/dt at an instant

```
function dobydt = firstorder (t, o, flag, I, K, T)
% Function to calculate do/dt for first order system
% t is time, o is current output, flag is dummy variable
% I is input, K and T are final value and time const
dobydt = (I * K - o) / T;
```


Then Run ODE45 from>> prompt

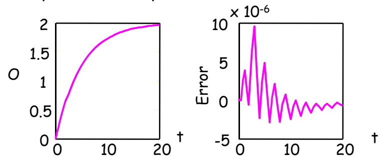
```
>> [t,y] = ode45('firstorder', [0 20], 0,[], 1, 2, 5);
```

call ode45 with name of do/dt file,

[0 20] means run from t = 0 to 20,

0 is initial value of O, [] is dummy, 1 2 5 are I, K, T

```
>> plot(t,y);                          % to plot how y varies with t
>> plot(t, y - (2 - 2 * exp(-t/5) ) ); % Plot error also
```



y – ytheory Much smaller

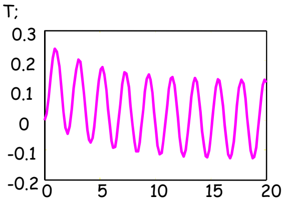
M-file using t – if input is sin(wt)

```
function dobydt = firstsin (t, o, flag, w, K, T)
% Function to calculate do/dt for first order system
% t is time, o is current output
% w is ang freq of input, K and T are paras of system
% Prof Richard Mitchell, 31/3/11
dobydt = (sin(w*t) * K - o) / T;
```

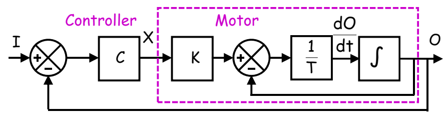
```
>> [t,y]=ode45('firstsin',
   [0 20], 0,[], 3, 2, 5);
>> plot(t,y)
```

NB y of form

$A\sin(wt-B) + Ce^{-t/T}$



Lecture 17 – In Class Exercise

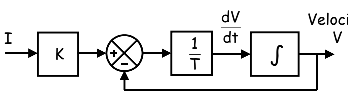


Complete the following function so as to simulate the above

```
function dobydt = firstorder (t, o, flag, I, C, K, T)
% Function to calculate do/dt for motor control system
```


Second Order Systems – 2 integrators

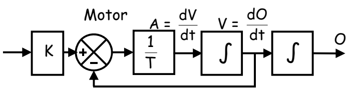
A first order system has one integrator ... Eg for motor velocity



Write m file return dV/dt

$$\frac{dV}{dt} = \frac{K*I - V}{T}$$

To model output position, O, add integrator – a second order system



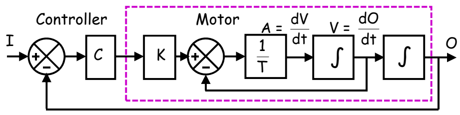
$$\frac{dV}{dt} = \frac{K*I - V}{T}$$

$$\frac{dO}{dt} = V$$

Control of Motor Position

Another second order system – one which controls motor position

Take motor position model, put in feedback system



Describe by two equations In terms of O and V

$$V = \frac{dO}{dt} \qquad \frac{dV}{dt} = \frac{C*K*(I-O) - V}{T}$$

We simulate by writing an m file to return both dV/dt and dO/dt

Simulate – again using ode45

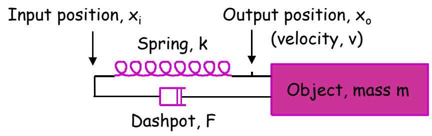
Now have vector for O and V and see how both change ...

$$ov = \begin{bmatrix} O \\ V \end{bmatrix} \qquad \frac{dov}{dt} = \begin{bmatrix} \frac{dO}{dt} \\ \frac{dV}{dt} \end{bmatrix} = \begin{bmatrix} V \\ \frac{CK(I - O) - V}{T} \end{bmatrix}$$

In MatLab, ov is vector = [O; V] . ov(1) is O, ov(2) is V ... then

```
function dovbydt = motorpos (t, ov, flag, I, C, K, T)
% Function to calculate do/dt for first order system
% t is time, ov(1) is output O, ov(2) is velocity V;  flag is dummy
% I is input, C  is controller, K and T motor parameters
dovbydt = [ov(2); ( (I – ov(1) ) * C * K – ov(2)) / T];
```


Another Example - Mass Spring System



This has similarities with a car suspension

Pull one end of spring, object at other end moves, friction exists

Spring extended : force generated is $k * (x_i - x_o)$

Friction force opposes this, is F * v   v is diff. of x_o

Net Force is thus $k * (x_i - x_o) - F * v$

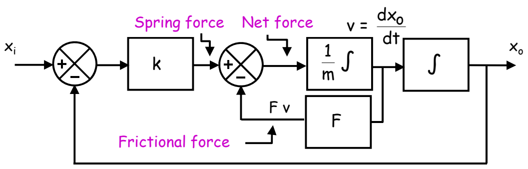
This must equal m * acceleration = m dv/dt

Block Diagram and ODEs

Have x_i, assume x_o, $(x_o - x_i) * k$ gives Spring force SF

Assume velocity, get frictional force FF , then net force = SF- FF

Divide by m and integrate to get velocity; integrate again for x_o



$$\frac{dx_o}{dt} = v; \text{ and } \frac{dv}{dt} = \frac{\text{Spring F - Frictional F}}{m} = \frac{k(x_i - x_o) - Fv}{m}$$

MatLab Code

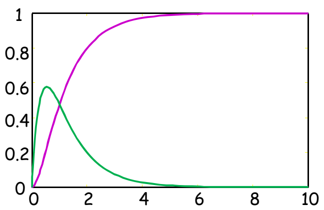
$$\frac{dx_o}{dt} = v; \text{ and } \frac{dv}{dt} = \frac{k(x_i - x_o) - Fv}{m}$$

```
function dxvbydt = massspring (t, xv, flag, xi, k, F, m)
% Function to calculate dxv/dt for first order system
% t is time, xv(1) is output xo, xv(2) is velocity;  flag is dummy
% xi is input, k is spring constant; F is friction; m is mass
dxvbydt = [xv(2); (k * (xi - xv(1)) - F * xv(2)) / m];
```

Now to use m file, call ODE45 at prompt:

Have defined m file, now use it

```
>> [t,xv] = ode45('massspring', [0 10], [0;0], [], 1, 3, 4, 1);
>> plot(t, xv);                    % plots xo v t and v vs t
```



$$\frac{dx_o}{dt} = v$$

$$\frac{dv}{dt} = \frac{3(x_i - x_o) - 4v}{1}$$

As $t \to \infty$, $v \to 0$ then

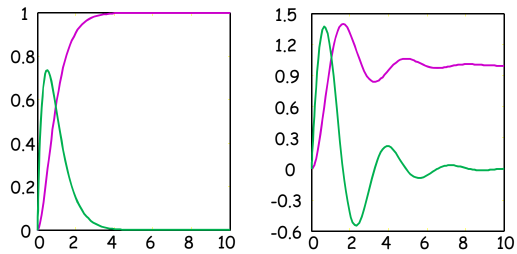
$$0 = \frac{3(x_i - x_o) - 0}{1}$$

So $x_i = x_o$

Looks like x_o very close to 1 by t = 6

Two other results

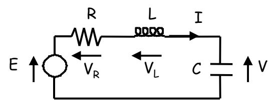
k now 4 (not 3); F still 4, m still 1   k still 4; F now 1.12, m still 1



x_o very close to 1 at t = 4 : faster   Damped oscillation ...

We will explain these different responses next week

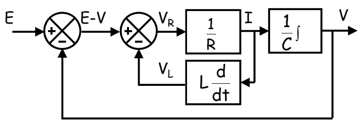
Another example – L C R circuit



$I = \frac{V_R}{R} = \frac{E - V - V_L}{R}$

$V_L = L\frac{dI}{dt} = L\,sI$

$V = \frac{1}{C}\int I\,dt = \frac{I}{sC}$



Again need 2 variables

Say V and I

$\frac{dV}{dt} = \frac{1}{C}I \qquad V_R = IR = E - V - L\frac{dI}{dt} \Rightarrow \frac{dI}{dt} = \frac{1}{L}(E - V - IR)$

In MatLab

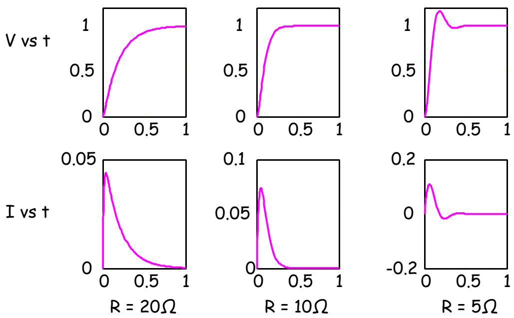
Again have 2 element vectors

Now variable is vector $\begin{bmatrix} V \\ I \end{bmatrix}$ and m-file finds $\begin{bmatrix} \frac{dV}{dt} \\ \frac{dI}{dt} \end{bmatrix} = \begin{bmatrix} \frac{1}{C} * I \\ \frac{E - V - I*R}{L} \end{bmatrix}$

In MatLab, vi is vector, vi = [V; I]

```
function dvibydt = LCRCircuit (t, vi, flag, E, R, C, L)
% Function to calculate dvi/dt for circuit
% t is time, vi(1) is  voltage V, vi(2) is current I;  flag is dummy
% E is input, R, C and L are components
dvibydt = [vi(2) / C; (E – vi(1) – vi(2)*R) / L];
```


Some Runs: C = 0.01F; L = 0.25H



Summary

We have seen how to simulate systems

- Simple Euler integration is ok, but there are errors
- The ode45 function is much more accurate
- You just need to write a function define dO/dt

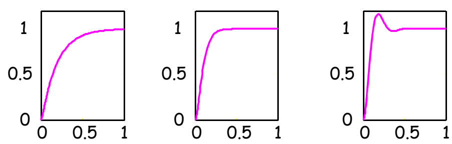
We have looked at some second order systems

- Different examples have been described
- Shown how ode45 simulates these – as two first orders
- Whilst first order step response has same shape, different shapes occur for second order
- Next week we find out why …

18 – Second Order Systems vs Time

Last week we looked at some second order systems,

We saw different step responses depending on parameter values

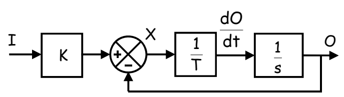


This week we find out why.

By extending the analysis of first order systems, where

steady state and transient responses found from transfer function

Reminder – first order

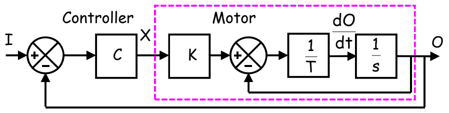


$\frac{O}{I} = \frac{K}{1 + sT}$

Assuming I is a unit step :

- Steady state is found by setting s to 0 (ie no change) = K/(0+1)
- Transient : find 'value of s' (root) where denominator is 0:
  - Solve 1 + s T = 0   ie s = -1/T
  - Then transient is const * exp(st) = c * exp(–t/T)
    - So O = K + c exp(-t/T);
- If O = 0 at t = 0; 0 = K + c so c = -K, and hence $O = K - K\,e^{-t/T}$

And for this in control system



$$\frac{O}{I} = \frac{CK\frac{1}{T}\frac{1}{s}}{1 - -CK\frac{1}{T}\frac{1}{s} - -\frac{1}{T}\frac{1}{s}} = \frac{CK}{sT + CK + 1} \qquad O_{ss} = \frac{CK}{0 + CK + 1} = \frac{CK}{1 + CK}$$

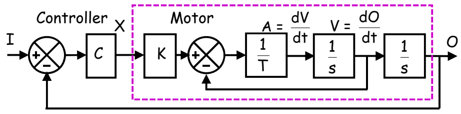
Denominator root where sT + CK + 1 = 0: ie $-\frac{CK+1}{T}$ $\quad O_t = c \exp^{-\frac{CK+1}{T}t}$

$$\text{So } O = \frac{CK}{1+CK} + c \exp^{-\frac{CK+1}{T}t}$$

Example Second Order System

Consider control system for position of motor

Extra integrator means system second order



$$\frac{O}{I} = \frac{\text{Forward}}{1 - \sum \text{loops}} = \frac{C*K*\frac{1}{T}*\frac{1}{s}*\frac{1}{s}}{1 - -C*K*\frac{1}{T}*\frac{1}{s}*\frac{1}{s} - -\frac{1}{T}*\frac{1}{s}} = \frac{C*K}{Ts^2 + C*K + s}$$

Mult by Ts^2

Second Order Unit Step Response

$$\frac{O}{I} = \frac{C*K}{Ts^2 + s + C*K}$$

Steady State, set s to 0 (as I step), in transfer function

$$O_{ss} = \frac{CK}{T*0 + 0 + CK} = 1$$

Transients found by values of s where denominator is 0 (roots)

Three response types, depending on roots of $Ts^2 + s + C*K = 0$

If 1 > 4CKT, two real roots,	overdamped response
If 1 = 4CKT, two identical roots,	critically damped
If 1 < 4CKT, two complex roots,	underdamped response

Example Overdamped Response

If T = 0.01, C = 9, K = 1; $(0.01s^2 + s + 9) = (0.1s + 1)(0.1s + 9)$

Transfer function can be expressed as $\frac{K_1}{0.1s + 1} + \frac{K_2}{0.1s + 9}$

Roots are $-\frac{1}{0.1} = -10$ and $-\frac{9}{0.1} = -90$;

Each term contributes an exponential

So Transient has form $K_1e^{-10t} + K_2e^{-90t}$

Thus complete response is

$$O(t) = 1 + K_1e^{-10t} + K_2e^{-90t}$$

Graph and Full Response

$O(t) = 1 + K_1e^{-10t} + K_2e^{-90t}$

As $t \to \infty$, exponentials decay, $O(t) \to 1$

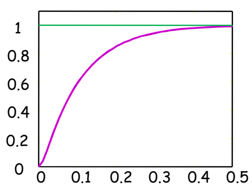
K's depend on initial conditions,

but if $O = \frac{dO}{dt} = 0$ at t = 0

$$O(t) = 1 - \frac{9}{8}e^{-10t} + \frac{1}{8}e^{-90t}$$

NB, slope of curve = 0 at t = 0

Also, O never exceeds O_{ss}



For interest: Finding Values of Ks

$O(t) = 1 + K_1e^{-10t} + K_2e^{-90t}$

Assume $O = \frac{dO}{dt} = 0$ at t = 0

$$\frac{dO(t)}{dt} = -10\ K_1e^{-10t} - 90\ K_2e^{-90t}$$

At t = 0, $O(t) = 1 + K_1 + K_2 = 0$

At t = 0, $\frac{dO(t)}{dt} = -10\ K_1 - 90\ K_2 = 0$

$K_1 = -9\ K_2$

So $1 - 9K_2 + K_2 = 0$

Hence $K_2 = \frac{1}{8}$ and so $K_1 = -\frac{9}{8}$

Lecture 18 – In Class Exercise

$$\frac{O}{I} = \frac{C^*K}{Ts^2 + s + C^*K} \quad \text{where } T = 0.02, C = 6, K = 2$$

Denominator is $(0.02s^2 + s + 12) = (0.2s + 4)(0.1s + 3)$

Determine O if I is unit step in form $x + K_1 e^{-yt} + K_1 e^{-zt}$

NB find x, y and z, but not K's

A Critically Damped Response

If $T = 1/36$, $C = 9$, $K = 1$; $(s^2/36 + s + 9) = (s/6 + 3)^2$

Transfer function in form $\frac{K_1}{s/6 + 3} + \frac{K_2}{(s/6 + 3)^2}$

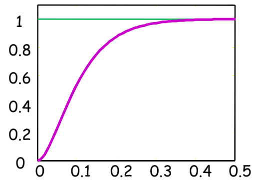
Repeated root of -18;

Transient of form $K_1e^{-18t} + K_2te^{-18t}$

So $O(t) = 1 + (K_1 + K_2t)e^{-18t}$

If $O = \frac{dO}{dt} = 0$ at $t = 0$ can show

$O(t) = 1 - (1 + 18^*t)\, e^{-18t}$



O never exceeds O_{ss} Fastest response without exceeding O_{ss}.

Underdamped Response

If $T = 1/4$, $C = 10$, $K = 1$; Denominator $s^2/4 + s + 10$

By completing the square this is $(s/2 + 1)^2 + 3^2$

Hence have complex roots $-2 \pm j6$;

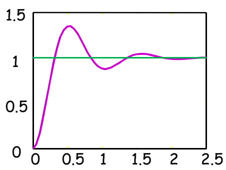
For why see after last slide

Hence $O(t) = 1 + K_1e^{-2t}\cos(6t) + K_2e^{-2t}\sin(6t)$

K's depend on initial conditions,

but if $O = \frac{dO}{dt} = 0$ at $t = 0$

$O(t) = 1 - e^{-2t}\cos(6t) - \frac{1}{3}e^{-2t}\sin(6t)$



O exceeds O_{ss} before oscillating back

Faster rise than critically damped

Oscillations decay away – they are damped.

Even More Underdamped

If $T = 1/4$, $C = 50$, $K = 1$; $(s^2/4 + s + 50) = (s/2 + 1)^2 + 7^2$

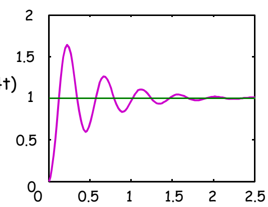
Has complex roots : $-2 \pm j14$;

Transient of form $K_1e^{-2t}\cos(14t) + K_2e^{-2t}\sin(14t)$

So $O(t) = 1 + K_1e^{-2t}\cos(14t) + K_2e^{-2t}\sin(14t)$

If $O = \frac{dO}{dt} = 0$

$O(t) = 1 - e^{-2t}\cos(14t) - \frac{1}{7}e^{-2t}\sin(14t)$



Higher C gain, faster rise, but more oscillatory, longer to settle

How underdamped

$s^2/4 + s + 10$ has roots $-2 \pm j6$ slightly underdamped

$s^2/4 + s + 50$ has roots $-2 \pm j14$ more underdamped

$s^2T + s + CK$ underdamped if $1 < 4CKT$

The more $1 <$ 4CKT, the more underdamped the response

So, for control system, if given K and T, system can be made less oscillatory by reducing C.

Let's consider another example where there are oscillations

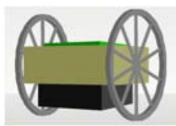
And how these are damped ...

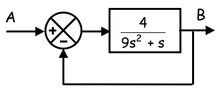
Rocking Robot

When ERIC accelerates, it pivots about axis between wheels

A = acceleration B = body angle

Model is in effect motor inertia giving velocity, integrated for position in a loop





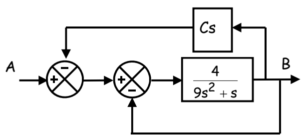
$$\frac{B}{A} = \frac{4}{9s^2 + s + 4}$$

Clearly $1 \ll 36$, so lots of oscillation

Continued

Solution is to measure the angle of the board and feedback

Key – feedback change in angle ... Differentiated angle



$$\frac{B}{A} = \frac{\frac{4}{9s^2 + s}}{1 - -\frac{4}{9s^2 + s} - -\frac{Cs}{9s^2 + s}}$$

$$\frac{B}{A} = \frac{4}{9s^2 + s + 4 + Cs}$$

If want to stop oscillation ... Make critically damped

$(C+1)^2 = 4*4*9$ ... $C+1 = 12$ ... $C = 11$.

Summary

In this lecture : the step response of second order systems.

Again, steady state response is found by setting s to 0.

And transient is determined by the roots of the denominator

- Overdamped (does not exceed Oss) if two real roots
- Critically damped (fastest for no overshoot) if repeated roots
- Underdamped (goes pass Oss, oscillates) if complex roots

Saw also how velocity feedback can dampen oscillations

$\frac{O}{I} = \frac{40}{s^2 + 7s + 10}$ I unit step. What is O_{ss} and what form is O_t?

Next week – formalise damping – and relate to Q factor

For Interest – why this answer

If T = 1/4, C = 10, K = 1; Denominator $s^2/4 + s + 10 = (s/2 + 1)^2 + 3^2$

Hence have complex roots $-2 \pm j6$; so transient $K e^{-2 \pm j6}$

But $e^{(-2 + j6)t} = e^{-2t}(\cos 6t + j \sin 6t)$

Suggesting transient is

$K_1' e^{-2t}(\cos(6t) + j \sin(6t)) + K_2' e^{-2t}(\cos(6t) - j \sin(6t))$

But must be purely real.

Can show (next slide) $K_1 = \frac{1}{2}\left(K_1' + K_2'\right)$ and $K_2 = \frac{1}{2}\left(jK_1' - jK_2'\right)$

So transient of form $K_1 e^{-2t}\cos(6t) + K_2 e^{-2t}\sin(6t)$

Hence $O(t) = 1 + K_1 e^{-2t}\cos(6t) + K_2 e^{-2t}\sin(6t)$

Why these values of K_1 and K_2

$K_1' e^{-2t}(\cos(6t) + j \sin(6t)) + K_2' e^{-2t}(\cos(6t) - j \sin(6t))$ must be real

Assume $K_1' = a + jb$ and $K_2' = c + jd$, ignore common e^{-2t} term:

$(a+jb)(\cos(6t) + j \sin(6t)) = a \cos(6t) - b \sin(6t) + jb \cos(6t) + ja \sin(6t)$

$(c+jd)(\cos(6t) - j \sin(6t)) = c \cos(6t) + d \sin(6t) + jd \cos(6t) - jc \sin(6t)$

Add these. Result must be real, so b = -d and a = c, then get

$a \cos(6t) - b \sin(6t) + c \cos(6t) + d \sin(6t) = 2a \cos(6t) + 2b \sin(6t)$

Hence $K_1' = a + jb$ and $K_2' = a - jb \Rightarrow K_1' + K_2' = 2a$ and $K_1' - K_2' = 2jb$

So indeed $K_1 = \frac{1}{2}\left(K_1' + K_2'\right)$ and $K_2 = \frac{1}{2}\left(jK_1' - jK_2'\right)$

So transient is $K_1 e^{-2t}\cos(6t) + K_2 e^{-2t}\sin(6t)$

19 : Damping and Second Order Freq

We saw last week the step response of second order system, eg

$$\frac{O}{I} = \frac{C*K}{Ts^2 + s + C*K}$$

From the transfer function we can

- easily assess the steady state output : CK/CK = 1
- the transient response varies depending on the TF denominator
  - two real roots, one repeated root or complex roots

This week we assess these responses using 'damping ratio'

This in fact relates to the Q factor used in electronics

- Which naturally leads to second order frequency response

Transfer Function and Damping

$$\frac{O}{I} = \frac{C*K}{Ts^2 + s + C*K}$$

We have had over-, critically- and under- damped systems.

Control Engineers express systems in terms of 'damping' ratio ζ

$$\frac{O}{I} = \frac{\frac{CK}{T}}{s^2 + \frac{1}{T}s + \frac{CK}{T}} = \frac{K'}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \omega_n = \sqrt{\frac{CK}{T}}; \ \zeta = \frac{1}{\sqrt{4CKT}}$$

ζ is 1 when 4CKT = 1, system critically damped : denom $(s + \omega_n)^2$

If $\zeta > 1$, system is overdamped : denom has form $(s + a)(s + b)$

If $\zeta < 1$, system is underdamped : roots of denom are complex

As we shall see, this relates to Q factor in electronics

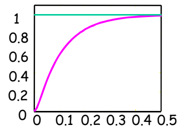
Damping Ratio For Last Week's Examples

T = 0.01, C = 9, K = 1;

$$\frac{900}{s^2 + 100s + 900}$$

$$\zeta = \frac{1}{\sqrt{4*9*1*0.01}} = \frac{1}{0.6} = \frac{5}{3};$$

$$\omega_n = \sqrt{900} = 30$$



No oscillations, Slowest to Rise

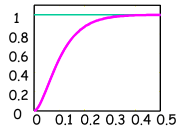
Approach Steady State at 0.5s

T = 1/36, C = 9, K = 1;

$$\frac{324}{s^2 + 36s + 324}$$

$$\zeta = \frac{1}{\sqrt{4*9*1/36}} = 1;$$

$$\omega_n = \sqrt{324} = 18$$



Fastest with no oscillations

Approach SS at 0.4s

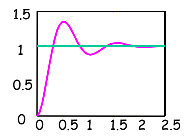
UnderDamped Examples

T = 1/4, C = 10, K = 1;

$$\frac{40}{s^2 + 4s + 40}$$

$$\zeta = \frac{1}{\sqrt{4*10*1*1/4}} = \frac{1}{\sqrt{10}} = 0.316$$

$$\omega_n = \sqrt{40} = 6.32 \text{ rad/s}$$



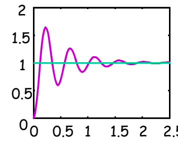
Faster to rise, but oscillates

T = 1/4, C = 50, K = 1;

$$\frac{200}{s^2 + 4s + 200}$$

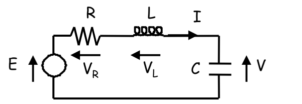
$$\zeta = \frac{1}{\sqrt{4*50*1*1/4}} = \frac{1}{\sqrt{50}} = 0.14$$

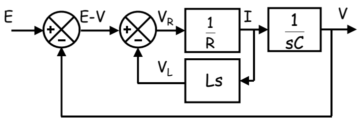
$$\omega_n = \sqrt{200} = 14.1 \text{ rad/s}$$



Faster, more oscillatory

LCR Circuit





$$\text{So } \frac{V_o}{V_s} = \frac{\text{Forward}}{\text{1-Loops}} = \frac{\frac{1}{R} * \frac{1}{sC}}{1 - -\frac{1}{R} * \frac{1}{sC} - -sL\frac{1}{R}} = \frac{1}{sCR + 1 + s^2LC}$$

In Class Exercise

$$\text{LCR } \frac{V_o}{V_s} = \frac{1}{s^2LC + sCR + 1} = \frac{1/LC}{s^2 + sR/L + 1/LC}$$

$$\omega_n^2 = \frac{1}{LC}; \text{ so } \omega_n = \sqrt{\frac{1}{LC}}; \; 2\zeta\omega_n = \frac{R}{L}; \text{ so } \zeta = \frac{R}{2L\omega_n} = \frac{R}{2}\sqrt{\frac{C}{L}}$$

For the following, calculate ω_n and ζ and state if overdamped, etc

L	C	R	$\omega_n = \frac{1}{\sqrt{LC}}$	$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$	Status
400mH	2.5μF	200Ω			
400mH	2.5μF	800Ω			
400mH	2.5μF	40Ω			
400mH	2.5μF	3200Ω			

Frequency Response of RC Circuit

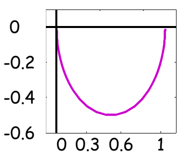
$$\frac{V_o}{V_s} = \frac{1}{1 + j\omega CR}$$

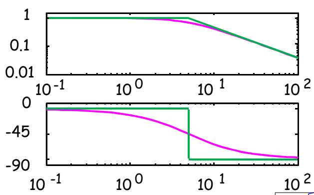
Freq Resp : how varies with ω, polar/Bode plot

For Bode, plot log(gain) vs log(ω) and phase vs log(ω)

Saw asymptotic approximations before/after corner freq 1/CR

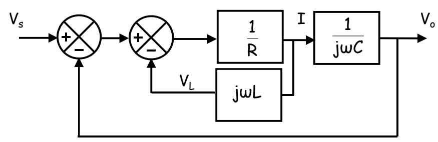
At Corner Freq, phase half way between 0 and -90… 1 = ωCR





Second Order Frequency Response

Now consider LCR circuit: earlier diagram but ∫ or 1/s = 1/jω



$$\frac{V_o}{V_s} = \frac{\frac{1}{R} * \frac{1}{j\omega C}}{1 - -\frac{1}{R} * \frac{1}{j\omega C} - -j\omega L\frac{1}{R}} = \frac{1}{j\omega CR + 1 + j^2\omega^2LC} = \frac{1}{1 - \omega^2LC + j\omega CR}$$

$$\text{At low } \omega, \; \frac{V_o}{V_s} = \frac{1}{1} = 1\angle 0; \text{ At high } \omega \; \frac{V_o}{V_s} = \frac{1}{j^2\omega^2LC} = \frac{1}{\omega^2LC}\angle -180$$

Corner Frequency and Q

$$\frac{V_o}{V_s} = \frac{1}{1 - \omega^2 LC + j\omega CR}$$

Consider what happens when $1 - \omega^2 LC = 0$ or $\omega = \frac{1}{\sqrt{LC}}$

$$\frac{V_o}{V_s} = \frac{1}{j\omega CR} \text{ so Gain} = \frac{1}{\omega CR} \text{ Phase} = -90^0$$

Phase half way between 0 and -180 … so this is corner freq
(first order system, corner freq where phase half way 0..-90)

Specifically $\omega = \frac{1}{\sqrt{LC}}$, Gain $= \frac{1}{\omega CR} = \frac{1}{R}\sqrt{\frac{L}{C}} = Q$; Phase $= -90^0$

This is angular frequency ω_o, when gain is Q (quality factor)

Polar Plot

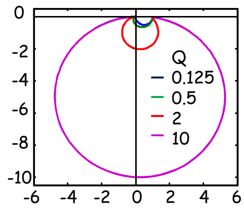
L	C	R	$\omega_n = \frac{1}{\sqrt{LC}}$	$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$
400mH	2.5μF	200Ω	1000 rad/s	2
400mH	2.5μF	800Ω	1000 rad/s	0.5
400mH	2.5μF	40Ω	1000 rad/s	10
400mH	2.5μF	3200Ω	1000 rad/s	0.125

```
>> tf = 1./(-L*C*w.^2 + w*C*R+1);
>> plot(real(tf),imag(tf)) ;
```

Plot varies 1 ∠ 0° .. 0 ∠ -180°

Q is gain when phase -90°

NB plot shape changes with Q



2nd Order Asymptotic Analysis

$$\frac{V_o}{V_s} = \frac{1}{1 - \omega^2 LC + j\omega CR}$$

At very low ω, $\frac{V_o}{V_s} \approx 1$, so Gain = 1. Phase = 0.

So Gain asymptote is straight line, slope 0

At very high ω, $\frac{V_o}{V_s} \approx \frac{1}{j^2\omega^2 LC}$; Gain $\frac{1}{\omega^2 LC}$ and Phase = -180^0

plot $\log\left(\frac{1}{\omega^2 LC}\right) = \log\left(\frac{1}{LC} * \omega^{-2}\right) = \log\left(\frac{1}{LC}\right) - 2\log(\omega)$

So Gain asymptote is straight line, slope -2

Asymptotes meet when $1 - \omega^2 LC = 0$: $\omega = \sqrt{\frac{1}{LC}}$

Bode Plot Asymptotes plus actual

Suppose

L = 400mH,

R = 200Ω and

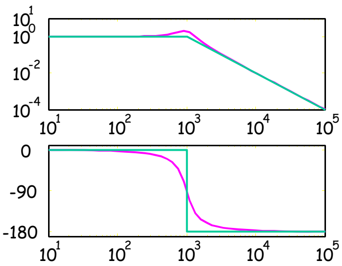
C = 2.5 μF ; (Q=2)

Asymptotes meet when

$\omega = \sqrt{\frac{1}{LC}}$

$= \frac{1}{\sqrt{400*10^{-3}*2.5*10^{-6}}}$

$= 10^3$

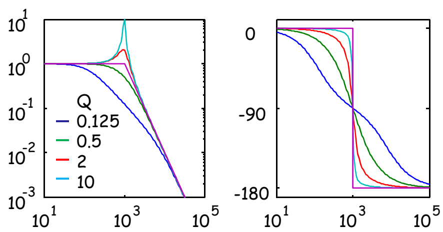


NB, here gain rises from initial value, as Q = 2. (earlier slide)

How Q affect Bode Plots

Q affects how gain/phase move between asymptotes

Plots below for same LCR examples given earlier



NB Gain = Q at ω_o

NB Peak gain before ω_o

Higher Q – higher peak … Faster change of phase.

On Bode Peaking and Underdamped

$$\frac{V_o}{V_s} = \frac{1}{1 - \omega^2 LC + j\omega CR}$$

When $1 - \omega^2 LC = 0$,

$$\frac{V_o}{V_s} = \frac{1}{j\omega CR} = \frac{\sqrt{LC}}{jCR} = \frac{1}{j}\sqrt{\frac{L}{CR^2}}$$

If $\sqrt{\frac{L}{CR^2}} > 1$, $\left|\frac{V_o}{V_s}\right|$ rises from 1 to a peak

To test if a system is underdamped we solve $s^2LC + sCR + 1 = 0$

Complex roots if $C^2R^2 < 4LC$ or $4\frac{L}{CR^2} > 1$ or $2\sqrt{\frac{L}{CR^2}} > 1$

So if $\sqrt{\frac{L}{CR^2}} > 1$, V_o will be underdamped

The bigger $\sqrt{\frac{L}{CR^2}}$ the bigger the peak, the more underdamped

Q factor and Damping Ratio

In Electronics, Q factor is used. In Control Damping ratio ...

In electronics, TF denominator of form $1 + j\omega / \omega_0 Q - \omega^2 / \omega_0^2$

For LCR $\quad \omega_0 = \frac{1}{\sqrt{LC}} \quad Q = \frac{1}{R}\sqrt{\frac{L}{C}}$

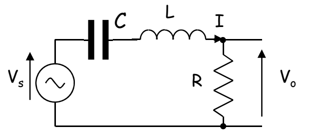
In control, express TF denominator as $1 + j2\zeta\omega/\omega_n - \omega^2/\omega_n^2$

For LCR $\quad \omega_n = \frac{1}{\sqrt{LC}} \quad \frac{2\zeta}{\omega_n} = 2\zeta\sqrt{LC} = CR$ so $\zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$

$\omega_n = \omega_0 \quad \zeta = \frac{1}{2Q} \quad \zeta$ is damping ratio

Underdamped if $\zeta < 1$. Note factor of 2.

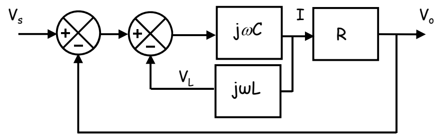
Band Pass Filter Example



$V_o = I*R$

$V_L = j\omega L * I$

$I = j\omega C*(V_s - V_o - V_L)$



So $\frac{V_o}{V_s} = \frac{\text{Forward}}{\text{1-Loops}} = \frac{j\omega C*R}{1 - j\omega C*R - j\omega L*j\omega L} = \frac{j\omega C*R}{1 + j\omega CR + j^2\omega^2 LC}$

Asymptotes

So $\frac{V_o}{V_s} = \frac{j\omega CR}{1 + j\omega CR - \omega^2 LC}$

At very low ω, $\frac{V_o}{V_s} \approx j\omega CR$. Gain asymptote slope +1

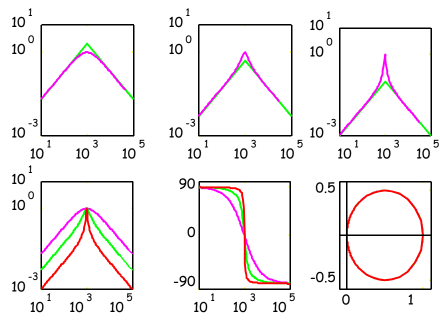
At very high ω, $\frac{V_o}{V_s} \approx \frac{j\omega CR}{j^2\omega^2 LC} = \frac{R}{j\omega L}$. Gain asymptote slope -1

Asymptotes meet when $\omega CR = \frac{R}{\omega L}$: $\omega = \omega_0 = \sqrt{\frac{1}{LC}}$

Then asymptotically gain $= \frac{R}{L} * \sqrt{LC} = R * \sqrt{\frac{C}{L}} = \frac{1}{Q}$

Actual gain at $\omega_o = 1$

Plots : same set of Q values

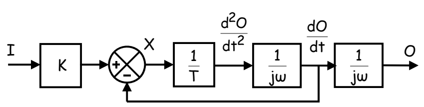


L = 400mH
C = 2.5μF
R = 800, 200 or 40Ω

Separate Gain plots + asymptotes

Combined gain, phase plots + polar

Frequency Model of Motor Position



$\frac{O}{I} = \frac{\text{Forward}}{\text{1 - loop}} = \frac{K * \frac{1}{T} * \frac{1}{j\omega} * \frac{1}{j\omega}}{1 - \frac{1}{T} * \frac{1}{j\omega}} = \frac{K}{(j\omega)^2 T + j\omega}$

Small ω : $\frac{O}{I} \sim \frac{K}{j\omega}$

Gain slope -1;

Phase is -90^O

Large ω : $\frac{O}{I} \sim \frac{K}{(j\omega)^2 T}$

Gain slope -2;

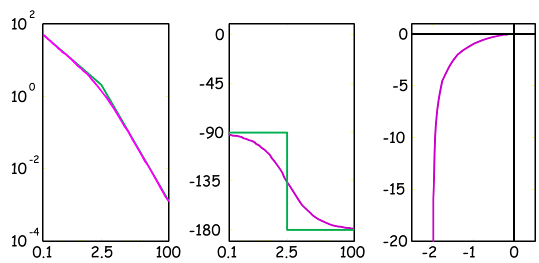
Phase is -180^O

Gain asyms meet

$\frac{K}{\omega^2 T} = \frac{K}{\omega}$

ie $\omega = 1/T$

Bode and Polar Plots K = 5, T = 0.4



As ω → 0, Gain → ∞, polar plot → ∞ on imaginary axis

Summary

We have looked more at second order systems

The step response has different forms

Which can be defined in terms of damping ratio

The frequency response also has different forms

Which can be defined in terms of Q factor

They are related : $Q = 1/2\zeta$

We have reaffirmed the asymptotic approximations

Used to plot the band pass filter

And motor position …

Next week we tidy up the course : modelling of animal systems

20 : Modelling Animal Systems

Cybernetics, control and communication in animal and machine…

Consider a Population model, if P is the population size:

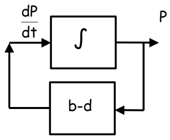
Change of Pop is a function of P : more P, more kiddies

$$\frac{dP}{dt} = P * (b - d)$$ b and d are birth and death rates

Population model is a feedback system with integrator.

If b > d, P increases without limit.

If b < d, P decreases to zero.



Using ODE45

As before, need function, say in file pop.m, to generate derivative

```
function dpbydt = pop (t, p, flag, b, d);
dpbydt = (b - d)*p;          % compute change in P
```

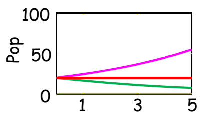
[flag is dummy value here, time t here not used]

Then invoke ode45 with m file, init p, max t and b and d:

```
>> [t1,p1] = ode45('pop', [0,5],20,[], 0.6, 0.4);    % b > d
>> [t2,p2] = ode45('pop', [0,5],20,[], 0.3, 0.5);    % b < d
>> plot(t1,p1,t2,p2);
```

If do so, get graph as shown

either $P \rightarrow 0$ or $P \rightarrow \infty$



More Complicated But Realistic Model

The above is very simple & unrealistic. Need better model

So define birth rate as $b_1 - b_2 * P$

And death rate as $d_1 + d_2 * P$.

Then model becomes $\frac{dP}{dt} = (b_1 - d_1 - (b_2 + d_2) * P) * P$

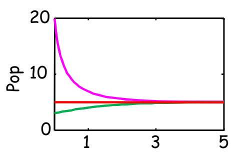
Population stablises when above = 0, ie when P = 0 (boring)

or $b_1 - d_1 - (b_2 + d_2)P = 0$ i.e. when $P = \frac{b_1 - d_1}{b_2 + d_2}$

```
function dpbydt = pop (t, p, flag, vals)   % m file
% vals has [b1,d1,b2,d2]
dpbydt = (vals(1) - vals(2) - (vals(3) + vals(4))*p) * p;
```


MATLAB Session + Results

```
>> v = [10, 9, 0.15, 0.05];             % define rates
>> [t1,p1]=ode45('pop', [0,5],20, v);   % run from 20
>> [t2,p2]=ode45('pop', [0,5],3, v);    % run from 3
>> plot(t1,p1, t2,p2);
```



As expected, P stabilises at (10 - 9)/(0.15 + 0.05) = 5

If start above 5,

decay down to 5,

Else

rise up to 5.

Classic Foxes and Rabbits Example

Now consider multiple interacting species:

can be mutualistic, competitive or predator prey.

Here do classic predator prey example – foxes & rabbits

Let F be number of foxes and R be number of rabbits.

System model, as follows, where a, b, c, d are constants:

$$\frac{dR}{dt} = a*R - b*R*F \qquad \frac{dF}{dt} = c*R*F - d*F$$

Note positive and negative feedback in dR/dt expression

First term +ve fb: more R, dR/dt +ve, so more R

Second term -ve fb: more F, so dR/dt -ve, so less R.

Stable Foxes and Rabbits?

$\frac{dR}{dt}$ = a*R-b*R*F $\qquad$ $\frac{dF}{dt}$ = c*R*F-d*F

Population stable when change of both populations 0:

a * R - b * R * F = 0 $\quad$ i.e R = 0 (boring) or F = a/b

and $\quad$ c * R * F - d * F = 0 $\quad$ i.e. F = 0 or R = d/c

In fact populations cycle round these values.

We can show this in MATLAB

Again need function to return change in population

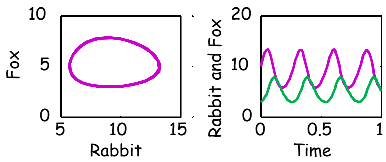
Now have two species, so P is column vector as is dP/dt

MATLAB Simulation

```
function dpbydt = pop (t, p, flag, vals);
% vals is [a b c d], p(1) = R; p(2) = F
dpbydt = [vals(1)*p(1) - vals(2)*p(1)*p(2); ...
          vals(3)*p(1)*p(2) - vals(4)*p(2)];
```

For a = 20, b = 4, c = 3 & d = 27; plot R v F; R and F v time

R v F is 'phase plane'



Note cyclic response: populations rise and fall

Foxes, Rabbits and Triffids

Novel extension of method (thanks to Dave Keating)

Classic: Rabbits eat grass, Foxes eat Rabbits

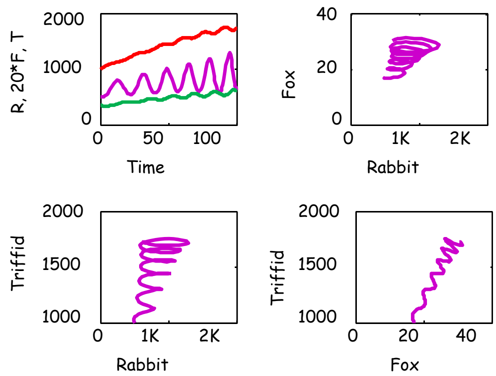
Extension: Rabbits eat Triffids (plant), Triffids eat Foxes

$$\frac{dR}{dt} = 0.001*R*T-0.06*R*F$$

$$\frac{dF}{dt} = 0.0001*R*F-0.00005*F*T$$

$$\frac{dT}{dt} = 0.025*T + 0.00015*F*T-0.00003*R*T$$

In graph over vs time, Foxes scaled by 20 so can see



Lecture 20 - In Class Exercise

Population P model: $\frac{dP}{dt} = (b_1 - d_1 - (b_2 + d_2)*P) * P$

Suppose b_1 = 10, d_1 = 6, b_2 = 0.7 and d_2 = 0.3

At what population will P stabilise?

Sketch 2 graphs of P vs time superimposed on same axes: first has P = 10 initially (at time 0); second has P = 2 then.

Mutualistic Species

Where species assist each other - mutualists

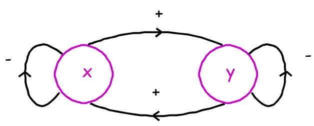
x affects y in a positive manner, and y affects x similarly

e.g. Hippo and Bird which eats weed round Hippo's teeth

Flowers and pollinating insects …

As both help each other, have run away positive feedback

So, need 'negative' feedback to limit x & y – eg food/land



Simple Mutualistic System

$\frac{dx}{dt}=(-13-2x^2+21y)*x$ $\frac{dy}{dt}=(-13+8x-3y^2)*y$

Formal analysis is tricky, but can estimate response.
Find values of x and y where x and y constant : diffs = 0
We then plot these on graphs of y vs x

$\frac{dx}{dt}=0$ where $-13-2x^2+21y = 0$ and where $x = 0$

$\frac{dy}{dt}=0$ where $-13+8x-3y^2=0$ and where $y = 0$

Stable where a $\frac{dx}{dt}=0$ line intersects with a $\frac{dy}{dt}=0$ line

Simple Mutualistic System

$\frac{dx}{dt}=(-13-2x^2+21y)*x$ $\frac{dy}{dt}=(-13+8x-3y^2)*y$

For dx/dt, one line is x = 0

In MATLAB for other dx/dt line, as has x^2 term, use

```
>> x = [2 : 6]; y = (13 + 2 * x .^ 2) / 21;
```

Do something similar for dy/dt

x	2	3	4	5	6
y	1	1.9	2.5	3	3.4

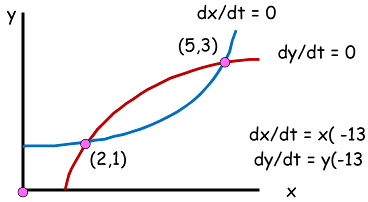
y	1	2	3	4	5
x	2	3.8	5	6	6.8

NB dx/dt and dy/dt both zero when x,y = 2,1 and 5,3

Both functions also zero when x,y = 0,0.

We plot these 'isoclines' on phase plane plot (x vs y)

Plot Zero Isoclines on Phase Plane



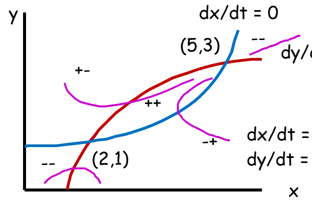
Populations stable
at x = 2, y = 1 and
at x = 5, y = 3 and
at x = 0, y = 0

$dx/dt = x(-13 - 2x^2 + 21y)$
$dy/dt = y(-13 + 8x - 3y^2)$

The iscolines for dx/dt are x = 0 and $-13 - 2x^2 + 21y = 0$
Those for dy/dt are y = 0 and $-12 + 8x - 3y^2 = 0$
Eq points: where a dx/dt isocline and a dy/dt isocline meet
Main iso's meet at 2,1 and 5,3; x = 0 and y = 0 meet at 0,0

Arguing Stability

But, one point not stable - if move from there don't return.
To argue this, label each region with sign of dx/dt & dy/dt
e.g. ++ = dx/dt > 0 & dy/dt > 0 -+ = dx/dt < 0 & dy/dt > 0

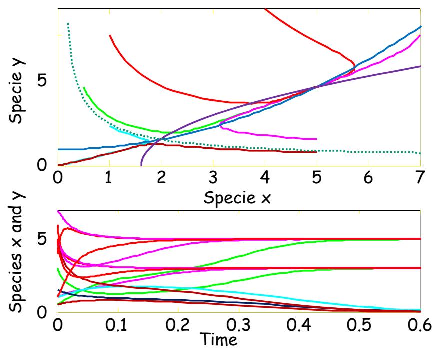


Now add trajectories from initial x,y

$dx/dt = x(-13 - 2x^2 + 21y)$
$dy/dt = y(-13 + 8x - 3y^2)$

Go to 5,3 or 0,0 – stable points

MATLAB Graphs: Phase Plane + Time



Phase plane plot y v x

Time plot y v t and x v t

Summary

We have looked at modelling feedback systems
- control systems
- population models of single or multiple species

Now course ends,
- an introduction has been given to feedback systems
- showing variety and application
- There will be revision lecture in Summer
  - You are recommended to do the following exercise