Step response using Laplace transform

First order systems

Problem:

\[
\frac{1}{a} \frac{dy}{dt} + y = u(t) \tag{1}
\]

Solve for \(y(t)\) if all initial conditions are zero. Show that \(y(\infty) = 1\). Use the Laplace transform.

Solution. Use Table A and Table B. If all initial conditions are zero, applying Laplace transform, we have

\[
Y(s) = \frac{a}{s(s + a)} = \frac{1}{s} - \frac{1}{s + a}
\]

So

\[
y(t) = 1 - e^{-at}
\]

Use Final value theorem (Item 11 Table B)

\[
y(\infty) = \lim_{{s \to 0}} sY(s) = \lim_{{s \to 0}} \left(1 - \frac{s}{s + a}\right) = 1
\]
General second order systems

Problem: If all initial conditions are zero. Find the general function form of \( y(t) \) for the general second order system

\[
\frac{1}{\omega_n^2} \left( \frac{d^2y}{dt^2} + 2\zeta \omega_n \frac{dy}{dt} + \omega_n^2 y \right) = u(t)
\]

where \( \omega_n \) is called the natural frequency. \( \zeta \) is the damping ratio. Solve the problem in each of the following cases:

1. \( \zeta > 1 \) (overdamped)

2. \( \zeta = 1 \) (critically damped)

3. \( 0 < \zeta < 1 \) (underdamped)

4. \( \zeta = 0 \) (undamped)

respectively.
Solution: Use Table A and Table B. If all initial conditions are zero, applying Laplace transform, we have $Y(s)$

$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta \omega_n s + \omega_n^2)}$$

(i) $\zeta > 1$ (overdamped)

$$s^2 + 2\zeta \omega_n s + \omega_n^2 = (s + \sigma_1)(s + \sigma_2)$$

where

$$\sigma_1 = \zeta \omega_n + \sqrt{\zeta^2 - 1}\omega_n$$

$$\sigma_2 = \zeta \omega_n - \sqrt{\zeta^2 - 1}\omega_n$$

are real and distinct (Case 1).

$$Y(s) = \frac{K_1}{s} + \frac{K_2}{s + \sigma_1} + \frac{K_3}{s + \sigma_2}$$

The general form of $y(t)$ is

$$y(t) = K_1 + K_2 e^{-\sigma_1 t} + K_3 e^{-\sigma_2 t}$$
(ii) $\zeta = 1$ (critically damped)

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = (s + \sigma)^2$$

where

$$\sigma = \omega_n$$

are real and repeated (Case 2).

$$Y(s) = \frac{K_1}{s} + \frac{K_2}{(s + \sigma)^2} + \frac{K_3}{s + \sigma}$$

The general form of $y(t)$ is

$$y(t) = K_1 + K_2te^{-\sigma t} + K_3e^{-\sigma t}$$
(iii) $0 < \zeta < 1$ (Undamped)

$$s^2 + 2\zeta \omega_n s + \omega_n^2 = (s + \sigma_1)(s + \sigma_2)$$

where

$$\sigma_1 = \zeta \omega_n + j\sqrt{1 - \zeta^2 \omega_n}$$

$$\sigma_2 = \zeta \omega_n - j\sqrt{1 - \zeta^2 \omega_n}$$

are complex (Case 3).

Partial fraction expansion of $Y(s)$

$$\frac{\omega_n^2}{s(s^2 + 2\zeta \omega_n s + \omega_n^2)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta \omega_n s + \omega_n^2}$$

Find $K_1, K_2, K_3$

$$K_1 = \left. \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \right|_{s \to 0} = 1$$
\[ \omega_n^2 = (s^2 + 2\zeta \omega_n s + \omega_n^2) + s(K_2 s + K_3) \]

Balancing the coefficients for \( s^1 \) and \( s^2 \),
\[ s^1 : \quad 0 = 2\zeta \omega_n + K_3 \]
\[ s^2 : \quad 0 = 1 + K_2 \]

\( K_2 = -1, \; K_3 = -2\zeta \omega_n \). Then we have
\[ Y(s) = \frac{1}{s} - \frac{s + 2\zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_n^2 (1 - \zeta^2)} \]
\[ = \frac{1}{s} - \frac{s + \zeta \omega_n + (\omega_n \sqrt{1 - \zeta^2}) \frac{\zeta}{\sqrt{1 - \zeta^2}}}{(s + \zeta \omega_n)^2 + \omega_n^2 (1 - \zeta^2)} \]

The general form of \( y(t) \) is
\[ y(t) = 1 - e^{-\zeta \omega_n t} \left( \cos(\omega_n \sqrt{1 - \zeta^2} t) + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin(\omega_n \sqrt{1 - \zeta^2} t) \right) \]
\[ = 1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \cos(\omega_n \sqrt{1 - \zeta^2} t - \phi) \]

where \( \phi = \arctan\left( \frac{\zeta}{\sqrt{1 - \zeta^2}} \right) \).
(iv) $\zeta = 0$ (Undamped)

$$s^2 + 2\zeta \omega_n s + \omega_n^2 = s^2 + \omega_n^2$$

where two complex roots are $+j\omega_n$ and $-j\omega_n$, still Case 3.

Partial fraction expansion of $Y(s)$

$$\frac{\omega_n^2}{s(s^2 + \omega_n^2)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + \omega_n^2}$$

Find $K_1 = 1$ as usual. Find $K_2, K_3$

$$\omega_n^2 = (s^2 + \omega_n^2) + s(K_2 s + K_3)$$

Balancing the coefficients for $s^1$ and $s^2$,

$$s^1 : \quad 0 = K_3$$
$$s^2 : \quad 0 = 1 + K_2$$

$K_2 = -1, \ K_3 = 0$. Then we have

$$Y(s) = \frac{1}{s} - \frac{s}{s + \omega_n^2}$$

$$y(t) = 1 - \cos \omega_n t$$
Figure above: Second order system response as a function of damping ratio.
Underdamped second order systems

Figure below: Second order underdamped re-
response $y(t)$ for damping ratio values.

$$y(t) = 1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \cos(\omega_n \sqrt{1 - \zeta^2} t - \phi)$$

where $\phi = \arctan(\frac{\zeta}{\sqrt{1 - \zeta^2}})$. Peak time $T_p$ is found by differentiating $y(t)$ and then the first zero crossing after $t = 0$.

$$\mathcal{L}\left[\frac{dy(t)}{dt}\right] = sY(s) = \frac{\omega_n^2}{(s^2 + 2\zeta \omega_n s + \omega_n^2)}$$

$$= \frac{\omega_n \sqrt{1 - \zeta^2}}{(s + \zeta \omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$
\[
\frac{d}{dt} y(t) = \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin(\omega_n \sqrt{1 - \zeta^2} t) = 0
\]

\[
\omega_n \sqrt{1 - \zeta^2} t = \pi
\]

\[
T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}
\]

The maximum \( y_{max} \) is

\[
y(T_p) = 1 - e^{-\zeta \pi / \sqrt{1 - \zeta^2}}
\]

The final value \( y(\infty) = 1 \). Thus

\[
\%OS = \frac{y_{max} - y(\infty)}{y(\infty)} = e^{-\zeta \pi / \sqrt{1 - \zeta^2}} \times 100
\]

We can similarly found that the settling time (defined as the time required to settle with \( \pm 2\% \) of the final value) can be approximated by \( T_s = \frac{4}{\zeta \omega_n} \).
Figure above: Step responses of second order underdamped systems. (a) with constant real part; (b) with constant imaginary part and (c) with constant damping ratio.