• Complex Form of Fourier Series

For a real periodic function $f(t)$ with period T, fundamental frequency f_0

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}$$

where

$$c_n = \frac{1}{T}\int_T f(t)e^{-jn\omega_0 t}dt$$

is the ‘complex amplitude spectrum’.

The coefficients are related to those in the other forms of the series by

$$c_0 = a_0 = A_0$$

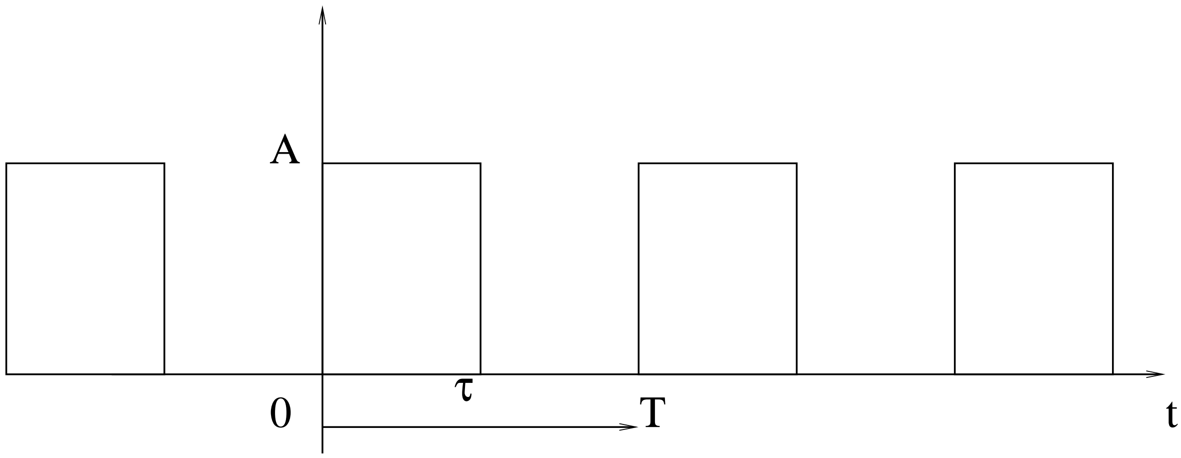
$$c_n = \frac{1}{2}(a_n - jb_n) = \frac{1}{2}A_n e^{j\phi_n} \quad \text{for} \quad n \geq 1$$

$$c_{-n} = c_n^*$$

Amplitude spectrum: $|c_n|$

Phase spectrum: $\arg(c_n)$

Example: Derive complex Fourier Series for the rectangular form in the Figure below, and the amplitude and phase spectrum.

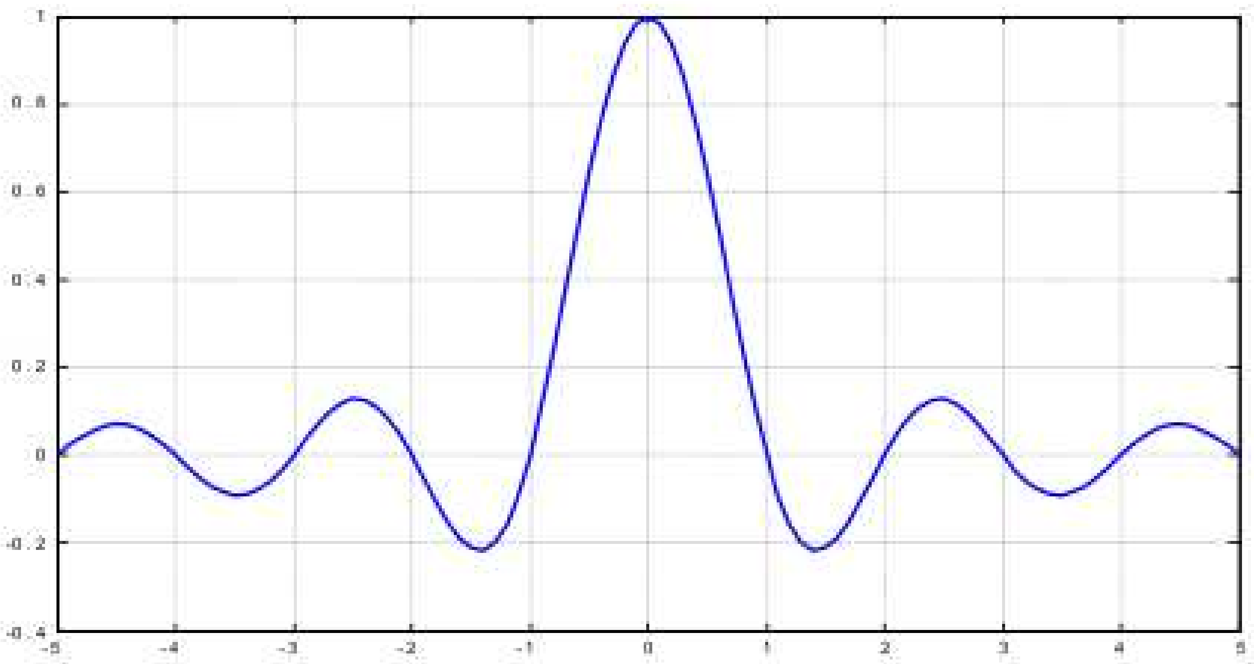


$$
\begin{aligned}
c_n &= \frac{1}{T}\int_0^{\tau} A e^{-jn\omega_0 t} dt \\
&\quad \longleftarrow \int e^{-jn\omega_0 t} dt = \frac{1}{-jn\omega_0} e^{-jn\omega_0 t} + c \\
&= \frac{A}{T}\left(\frac{1}{-jn\omega_0}\right)[e^{-jn\omega_0 \tau} - e^{-jn\omega_0 \times 0}] \\
&\quad \longleftarrow e^0 = 1 \\
&= \frac{A}{T}\left(\frac{1}{-jn\omega_0}\right)[e^{-jn\omega_0 \tau} - 1]
\end{aligned}
$$

$$
\begin{aligned}
c_n &= \frac{A}{T}(\frac{1}{-jn\omega_0})[e^{-jn\omega_0\tau} - 1] \\
&= \frac{A}{T}(\frac{1}{-jn\omega_0})[e^{\frac{-jn\omega_0\tau}{2} - \frac{jn\omega_0\tau}{2}} - e^{\frac{-jn\omega_0\tau}{2} + \frac{jn\omega_0\tau}{2}}] \\
& \quad \longleftarrow e^{a+b} = e^a e^b \\
&= \frac{A}{T}(\frac{1}{-jn\omega_0})[e^{\frac{-jn\omega_0\tau}{2}} e^{-\frac{jn\omega_0\tau}{2}} - e^{\frac{-jn\omega_0\tau}{2}} e^{\frac{jn\omega_0\tau}{2}}] \\
&= \frac{A}{T}(\frac{1}{-jn\omega_0}) e^{\frac{-jn\omega_0\tau}{2}} [e^{-\frac{jn\omega_0\tau}{2}} - e^{\frac{jn\omega_0\tau}{2}}] \\
&= \frac{A}{T}(\frac{1}{-jn\omega_0}) e^{\frac{-jn\omega_0\tau}{2}} [-2j] \frac{[e^{\frac{jn\omega_0\tau}{2}} - e^{-\frac{jn\omega_0\tau}{2}}]}{2j} \\
& \quad \longleftarrow \frac{[e^{\frac{jn\omega_0\tau}{2}} - e^{-\frac{jn\omega_0\tau}{2}}]}{2j} = \sin[\frac{n\omega_0\tau}{2}] \\
&= (\frac{2A}{n\omega_0 T}) e^{\frac{-jn\omega_0\tau}{2}} \sin[\frac{n\omega_0\tau}{2}] \\
& \quad \longleftarrow \omega_0 T = 2\pi \\
&= \frac{A}{n\pi} e^{\frac{-jn\pi\tau}{T}} \sin[\frac{n\pi\tau}{T}] \\
&= \frac{A(\frac{\tau}{T})}{n\pi(\frac{\tau}{T})} e^{\frac{-jn\pi\tau}{T}} \sin[\frac{n\pi\tau}{T}] \\
& \quad \longleftarrow \text{sinc}[\frac{n\tau}{T}] = \frac{\sin[\frac{n\pi\tau}{T}]}{[\frac{n\pi\tau}{T}]} \\
&= \frac{A\tau}{T} \text{sinc}[\frac{n\tau}{T}] e^{\frac{-jn\pi\tau}{T}}
\end{aligned}
$$

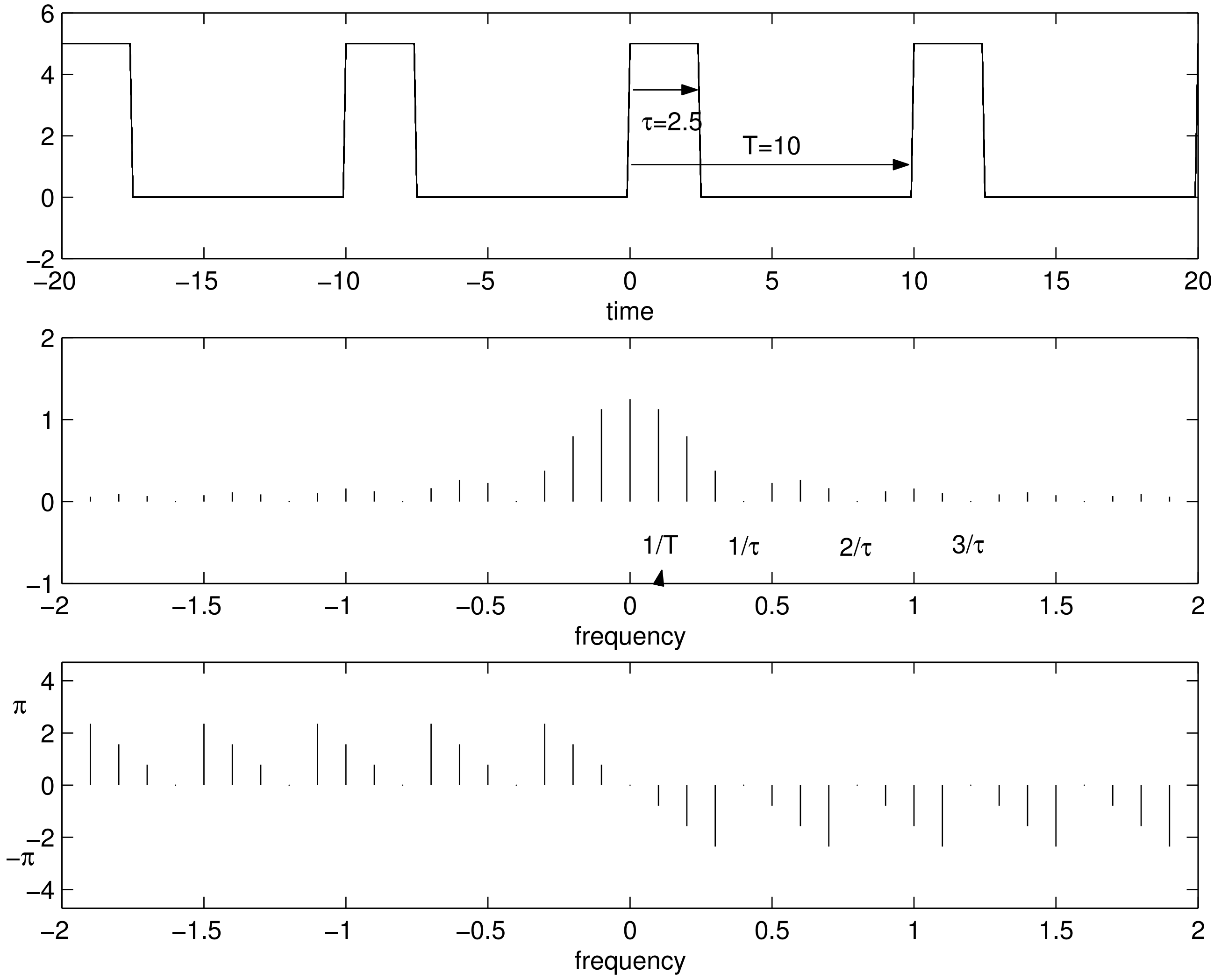
Note that the sinc function is given by

$$\text{sinc}(x) = \frac{\sin \pi x}{\pi x}$$



$$c_n = \frac{A\tau}{T}\text{sinc}[\frac{n\tau}{T}]e^{\frac{-jn\pi\tau}{T}}$$

$$|c_n| = \frac{A\tau}{T}\text{sinc}[\frac{n\tau}{T}], \quad \arg(c_n) = -\frac{n\pi\tau}{T}$$



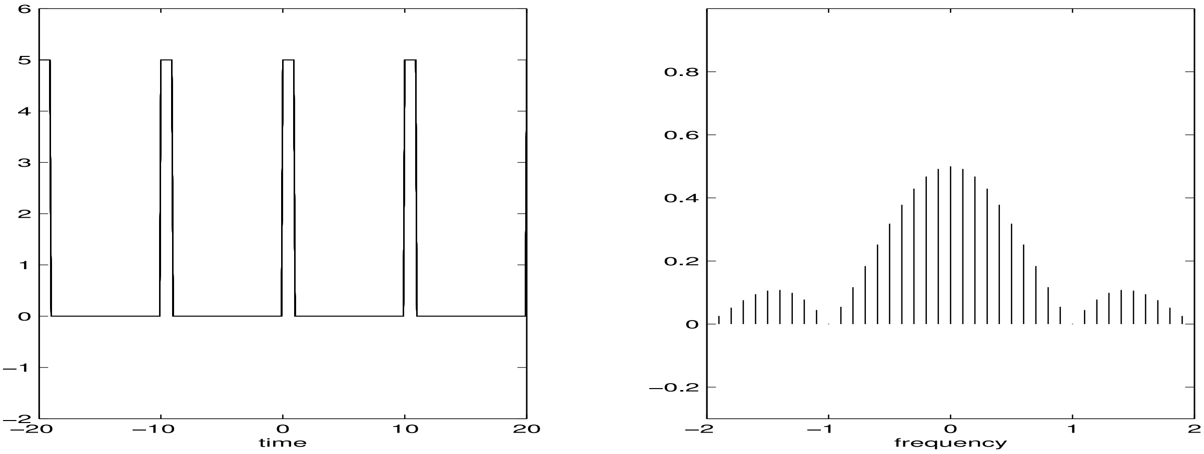
The harmonics are placed at intervals of $1/T$, their envelop following the (modulus) of the sinc function. A zero amplitude occurs whenever sinc($\frac{n\tau}{T}$) is integral so with $\frac{T}{\tau} = 4$, the fourth, eighth, twelfth lines etc. are zero. These zeros occurs at frequencies $1/\tau$, $2/t$, $3/\tau$ etc..

The repetition of the waveform produces lines every $1/T$ Hz and the envelope of the spectrum is determined by the shape of the waveform.

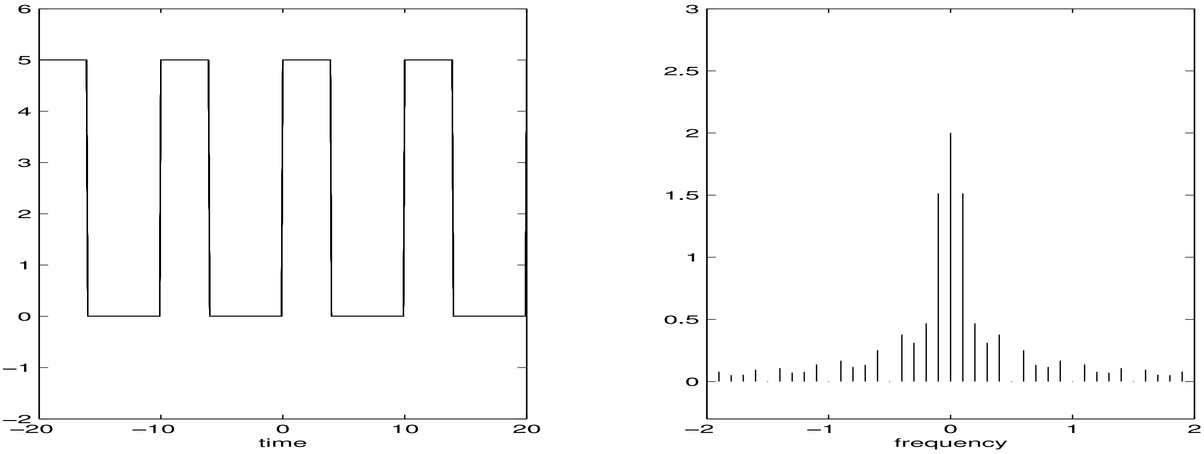
The term $e^{-\frac{jn\pi\tau}{T}}$ is a phase term dependent on the choice of origin and vanishes if the origin is in chosen in the center of a pulse. In general a shift of origin of θ in time produces a phase term of $e^{-jn\omega_0\theta}$ in the corresponding spectrum.

•Useful deductions:

(i) For a given period T , the value of τ determines the distribution of power in the spectrum.

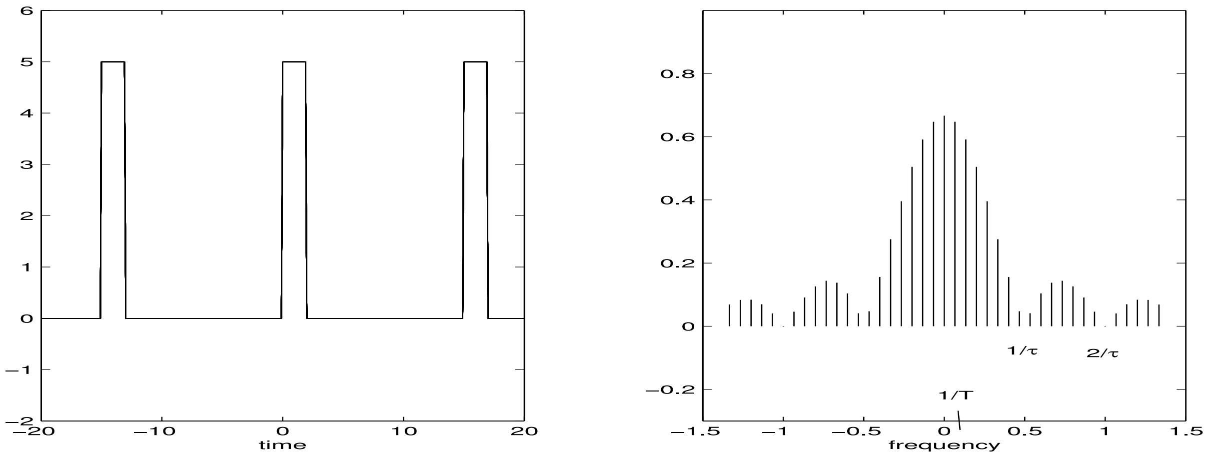


small τ

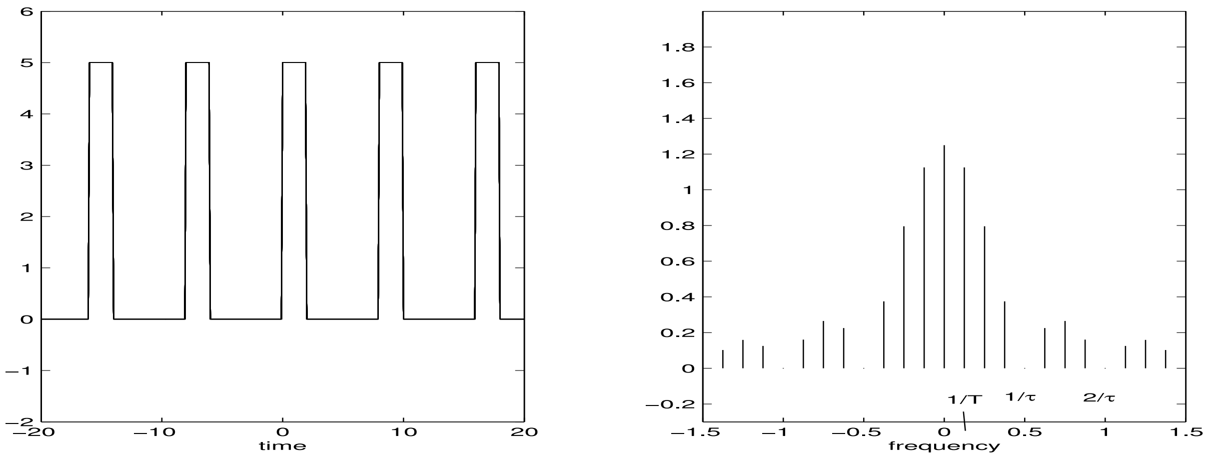


large τ

(ii) For a given value of pulse width τ, the period T similarly determines determines the power distribution.

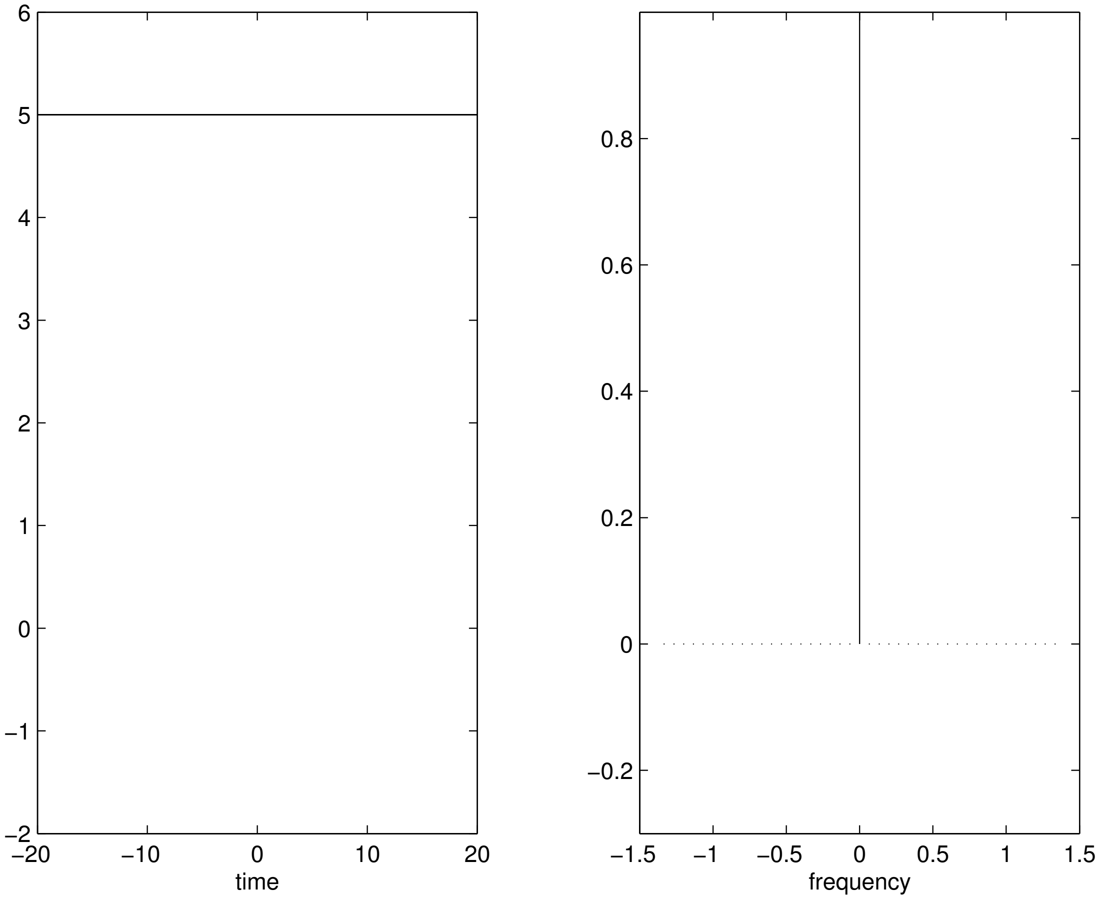


large T

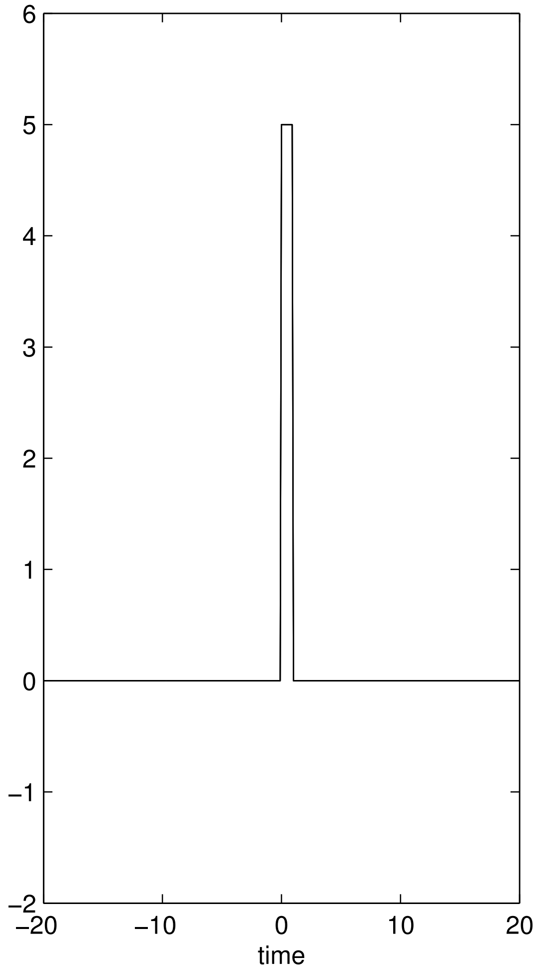


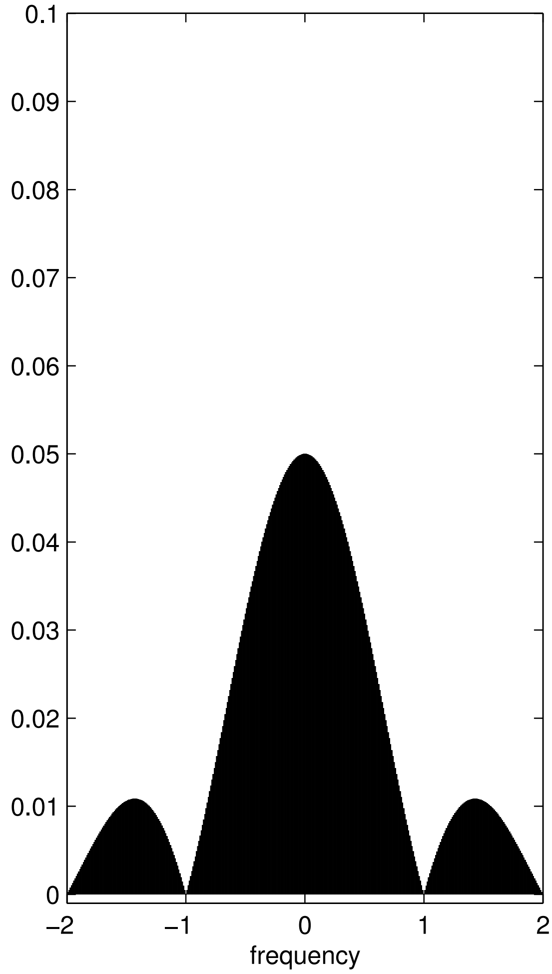
small T

(iii) If we put $T = \tau$, we get a constant (d.c) level. $|c_n|$ is then given by $A\text{sinc}(n)$, so a single spectral line of height A occurs at zero frequency.

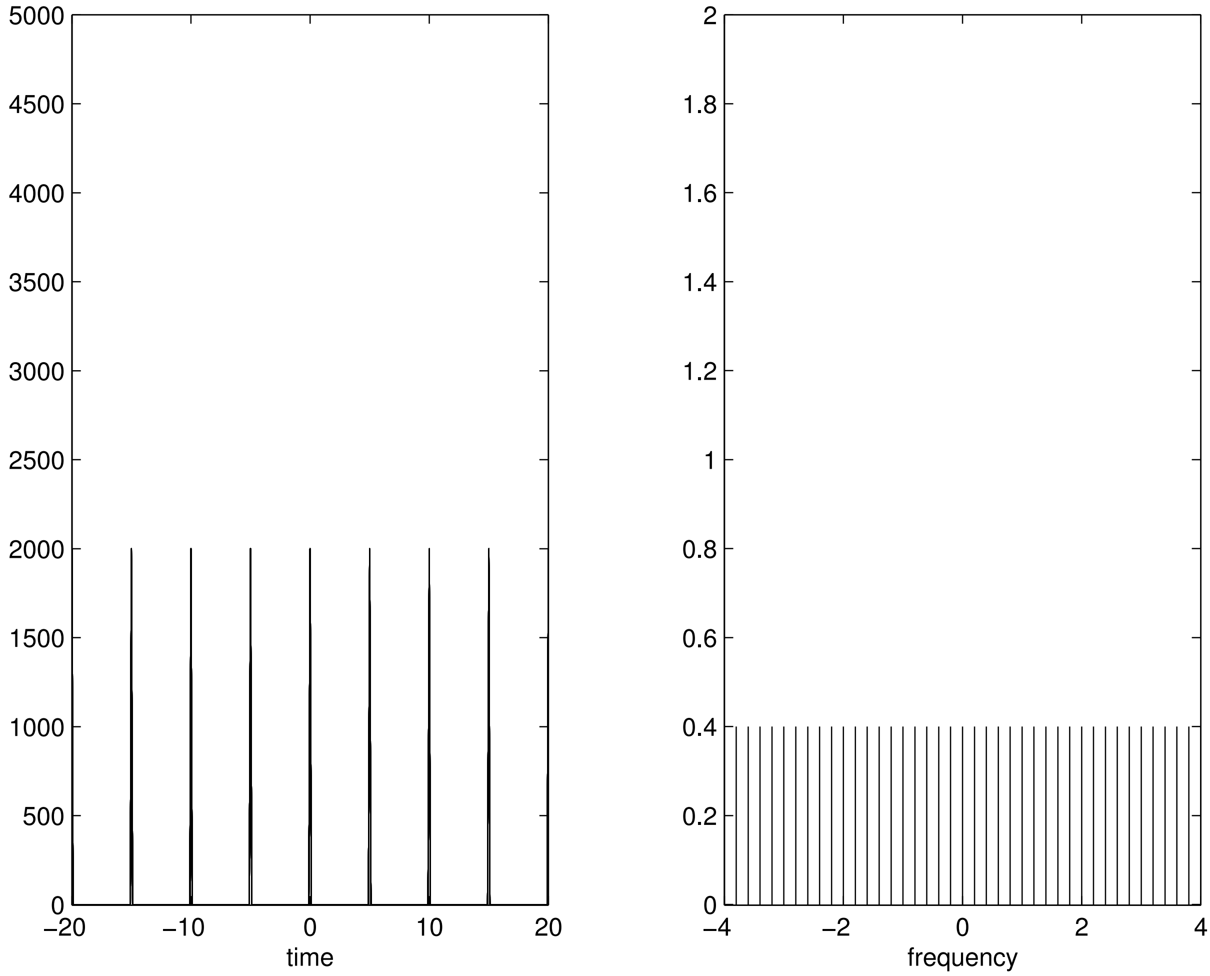


(iv) If we let the repetition period T become very large, the line spacing $1/T$ becomes very small. As T tends to infinity, the spacing tends to zero and we get a continuous spectrum. This is because $f(t)$ becomes a finite energy signal if T is infinite. Such signal has continuous spectra.





(v) Suppose we make τ small but keep the pulse area $A\tau$ constant. In the limit we get an impulse of strength $A\tau$, and the spectrum will simply be a set of lines of constants heights $\frac{A}{T}$.



(vi) Finally, it is clear tha a single impulse will have a constant but continuous spectrum.