# The convergence of incremental 4D-Var using non-tangent linear models

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The incremental formulation of 4D-Var allows the use of a linear model which is not exactly tangent to the discrete nonlinear model. One method of developing such a model is to discretize the continuous linear equations, forming a perturbation forecast model (PFM). Lawless *et al.* (2003) showed that a PFM can describe accurately the evolution of a perturbation in the discrete nonlinear model, provided that the perturbation is of a reasonable size, but for very small perturbations this is not the case. The present study considers the effects of using a PFM instead of a tangent linear model (TLM) within the inner loop of an incremental 4D-Var system.

The system used to test these effects is the one-dimensional shallow-water equations in the absence of rotation, which contains two variables, a wind field $u$ and a mass field $\phi$. The nonlinear model uses a semi-implicit semi-Lagrangian scheme, the TLM is derived directly from the discrete nonlinear model and the the PFM is derived by discretizing the continuous linearized equations of the system. The discretization is over 1000 grid points, with a distance $\Delta x = 0.01m$ between them. Further details of the model schemes can be found in Lawless *et al.* (2003).

We consider a situation in which a shock forms in the solution by the end of the assimilation window, so that we expect nonlinearities to arise and so differences between the two linear models to be highlighted. Identical twin experiments are performed using an incremental 4D-Var scheme, with both a TLM and a PFM. No background term is included in the cost function.

For the first experiment we run for a total of 12 outer loops and within each inner loop we converge until the norm of the gradient of the cost function is reduced by a factor of $10^4$, with a restriction on the maximum number of inner iterations. Figure 1 shows the convergence of the cost function and its gradient for the two assimilations. We find that the convergence of the assimilation using the PFM follows very closely that using the TLM, despite the fact that the two models behave differently as the perturbation size is reduced. In Figure 2 we show the true solution for the wind field $u$ at the end of the assimilation window and the analysis error from both assimilation runs. Even though the flow is highly nonlinear, both assimilations are able to analyse the true state to a high degree of accuracy. From the plot of analysis errors we see that both analyses are very close and the difference between them is no greater than would be expected from the convergence criterion used. The analyses for the $\phi$ field show similar errors to the $u$ field analyses.

In order to test the assimilation as the perturbations become very small, we increase the convergence tolerance in each inner loop, requiring that the gradient norm be reduced by a factor of $10^8$, again within a maximum number of inner iterations. Figure 3 shows a comparison of the convergence for the two different assimilations for this experiment. We see that the final value of the cost function is much reduced with respect to the first experiment. However, the two linear models continue to follow a very similar convergence pattern and again we find that the final analyses are very similar (not shown). Work is now in progress to develop a more complete theoretical explanation for these results.

## References

Lawless, A.S., Nichols, N.K. and Ballard, S.P. (2003) A comparison of two methods for developing the linearization of a shallow-water model, *Q.J.R.M.S.*, **129**, 1237-1254.

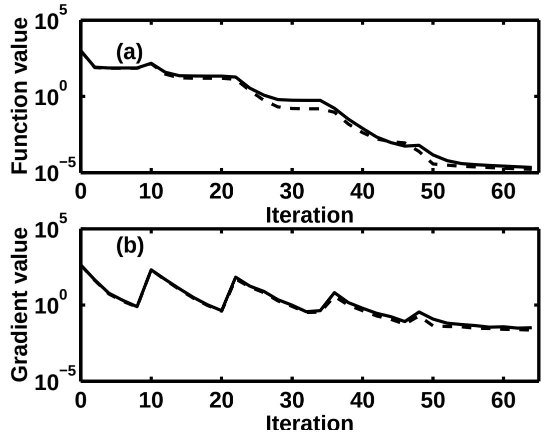


Figure 1: Convergence of (a) cost function and (b) gradient for assimilation using the tangent linear model (solid line) and the perturbation forecast model (dashed line).

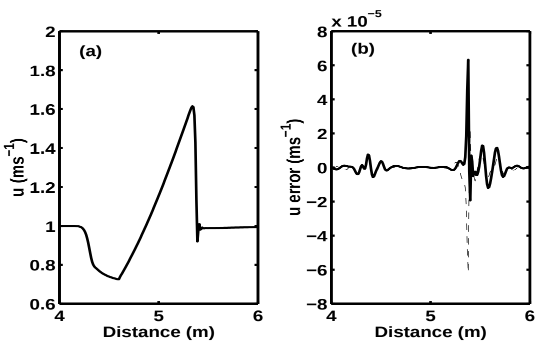


Figure 2: Analysis in $u$ field at end of assimilation window. Plot (a) shows the true solution and plot (b) shows the analysis error for the TLM assimilation (solid line) and the PFM assimilation (dashed line).

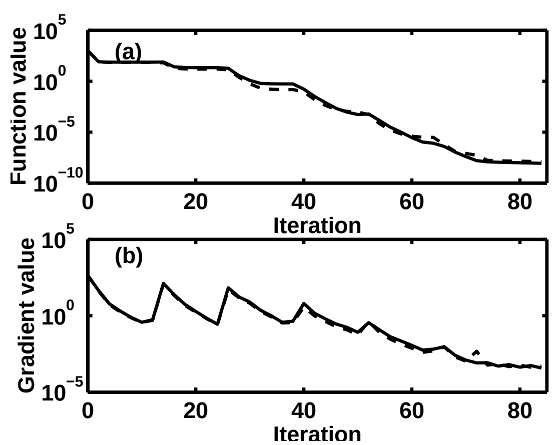


Figure 3: Convergence of (a) cost function and (b) gradient for assimilation using the tangent linear model (solid line) and the perturbation forecast model (dashed line).