Boundary value problems for the N-wave interaction equations

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Abstract

We consider boundary value problems for the N-wave interaction equations in one and two space dimensions, posed for $x \geq 0$ and $x, y \geq 0$ respectively. Following the recent work of Fokas, we develop an inverse scattering formalism to solve these problems by considering the simultaneous spectral analysis of the two ordinary differential equations in the associated Lax pair. The solution of the boundary value problems is obtained through the solution of a local Riemann-Hilbert problem in the one dimensional case, and a nonlocal Riemann-Hilbert problem in the two-dimensional case.

1 Introduction

In this paper, we analyse an initial-boundary value problem for the so-called N wave interaction equations:

$$\begin{aligned} (q_{ij})_t &= \alpha_{ij}(q_{ij})_x + \beta_{ij}(q_{ij})_y - \sum_{k \neq j}(\alpha_{ik} - \alpha_{kj}) q_{ik} q_{kj}, \quad x, y, t \in [0, \infty), \\ i &\neq j, \quad i, j, k \in [1, ..., N], \end{aligned} \tag{1.1}$$

where $q_{ij} = q_{ij}(x, y, t)$, α_{ij} and β_{ij} are real constants.
When the functions $q_{ij} = q_{ij}(x, t)$ do not depend on y, the problem is reduced to one space and one time variable - 1+1 dimensions. In general, however, we consider the problem in two space and one time variables - 2+1 dimensions.
These equations model important physical phenomena, and have been considered by many authors, see e.g. [5, 6, 9]. From a mathematical point of view, these equations are interesting because they are an example of *integrable* PDE systems. This means that it is possible to formulate them in terms of a *linear* spectral problem. Hence the initial value problem for (1.1) can be solved using the inverse scattering formalism. The integrability of these systems, and many of their important properties, were first established in [11, 12, 13]. A systematic and rigorous analysis of the Cauchy problem for one space dimension is given by Beals and Coifman [2, 3], while the Cauchy problem for $(x, y) \in \mathbb{R}^2$ was considered in [8, 10].The essential difference between these two cases is that in the $1 + 1$ case, the Riemann-Hilbert problem associated with the spectral Lax formulation is local, while in the $2 + 1$ case, this RH problem is nonlocal.
In this paper, we give the formal derivation of the solution of initial-boundary value problem, in 1+1 and 2+1 dimensions, under the assumption that all given data are Schwartz

functions, and that a solution exists. The requirement on the functional class of the prescribed data is stronger than necessary, and weaker conditions can be required, see [2]. The solution of the Cauchy problem, as considered in earlier works, is based on the spectral analysis of one of the two ODEs in the Lax pair, namely the one containing derivatives with respect to the space variable. The novelty of the work we present here lies in the fact that in order to analyse initial-boundary value problems, the spectral analysis of both ODEs in the Lax pair must be considered simultaneously. We also find under which conditions on the group velocities the initial and boundary conditions can be prescribed arbitrarily: teh conclusion, based on physical intuition, that this is not possible if all the initial waves travel with negative speed is confirmed by the mathematical formalism.

We characterise the matrix Q whose elements are the functions $q_{kl}(x,y,t)$ that satisfy the following boundary value problems:

(A) if there exist indices i,j such that $\alpha_{ij}<0$:

• 1+1 case:

$$(q_{ij})_t = \alpha_{ij}(q_{ij})_x - \sum_{k\neq j}(\alpha_{ik}-\alpha_{kj})q_{ik}q_{kj}, \quad x,t\in[0,\infty), \tag{1.2}$$

$$i\neq j, \quad i,j,k\in[1,...,N], \tag{1.3}$$

$$q_{ij}(x,0)=(q^0)_{ij}(x)\in\mathcal{S}([0,\infty)), \quad q_{ij}(0,t)=g_{ij}(t)\in\mathcal{S}([0,\infty)),$$

$$(q^0)_{ij}(0)=g_{ij}(0).$$

• 2+1 case:

$$(q_{ij})_t = \alpha_{ij}(q_{ij})_x + \beta_{ij}(q_{ij})_y - \sum_{k\neq j}(\alpha_{ik}-\alpha_{kj})q_{ik}q_{kj}, \quad x,y,t\in[0,\infty), \tag{1.4}$$

$$i\neq j, \quad i,j,k\in[1,...,N], \tag{1.5}$$

$$q_{ij}(x,y,0)=(q_0)_{ij}(x,y)\in\mathcal{S}([0,\infty)\times[0,\infty)), \quad q_{ij}(0,y,t)=f_{ij}(y,t)\in\mathcal{S}([0,\infty)\times[0,\infty)),$$

$$(q_0)_{ij}(0,y)=f_{ij}(y,0).$$

(B) if $\alpha_{ij}\geq 0$ for all i,j:

The same as (1.3)-(1.5) except that *no boundary conditions* are prescribed at $x=0$.

The rigorous validity of the formal representations of the solutions of these boundary value problems, and their scope, can be analysed in the same way as in [2, 3, 8, 10]. Therefore we give here only the final statements, and comment briefly on how the existing results can be generalised to the case of initial-boundary value problems.

However we do give full details of the construction of the solution representation, in both the one- and two-dimensional cases. Indeed, although the steps involved in this constuction are similar to the ones required to solve the initial value problem, full details are rarely presented in the literature. Moreover, we follow these steps to formulate a Riemann-Hilbert problem parametrised by both the space and time variable, as opposed to the usual case, when only the space variable is involved in the spectral analysis. We feel therefore that it might be useful to the reader, and as a general reference, to present all steps of the construction explicitly.

Notation

1. $M_N(\mathbb{C})$ denotes the set of all $N\times N$ matrices with complex entries
2. $M_N^{diag}(\mathbb{R})$ denotes the set of all $N\times N$ invertible diagonal matrices with real diagonal entries

3. $\mathcal{S}_N^0$ denotes the set of $M \in M_N(\mathbb{C})$ such that $M = M(x,y,t)$ and

   (i) M is off diagonal

   (ii) all components of M are Schwartz functions of their arguments

4. $[\cdot,\cdot]$ denotes the usual matrix commutator

5. For a given $J \in M_N^{diag}(\mathbb{C})$, we define

$$\hat{J}M = [J, M], \qquad \mathrm{e}^{z\hat{J}}M = \mathrm{e}^{zJ}M\mathrm{e}^{-zJ}, \quad M \in M_N(\mathbb{C}). \tag{1.6}$$

6. For $F \in M_N(\mathbb{C})$ we denote by $\Pi_D F$, $\Pi_U F$, $\Pi_L F$ the diagonal, strictly upper diagonal and strictly lower diagonal parts of F.

   We also denote by A_+F, A_-F the projection onto the elements F_{kl} such that the corresponding α_{kl} is positive or negative, respectively:

$$(A_+F)_{kl} = \begin{cases} F_{kl} & \alpha_{kl} > 0, \\ 0 & \alpha_{kl} < 0 \end{cases}, \quad (A_-F)_{kl} = \begin{cases} 0 & \alpha_{kl} > 0 \\ F_{kl} & \alpha_{kl} < 0, \end{cases}$$

   where α_{kl} are the coefficients of the x-derivatives in (1.1).

2 The problem in one space dimension

In this section, we derive a formal representation of the solution of the boundary value problem (1.3). This representation is given in terms of the prescribed initial and boundary conditions q_{ij}^0, $g(t)_{ij}$, $i, j \in [1, ..., N]$. We assume that all given conditions are Schwartz functions of their respective argument, namely

$$Q_0(x) = (q_{ij}^0) \in \mathcal{S}_N^o, \quad Q_1(t) = (g_{ij}) \in \mathcal{S}_N^o.$$

To derive formally the solution representation, we assume that the problem admits a unique solution. Hence we assume the existence of scalar functions $q_{ij}(x, y, t)$, $i \neq j, i, j \in [1, ..., n]$ that satisfy the 1-dimensional N-wave interaction equations (1.2) as well as the initial and boundary conditions (1.3).
The assumption of existence can be justified a-posteriori. Indeed, it can be proved that a solution of the initial-boundary value problem exists, and is given precisely by our construction, see propositions (2.2), (2.3).

The Lax pair formulation

Equations (1.2) can be written as the compatibility condition of the following Lax pair [8]:

$$\begin{cases} \mu_x - i\lambda[J, \mu] = Q\mu, \\ \\ \mu_t - i\lambda[A, \mu] = R\mu, \end{cases} \tag{2.1}$$

where $\mu = \mu(x, t, \lambda) \in M_N(\mathbb{C})$. The various other matrices involved are defined as follows:

- J and A are elements of $M_N^{diag}(\mathbb{R})$ with constant real entries

$$J = diag\,(j_1, \ldots, j_N), \quad A = diag\,(a_1, \ldots, a_N), \tag{2.2}$$

  where j_k, a_k are defined in terms of α_{kl} through the relations

$$\alpha_{kl} = \frac{a_k - a_l}{j_k - j_l}.$$

We assume the diagonal entries of J are all different, and ordered decreasingly:

$$j_1 > j_2 > \ldots > j_N.$$

Then

$$j_k - j_l > 0, \;\; k < l, \quad j_k - j_l < 0, \;\; k > l.$$

- $Q(x,t)$, $R(x,t)$ are the off-diagonal matrices

$$Q(x,t) = \begin{pmatrix} 0 & q_{12} & \cdots & q_{1n} \\ q_{21} & 0 & \cdots & q_{2n} \\ \vdots & & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & 0 \end{pmatrix}, \quad R(x,t) = \begin{pmatrix} 0 & \alpha_{12}q_{12} & \cdots & \alpha_{1n}q_{1n} \\ \alpha_{21}q_{21} & 0 & \cdots & \alpha_{2n}q_{2n} \\ \vdots & & \ddots & \vdots \\ \alpha_{n1}q_{n1} & \alpha_{n2}q_{n2} & \cdots & 0 \end{pmatrix}. \tag{2.3}$$

Remark 2.1. The above definition of the entries of the matrices J and A makes sense only if $\alpha_{kl} = \alpha_{lk}$ in the original equations. This is consistent with the physical interpretation of these constants.

Under the above assumptions, we prove the following result.

Proposition 2.1 (The formal solution representation)**.** *Assume that the boundary value problem (1.3) admits a unique solution.*
Then this solution is given by

$$Q(x,t) = -i \lim_{|\lambda|\to\infty} \lambda[J,\mu_-] = \frac{\hat{J}}{2\pi}\int_{\mathbb{R}} \mu_-(x,t,\lambda)\mathrm{e}^{i\lambda x\hat{J}+i\lambda t\hat{A}}(w(\lambda)-I)d\lambda \tag{2.4}$$

The function $w(\lambda)$ in this expression is defined by

$$w(\lambda): \mathbb{R} \to M_N(\mathbb{C}) \;\; \text{such that} \quad w(\lambda) = m_-^{-1}(\lambda)m_+(\lambda) \tag{2.5}$$

where $m_\pm(\lambda) = \mu_\pm(0,0,\lambda)$, and the matrices $\mu_\pm(x,t,\lambda)$ are defined as the unique solutions of the linear integral equations

$$\begin{aligned} \mu_+(x,t,\lambda) = I \quad &- \quad \left\{(\Pi_D + \Pi_L)\int_x^\infty - \Pi_U \int_0^x\right\} \mathrm{e}^{i\lambda\hat{J}(x-x')}(Q\mu_+)(x',t,\lambda)dx' + \\ &+ \quad \Pi_U\left\{A_+\int_0^t - A_-\int_0^\infty\right\}\mathrm{e}^{i\lambda\hat{J}x+i\lambda\hat{A}(t-t')}(R\mu_+)(0,t',\lambda)dt', \end{aligned} \tag{2.6}$$

$$\begin{aligned} \mu_-(x,t,\lambda) = I \quad &- \quad \left\{(\Pi_D + \Pi_U)\int_x^\infty - \Pi_L \int_0^x\right\} \mathrm{e}^{i\lambda\hat{J}(x-x')}(Q\mu_-)(x',t,\lambda)dx' + \\ &+ \quad \Pi_L\left\{A_+\int_0^t - A_-\int_0^\infty\right\}\mathrm{e}^{i\lambda\hat{J}x+i\lambda\hat{A}(t-t')}(R\mu_-)(0,t',\lambda)dt', \end{aligned} \tag{2.7}$$

where the matrices Q and R are defined in terms of the initial and boundary data by (2.3).

Remark 2.2. If the projection A_- is zero, i.e. if all coefficients α_{ij} are non-negative, the spectral functions $m^\pm$ do not depend on $(R\mu_\pm)(0,t,\lambda)$, hence they do not depend on the value of the solution at $x = 0$. It follows that in this case, the boundary values cannot be prescribed arbitrarily.

Remark 2.3. The matrix $\mu_-(x,t,\lambda)$ defined in (2.7) can be characterised also as the solution of the singular integral equation

$$\mu_-(x,t,\lambda) = I + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\mu_-(x,t,\lambda')\mathrm{e}^{i\lambda' x\hat{J}+i\lambda' t\hat{A}}(w(\lambda')-I)}{\lambda'-\lambda+i0}d\lambda'. \tag{2.8}$$

We will give the proof of propoosition 2.1 in the next section. However, we note here that the validity of the formal solution given in proposition 2.1 is guaranteed by the following result. The proof of an analogous result, under less restrictive assumptions on the prescribed data, can be found in the work of Beals and Coifman [2, 3].

Proposition 2.2 (The solution of the direct problem)**.** *The integral equations (2.6), (2.7) and (2.8) are always solvable if* $J,\ A \in M_N^{diag}(\mathbb{R})$*,* $Q(x,0) = (q_i^0 j) \in \mathcal{S}_N^0$ *and* $Q(0,t) = (g_{ij}) \in \mathcal{S}_N^0$*.*
In addition, the spectral function $w(\lambda)$ *defined by (2.5) belongs to the space* $\mathcal{S}_N^0$*.*

To complete the rigorous analysis of this problem, it is necessary to study the solution of the inverse problem. This means that it must be shown that under reasonable assumptions on $Q(x,0)$ and $Q(0,t)$, or, more generally, on the spectral function $w(\lambda)$, the matrix-valued function $Q(x,t)$ constructed by the above process is a solution of the given boundary value problem. This is summarised in the following proposition.

Proposition 2.3 (The solution of the inverse problem)**.** *Let* $w(\lambda) : \mathbb{R} \to \mathcal{S}_N^0$ *be a matrix-valued function such that* $\lim_{|\lambda|\to\infty} w(\lambda) = I$*, and that the singular integral equation*

$$m_{(x,t)}(\lambda) = I + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{m_{(x,t)}(\lambda')\mathrm{e}^{i\lambda' x\hat{J}+i\lambda' t\hat{A}}(w(\lambda')-I)}{\lambda'-\lambda+i0}d\lambda' \tag{2.9}$$

admits a unique solution. Then the matrix $Q(x,t)$ *defined by*

$$Q(x,t) = -i\lim_{|\lambda|\to\infty}\lambda[J, m_{(x,t)}] = \frac{\hat{J}}{2\pi}\int_{\mathbb{R}} m_{(x,t)}(\lambda)\mathrm{e}^{i\lambda x\hat{J}+i\lambda t\hat{A}}(w(\lambda)-I)d\lambda \tag{2.10}$$

is in $\mathcal{S}_N^0$*, and it satisfies equation (1.1).*
In particular, let $Q(x,0),\ Q(0,t) \in M_N(\mathbb{C})$ *be the matrices whose components are the prescribed functions* $q_{ij}^0(x)$*,* $g_{ij}(t)$ *of (1.3b). If* $w(\lambda)$ *is defined in terms of these matrices by expression (2.5), then* $w(\lambda)$ *satisfies the above assumptions, and the matrix* $Q(x,t)$ *defined in terms of* $w(\lambda)$ *by (2.4) satisfies the boundary value problem (1.3).*

As with the proof of the direct problem, the proof of this rigorous result follows from the proof of the more general result of Beals and Coifman in [2].

2.1 The spectral analysis in one space dimension

In this section, we construct the representation (2.4) under the assumption of existence of a solution of the given boundary value problem.
In component form, the Lax pair (2.1) is written as

$$\begin{cases} (\mu_{kl})_x - i\lambda(j_k - j_l)\mu_{kl} = (Q\mu)_{kl}, \\ (\mu_{kl})_t - i\lambda(a_k - a_l)\mu_{kl} = (R\mu)_{kl}. \end{cases} \tag{2.11}$$

The system (2.11) admits the following particular solutions:

$$(\mu_1)_{kl}(x,t,\lambda) = \int_0^x \mathrm{e}^{i\lambda(j_k-j_l)(x-x')}(Q\mu_1)_{kl}(x',t,\lambda)dx' + \mathrm{e}^{i\lambda(j_k-j_l)x}\int_0^t \mathrm{e}^{i\lambda(a_k-a_l)(t-t')}(R\mu_1)_{kl}(0,t',\lambda)dt', \tag{2.12}$$

$$(\mu_2)_{kl}(x,t,\lambda) = \int_0^x \mathrm{e}^{i\lambda(j_k-j_l)(x-x')}(Q\mu_2)_{kl}(x',t,\lambda)dx' - \mathrm{e}^{i\lambda(j_k-j_l)x}\int_t^\infty \mathrm{e}^{i\lambda(a_k-a_l)(t-t')}(R\mu_2)_{kl}(0,t',\lambda)dt', \tag{2.13}$$

$$(\mu_3)_{kl}(x,t,\lambda) = -\int_x^\infty \mathrm{e}^{i\lambda(j_k-j_l)(x-x')}(Q\mu_3)_{kl}(x',t,\lambda)dx' \qquad (2.14)$$

Integrating by parts the definition (2.12)-(2.14), we see that these functions satisfy

$$\mu_i(x,t,\lambda) = O\left(\frac{1}{\lambda}\right), \quad |\lambda| \to \infty. \qquad i = 1,2,3. \qquad (2.15)$$

The decay of the exponentials appearing in the integrand is dictated by the real part of the exponent, given by

$$Re[i\lambda(j_k - j_l)(x-x')] = -Im(\lambda)[(j_k - j_l)(x-x')],$$
$$Re[i\lambda(a_k - a_l)(t-t')] = -Im(\lambda)[(j_k - j_l)\alpha_{kl}(t-t')].$$

It follows that the two integral terms in $(\mu_1)_{kl}$ can only be bounded in the same region if $\alpha_{kl} > 0$, and similarly, the two integral terms in $(\mu_2)_{kl}$ can only be bounded in the same region if $\alpha_{kl} < 0$. Note also that

$$\lim_{x\to\infty} (\mu_3)_{kl}(x,t,\lambda) = 0,$$

and by the dominated convergence theorem, since all elements of Q and R are Schwartz functions of x and t, hence in particular bounded and integrable, we also have for each fixed λ

$$\lim_{x\to\infty} (\mu_1)_{kl}(x,t,\lambda) = \lim_{x\to\infty} (\mu_2)_{kl}(x,t,\lambda) = 0 \quad for \left\{ \begin{array}{ll} k \le l & if\ \lambda \in \mathbb{C}^+, \\ k > l & if\ \lambda \in \mathbb{C}^-. \end{array} \right. \qquad (2.16)$$

Motivated by these observations, we define the following two solutions of (2.11):

$$\mu_+(x,t,\lambda) = ((\mu_+)_{kl}): \qquad (\mu_+)_{kl} = \left\{ \begin{array}{ll} 1 + (\mu_3)_{kk} & k = l \\ (\mu_3)_{kl} & k > l, \\ (\mu_1)_{kl} & k < l,\ \alpha_{kl} > 0 \\ (\mu_2)_{kl} & k < l,\ \alpha_{kl} < 0, \end{array} \right. \qquad (2.17)$$

$$\mu_-(x,t,\lambda) = ((\mu_-)_{kl}): \qquad (\mu_-)_{kl} = \left\{ \begin{array}{ll} 1 + (\mu_3)_{kk} & k = l \\ (\mu_3)_{kl} & k < l, \\ (\mu_1)_{kl} & k > l,\ \alpha_{kl} > 0 \\ (\mu_2)_{kl} & k > l,\ \alpha_{kl} < 0, \end{array} \right. \qquad (2.18)$$

We can now give the proof of our first result.

Proof of proposition 2.1

We consider the two solutions of (2.1) given by (2.17)-(2.18), and write them in the form

$$\mu_+(x,t,\lambda) = I + (\Pi_D + \Pi_L)\mu_3 + \Pi_U A_+\mu_1 + \Pi_U A_-\mu_2 \qquad (2.19)$$
$$\mu_-(x,t,\lambda) = I + (\Pi_D + \Pi_U)\mu_3 + \Pi_L A_+\mu_1 + \Pi_L A_-\mu_2 \qquad (2.20)$$

where the operators $\Pi_.$, $A_\pm$ are defined in Notations (6.).
By construction, $\mu^\pm$ is an analytic bounded function of λ in $\mathbb{C}^\pm$ respectively. Using (2.16) we find that these functions satisfy the asymptotic condition

$$\lim_{x\to\infty} \mu_+ = \lim_{x\to\infty} \mu_- = I.$$

This implies that these matrices have unit determinant, and hence are invertible.
Indeed, if μ is any solution of (2.1), then $\Psi = \mu \mathrm{e}^{i\lambda Jx + i\lambda At}$ satisfies

$$\Psi_x = (i\lambda J + Q)\Psi \Rightarrow (\det\Psi)_x = \mathrm{Tr(i\lambda J + Q)}\det\Psi$$

$$\Psi_t = (i\lambda A + R)\Psi \Rightarrow (\det \Psi)_t = \mathrm{Tr(i\lambda A + R)} \det \Psi$$

and since $\mathrm{Tr\,Q} = \mathrm{Tr\,R} = 0$, we obtain $\det \Psi = C(\lambda)\mathrm{e}^{i\lambda \mathrm{TrJx} + \mathrm{i}\lambda \mathrm{TrAt}}$ and hence

$$\det \mu = \det \Psi \mathrm{e}^{-i\lambda \mathrm{TrJx} - \mathrm{i}\lambda \mathrm{TrAt}} = C(\lambda).$$

Evaluating $C(\lambda)$ as $x \to \infty$ yields the desired conclusion.
Note that $\nu(x,t,\lambda) = \mu_-^{-1}\mu_+$ satisfies

$$\nu_x = i\lambda[J,\nu], \quad \nu_t = i\lambda[A,\nu], \quad \lambda \in \mathbb{R}.$$

Hence ν is of the form

$$\nu = \mu_-^{-1}\mu_+ = \mathrm{e}^{i\lambda x\hat{J} + i\lambda t\hat{A}} w(\lambda), \quad w(\lambda) = \mu_-^{-1}\mu_+(0,0,\lambda) = m_-^{-1}m_+(\lambda), \tag{2.21}$$

where $\mu_\pm$ are defined by (2.6)-(2.7).
In particular, since

$$\begin{aligned} m_+(\lambda) &= I - (\Pi_D + \Pi_L)\int_0^\infty \mathrm{e}^{-i\lambda\hat{J}x} Q\mu_+(x,0,\lambda)dx - \Pi_U A_- \int_0^\infty \mathrm{e}^{-i\lambda\hat{A}t} R\mu_+(0,t,\lambda)dt, \\ m_-(\lambda) &= I - (\Pi_D + \Pi_U)\int_0^\infty \mathrm{e}^{-i\lambda\hat{J}x} Q\mu_-(x,0,\lambda)dx - \Pi_L A_- \int_0^\infty \mathrm{e}^{-i\lambda\hat{A}t} R\mu_-(0,t,\lambda)dt, \end{aligned}$$

the definition of the function $w(\lambda)$, $\lambda \in \mathbb{R}$, depends only on the given initial and boundary conditions, $Q(x,0)$ and $Q(0,t)$, unless the projection operator A_- is the null projector, in which case this definition depends only on the initial condition $Q(x,0)$.
Rewriting this relation in additive form, we obtain

$$\mu_+ - \mu_- = \mu_- \mathrm{e}^{i\lambda x\hat{J} + i\lambda t\hat{A}}(w(\lambda) - I). \tag{2.22}$$

The relation (2.22) together with the asymptotic condition

$$\mu \sim I + O\left(\frac{1}{\lambda}\right), \quad |\lambda| \to \infty,$$

determines a Riemann-Hilbert problem on the real axis, whose unique solution is given by

$$\mu(x,t,\lambda) = I + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\mu_-(x,t,\lambda')\mathrm{e}^{i\lambda' x\hat{J} + i\lambda' t\hat{A}}(w(\lambda') - I)}{\lambda' - \lambda} d\lambda', \quad \lambda \in \mathbb{C}. \tag{2.23}$$

Taking the limit as $\lambda \to \mathbb{R}$ from the lower half λ plane, this gives the following singular integral equation (2.8) for μ_-:

$$\mu_-(x,t,\lambda) = I + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\mu_-(x,t,\lambda')\mathrm{e}^{i\lambda' x\hat{J} + i\lambda' t\hat{A}}(w(\lambda') - I)}{\lambda' - \lambda + i0} d\lambda', \quad \lambda \in \mathbb{R}.$$

The analysis of the solvability of this equation, in substantially more general conditions, is given in [2]. Under our hypothesis, this equation, and equation (2.23) always admits a unique solution.

We now indicate how to solve formally the inverse problem, hence how to represent $Q(x,t)$ in terms of the spectral function $w(\lambda)$.
Since $\mu_- = I + O\left(\frac{1}{\lambda}\right)$, $\lambda \to \infty$, we can assume that

$$\mu_- = I + \frac{\mu_*}{\lambda} + O\left(\frac{1}{\lambda^2}\right).$$

Computing $(\mu_-)_x$ and equating powers of λ, we find $i[J,\mu_*]+Q=0$, hence

$$Q(x,t)=-i[J,\mu_*].$$

On the other hand, since $\mu_* = \lim_{|\lambda|\to\infty}\lambda(\mu_- - I)$, using (2.8) we find

$$\mu_* = -\frac{1}{2\pi i}\int_{\mathbb{R}}\mu_-(x,t,\lambda')\mathrm{e}^{i\lambda' x\hat{J}+i\lambda' t\hat{A}}(w(\lambda')-I)d\lambda'. \tag{2.24}$$

In summary,

$$Q(x,t)=-i[J,\mu_*]=\frac{1}{2\pi}[J,\int_{\mathbb{R}}\mu_-(x,t,\lambda)\mathrm{e}^{i\lambda x\hat{J}+i\lambda t\hat{A}}(w(\lambda)-I)d\lambda]. \tag{2.25}$$

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3 The problem in two space dimensions

In this section, we derive a formal representation of the solution of the boundary value problem (1.5), hence we consider the boundary value problem obtained when boundary values are specified for all times on the half-line $y\geq 0$, as given in (1.5). The solution representation is given in terms of the prescribed initial and boundary conditions q^0_{ij}, $g(t)_{ij}$, $i,j\in[1,...,N]$.
These boundary conditions are the natural candidates to obtain solvability of this problem. This follows from the fact that, although formally this problem is posed in two space dimensions and contains derivatives with respect to both x and y, the y derivative can be eliminated by a change of variables.
As in the previous section, we assume that the problem admits a unique solution. Hence we assume the existence of scalar functions $q_{ij}(x,y,t)$, $i\neq j, i,j\in[1,...,n]$ that satisfy the two-dimensional N-wave interaction equations (1.4) as well as the initial and boundary conditions (1.5).

Lax pair formulation in 2+1

The N-wave interaction equations (1.4) can be written as the compatibility condition of the following Lax pair [8]:

$$\begin{cases}\mu_x - i\lambda[J,\mu]=Q\mu+J\mu_y,\\ \mu_t - i\lambda[A,\mu]=R\mu+A\mu_y,\end{cases} \tag{3.1}$$

where $\mu=\mu(x,y,t,\lambda)\in M_N(\mathbb{C})$. The various other matrices involved are defined as follows:

- J and A are the diagonal matrices with constant real entries

  $$J=diag\,(J_1,\ldots,J_N),\quad A=diag\,(A_1,\ldots,A_N), \tag{3.2}$$

  where J_k, A_k are defined in terms of α_{kl}, β_{kl} through the relations

  $$\alpha_{kl}=\frac{A_k-A_l}{J_k-J_l},\quad \beta_{kl}=A_k-J_k\alpha_{kl}.$$

  (we use uppercase for the diagonal entries to distinguish this from the one-dimensional case).

  As in the 1+1 case, we assume the diagonal entries of J are ordered decreasingly:

  $$J_1>J_2>\ldots>J_N.$$

- $Q(x,y,t)$, $R(x,y,t)$ are the off-diagonal matrices (2.3), with q_{ij} now depending also on y.

We prove the following result.

Proposition 3.1 (The formal solution representation)**.** *Assume that there exists a unique solution of the boundary value problem (1.5).*
Then this solution is given by

$$Q(x,y,t) = -i \lim_{|\lambda|\to\infty} \lambda[J,\mu_0] =$$

$$= \frac{\hat{J}}{2\pi}\int_{R^2} \mu_0(x,y,t,l) e^{il(xJ+yI+tA)} [T^-(l,\lambda) - T^+(l,\lambda)] e^{-il(xJ+yI+tA)} d\lambda dl. \qquad (3.3)$$

The various matrices appearing in this expression are defined as follows.

- *the matrix-valued functions $T^{\pm}(l,\lambda)$ are defined by*

$$T^+(l,\lambda) = \frac{1}{2\pi}(\Pi_D + \Pi_L)\int_0^\infty\int_0^\infty \mathrm{e}^{-il(x'J+y'I)}(Q\mu^+)(x',y',0,\lambda)\mathrm{e}^{i\lambda(x'J+y'I)}dy'dx'$$
$$+\frac{1}{2\pi}\Pi_U A_-\int_0^\infty\int_0^\infty \mathrm{e}^{-il(t'A+y'I)}(R\mu^+)(0,y',t',\lambda)\mathrm{e}^{i\lambda(t'A+y'I)}dy'dt', \qquad (3.4)$$

$$T^-(l,\lambda) = \frac{1}{2\pi}(\Pi_D + \Pi_U)\int_0^\infty\int_0^\infty \mathrm{e}^{-il(x'J+y'I)}(Q\mu^-)(x',y',0,\lambda)\mathrm{e}^{i\lambda(x'J+y'I)}dy'dx'$$
$$+\frac{1}{2\pi}\Pi_L A_-\int_0^\infty\int_0^\infty \mathrm{e}^{-il(t'A+y'I)}(R\mu^-)(0,y',t',\lambda)\mathrm{e}^{i\lambda(t'A+y'I)}dy'dt', \qquad (3.5)$$

where μ^+ and μ^- solve of the linear integral equations

$$\mu^+(x,y,t,\lambda) = I + \left(\int_0^x - (\Pi_D+\Pi_L)\int_0^\infty\right)\mathrm{e}^{i\lambda(x-x')\hat{J}}(Q\mu^+)(x',y+(x-x')J,t,\lambda)dx' +$$

$$+\Pi_U\left\{A_+\int_0^t \mathrm{e}^{i\lambda x\hat{J}+i\lambda(t-t')\hat{A}}(R\mu^+)(0,y+xJ+(t-t')A,t',\lambda)dt'\right.$$
$$\left.- A_-\int_0^\infty \mathrm{e}^{i\lambda x\hat{J}+i\lambda(t-t')\hat{A}}(R\mu^+)(0,y+xJ+(t-t')A,t',\lambda)dt'\right\}, \qquad (3.6)$$

$$\mu^-(x,y,t,\lambda) = I + \left(\int_0^x - (\Pi_D+\Pi_U)\int_0^\infty\right)\mathrm{e}^{i\lambda(x-x')\hat{J}}(Q\mu^-)(x',y+(x-x')J,t,\lambda)dx' +$$

$$+\Pi_L\left\{A_+\int_0^t \mathrm{e}^{i\lambda x\hat{J}+i\lambda(t-t')\hat{A}}(R\mu^-)(0,y+xJ+(t-t')A,t',\lambda)dt'\right.$$
$$\left.- A_-\int_0^\infty \mathrm{e}^{i\lambda x\hat{J}+i\lambda(t-t')\hat{A}}(R\mu^-)(0,y+xJ+(t-t')A,t',\lambda)dt'\right\}. \qquad (3.7)$$

- *the matrix $\mu^0(x,y,t,\lambda)$ is the unique solution of the linear integral equation*

$$\mu^0(x,y,t,\lambda) = I + P_+\int_{\mathbb{R}} \mu^0(x,y,t,l)\mathrm{e}^{il(xJ+tA+yI)}T^-(l,\lambda)dl\,\mathrm{e}^{-i\lambda(xJ+tA+yI)}$$
$$+P_-\int_{\mathbb{R}} \mu^0(x,y,t,l)\mathrm{e}^{il(xJ+tA+yI)}T^+(l,\lambda)dl\,\mathrm{e}^{-i\lambda(xJ+tA+yI)} \qquad (3.8)$$

where $P_\pm$ are the Plemelj projection operators in λ.

The validity of the formal solution given above is guaranteed by the following results.

Proposition 3.2 (The solution of the direct problem)**.** *The integral equations (3.6), (3.7) and (3.8) are always solvable if* $J,\ A \in M_N^{diag}(\mathbb{R})$, $Q(x,y,0) = (q_{ij}^0) \in \mathcal{S}_N^0$ *and* $Q(0,y,t) = (g_{ij}) \in \mathcal{S}_N^0$.
In addition, the spectral functions $T^{\pm}(l,\lambda)$ *defined by (3.4), (3.5) satisfy* $T^{\pm} \in \mathcal{S}_N^0$.

Proposition 3.3 (The solution of the inverse problem)**.** *Let* $T^{\pm}(l,\lambda)$ *be given matrix-valued functions belonging to* $\mathcal{S}_N^o$*, such that the linear integral equation*

$$\begin{aligned} m_{(x,y,t)}(\lambda) = I + P_+ \int_{\mathbb{R}} m_{(x,y,t)}(l) \mathrm{e}^{il(xJ+tA+yI)} T^-(l,\lambda) dl\, \mathrm{e}^{-i\lambda(xJ+tA+yI)} \\ +P_- \int_{\mathbb{R}} m_{(x,y,t)}(l) \mathrm{e}^{il(xJ+tA+yI)} T^+(l,\lambda) dl\, \mathrm{e}^{-i\lambda(xJ+tA+yI)} \end{aligned} \quad (3.9)$$

admits a unique solution. Then the matrix $Q(x,y,t)$ *defined by*

$$Q(x,y,t) = \frac{\hat{J}}{2\pi} \int_{R^2} m_{(x,y,t)}(l) e^{il(xJ+yI+tA)} [T^-(l,\lambda) - T^+(l,\lambda)] e^{-il(xJ+yI+tA)} d\lambda dl \quad (3.10)$$

is in $\mathcal{S}_N^o$*, and it satisfies equation (1.4).*
In particular, let $Q(x,y,0),\ Q(0,y,t) \in \mathcal{S}_N^o$ *be the matrices whose components are the prescribed functions* $(q_0)_{ij}(x,y)$, $f_{ij}(y,t)$ *of (1.5b). If* $T^{\pm}(l,\lambda)$ *are defined in terms of the given data by (3.4), (3.5) respectively, then they satisfy the above assumptions, and the matrix-valued function* $Q(x,y,t)$ *defined by (3.10) satisfies the boundary value problem (1.5).*

The proof of these results follow the lines of the analogous proofs in [10].

3.1 The spectral analysis

In this section we construct the solution representation (3.3).
In component form, the Lax pair (3.1) is written

$$\begin{cases} (\mu_{kl})_x - J_k(\mu_{kl})_y - i\lambda(J_k - J_l)\mu_{kl} = (Q\mu)_{kl}, \\ (\mu_{kl})_t - A_k(\mu_{kl})_y - i\lambda(A_k - A_l)\mu_{kl} = (R\mu)_{kl}. \end{cases}$$

For fixed k, consider the change of variables $(x,y,t) \to (\xi, \eta_k, \tau)$ with

$$\xi = x, \quad \eta_k = y + J_k x + A_k t, \quad \tau = t, \quad k = 1,\dots,N \quad (3.11)$$

Let $M(\xi,\eta_k,\tau,\lambda)$ be the row vector

$$M(\xi,\eta_k,\tau,\lambda) = (\mu(\xi,\eta_k - J_k\xi - A_k\tau,\tau,\lambda)_{k1}, \dots, \mu(\xi,\eta_k - J_k\xi - A_k\tau,\tau,\lambda)_{kN}),$$

and let $M(\xi,\eta,\tau,\lambda)$ the matrix formed with these row vectors, $k = 1,\dots,N$.
Then the Lax pair satisfied by M can be written as

$$\begin{cases} M_\xi - i\lambda[J,M] = QM, \\ M_\tau - i\lambda[A,M] = RM, \end{cases} \quad (3.12)$$

with $Q = Q(\xi,\eta,\tau)$ and $R = R(\xi,\eta,\tau)$ and the notational convention (which we adhere to in the sequel) that the components of all matrices involved are computed as follows:

$$F_{kl}(\xi,\eta,\tau) = F_{kl}(\xi,\eta_k,\tau), \quad F \in M_N(\mathbb{C}).$$

In component form, any solution of (3.12) is of the form

$$M_{kl}(\xi,\eta_k,\tau,\lambda)=\zeta_{kl}(\eta_k,\lambda)\mathrm{e}^{i\lambda[(J_k-J_l)\xi+(A_k-A_l)\tau]}+\int_{\xi^0_{kl}}^{\xi}\mathrm{e}^{i\lambda(J_k-J_l)(\xi-\xi')}(QM)_{kl}(\xi',\eta_k,\tau,\lambda)d\xi'$$

$$+\int_{\tau^0_{kl}}^{\tau}\mathrm{e}^{i\lambda(A_k-A_l)(\tau-\tau')+i\lambda(J_k-J_l)(\xi-\xi^0_{kl})}(RM)_{kl}(\xi^0_{kl},\eta_k,\tau',\lambda)d\tau', \tag{3.13}$$

where the functions $\zeta_{kl}(\eta_k,\lambda)$ and the points ξ^0_{kl} and τ^0_{kl} are determined by the initial and boundary conditions.
In the original coordinates, this yields

$$\mu_{kl}(x,y,t,\lambda)=\zeta_{kl}(y+xJ_k+tA_k,\lambda)\mathrm{e}^{i\lambda[(J_k-J_l)x+(A_k-A_l)t]}$$

$$+\int_{x^0_{kl}}^{x}\mathrm{e}^{i\lambda[(J_k-J_l)(x-x')}(Q\mu)_{kl}(x',y+J_k(x-x'),t,\lambda)dx'$$

$$+\int_{t^0_{kl}}^{\tau}\mathrm{e}^{i\lambda(A_k-A_l)(t-t')+i\lambda(J_k-J_l)(x-x^0_{kl})}(R\mu)_{kl}(x^0_{kl},y+J_k(x-x^0_{kl})+A_k(t-t'),t',\lambda)dt'.$$

By choosing the points x^0_{kl} and t^0_{kl} and the arbitrary functions ζ_{kl} appropriately, in analogy with what done in the previous section for the problem in one space dimension, we define particular solutions of (3.1) as follows:

$$\mu^+(x,y,t,\lambda)=I+\left[\Pi_U\int_0^x+(\Pi_D+\Pi_L)\int_\infty^x\right]\mathrm{e}^{i\lambda(x-x')\hat{J}}(Q\mu^+)(x',y+(x-x')J,t,\lambda)dx'$$
$$+\Pi_U\left\{A_+\int_0^t\mathrm{e}^{i\lambda x\hat{J}+i\lambda(t-t')\hat{A}}(R\mu^+)(0,y+xJ+(t-t')A,t',\lambda)dt'\right.$$
$$\left.-A_-\int_t^\infty\mathrm{e}^{i\lambda x\hat{J}+i\lambda(t-t')\hat{A}}(R\mu^+)(0,y+xJ+(t-t')A,t',\lambda)dt'\right\}, \tag{3.14}$$

and

$$\mu^-(x,y,t,\lambda)=I+\left[\Pi_L\int_0^x+(\Pi_D+\Pi_U)\int_\infty^x\right]\mathrm{e}^{i\lambda(x-x')\hat{J}}(Q\mu^-)(x',y+(x-x')J,t,\lambda)dx'$$
$$+\Pi_L\left\{A_+\int_0^t\mathrm{e}^{i\lambda x\hat{J}+i\lambda(t-t')\hat{A}}(R\mu^-)(0,y+xJ+(t-t')A,t',\lambda)dt'\right.$$
$$\left.-A_-\int_t^\infty\mathrm{e}^{i\lambda x\hat{J}+i\lambda(t-t')\hat{A}}(R\mu^-)(0,y+xJ+(t-t')A,t',\lambda)dt'\right\}. \tag{3.15}$$

Remark 3.1. As in the one dimensional case, if the projection A_- is zero, i.e. if all coefficients α_{ij} are non-negative, the integral terms containing the functions $(R\mu_\pm)(0,t,\lambda)$ disappear, hence the spectral problem does not involve the value of the solution at $x=0$. It follows that in this case, the boundary values cannot be prescribed arbitrarily.

Proof of proposition 3.1 The functions (3.14)-(3.15) can be written in the form (3.6), (3.7), respectively, by using the definition of the operators Π_*, given in Notations (6.). By construction, $\mu^\pm$ is an analytic bounded function of λ in $(\mathbb{C}^*)^\pm$ respectively.
Using

$$f(y)=\frac{1}{2\pi}\int_{\mathbb{R}}\int_0^\infty\mathrm{e}^{il(y-y')}f(y')dy'dl$$

where $f(y)$ is a function of y supported in $[0,\infty)$, we write

$$Q\mu^+(x',y+(x-x')J,t,\lambda)=\frac{1}{2\pi}\int_{\mathbb{R}}\int_0^\infty\mathrm{e}^{il(y-y')I+il(x-x')J}Q\mu^+(x',y',t,\lambda)dy'dl.$$

and

$$R\mu^+(0,y+xJ+(t-t')A,t',\lambda)=\frac{1}{2\pi}\int_{\mathbb{R}}\int_0^\infty \mathrm{e}^{il(y-y')I+ilxJ+il(t-t')A}R\mu^+(0,y',t',\lambda)dy'dl.$$

Hence

$$\int_0^\infty \mathrm{e}^{i\lambda(x-x')\hat{J}}(Q\mu^+)(x',y+(x-x')J,t,\lambda)dx'=$$

$$=\frac{1}{2\pi}\int_{\mathbb{R}}\left[\int_0^\infty \mathrm{e}^{i\lambda(x-x')\hat{J}}\int_0^\infty \mathrm{e}^{il(y-y')I+il(x-x')J}Q\mu^+(x',y',t,\lambda)dy'dx'\right]dl$$

$$=\frac{1}{2\pi}\int_{\mathbb{R}}\left[\int_0^\infty\int_0^\infty \mathrm{e}^{il(y-y')I+i(\lambda+l)(x-x')J}Q\mu^+(x',y',t,\lambda)\mathrm{e}^{-i\lambda(x-x')J}dy'dx'\right]dl$$

$$=\frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{e}^{il'(xJ+yI)}\left[\int_0^\infty\int_0^\infty \mathrm{e}^{-il'(x'J+y'I)}Q\mu^+(x',y',t,\lambda)\mathrm{e}^{i\lambda(x'J+y'I)}dy'dx'\right]dl'\mathrm{e}^{-i\lambda(xJ+yI)}$$

where we used the change of variables $l'=l+\lambda$. Similarly

$$\int_0^\infty \mathrm{e}^{i\lambda x\hat{J}+i\lambda(t-t')\hat{A}}(R\mu^+)(0,y+xJ-t'A,t',\lambda)dt'=$$

$$=\frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{e}^{il'(xJ+yI+tA)}\left[\int_0^\infty\int_0^\infty \mathrm{e}^{-il'(y'I+t'A)}R\mu^+(0,y',t',\lambda)\mathrm{e}^{i\lambda(y'I+t'A)}dy'dt'\right]dl'\mathrm{e}^{-i\lambda(xJ+yI+tA)}.$$

Motivated by the above computation, we now define the spectral functions $T_0^\pm$, $T_1^\pm$ as

$$T_0^+(l,\lambda)=\frac{1}{2\pi}(\Pi_D+\Pi_L)\int_0^\infty\int_0^\infty \mathrm{e}^{-il(x'J+y'I)}Q\mu^+(x',y',0,\lambda)\mathrm{e}^{i\lambda(x'J+y'I)}dy'dx',\quad(3.16)$$

$$T_0^-(l,\lambda)=\frac{1}{2\pi}(\Pi_D+\Pi_U)\int_0^\infty\int_0^\infty \mathrm{e}^{-il(x'J+y'I)}Q\mu^-(x',y',0,\lambda)\mathrm{e}^{i\lambda(x'J+y'I)}dy'dx',\quad(3.17)$$

$$T_1^+(l,\lambda)=\frac{1}{2\pi}\Pi_UA_-\int_0^\infty\int_0^\infty \mathrm{e}^{-il(t'A+y'I)}R\mu^+(0,y',t',\lambda)\mathrm{e}^{i\lambda(t'A+y'I)}dy'dt',\qquad(3.18)$$

$$T_1^-(l,\lambda)=\frac{1}{2\pi}\Pi_LA_-\int_0^\infty\int_0^\infty \mathrm{e}^{-il(t'A+y'I)}R\mu^-(0,y',t',\lambda)\mathrm{e}^{i\lambda(t'A+y'I)}dy'dt'.\qquad(3.19)$$

Then the functions $T^\pm(l,\lambda)$ of (3.4)-(3.5) are given by $T^\pm=T_0^\pm+T_1^\pm$. These functions depend only on the given initial conditions $Q(x,y,0)$ and, unless $A_-=0$, they also depend on the prescribed boundary conditions $Q(0,y,t)$. Note however that if A_- is the null projector, then $T_1^\pm$ vanish.
We can now evaluate μ^+, μ^- along the line $x=t=0$ in terms of these spectral functions T as follows:

$$(\mu^\pm-I)(0,y,0,\lambda)\mathrm{e}^{i\lambda yI}=-\int_{\mathbb{R}}\mathrm{e}^{ilyI}T^\pm(l,\lambda)dl.\qquad(3.20)$$

We also define an auxiliary eigenfunction of the spectral problem as

$$\mu^0(x,y,t,\lambda)=I+\int_0^x \mathrm{e}^{i\lambda(x-x')\hat{J}}(Q\mu^0)(x',y+(x-x')J,t,\lambda)dx'$$

$$+\int_0^t \mathrm{e}^{i\lambda x\hat{J}+i\lambda(t-t')\hat{A}}(R\mu^0)(0,y+xJ+(t-t')A,t',\lambda)dt'.\qquad(3.21)$$

The function $\mu^0(x,y,t,\lambda)$, is the particular solution of (3.1) satisfying $\mu^0(0,y,0,\lambda) = I$. Although this function has no global analyticity properties, we show below that μ_0 satisfies a linear integral equation directly in terms of the spectral data $T_0^\pm$, $T_1^\pm$, and that, in addition, $Q(x,y,t)$ can be recostructed from the knowledge of μ_0.
Using (3.20) and the fact that the difference $\mu^\pm - \mu^0$ is also a solution of (3.1), we find

$$(\mu^\pm - \mu^0)(x,y,t,\lambda)\mathrm{e}^{i\lambda(xJ+tA+yI)} = -\int_{\mathbb{R}} \mu^0(x,y,t,l)\mathrm{e}^{il(xJ+tA+yI)}T^\pm(l,\lambda)dl. \quad (3.22)$$

We now consider the projections $P_\pm$ of (3.22) onto the space of functions analytic in the upper or lower half space respectively. Computing $P_+(\mu^- - \mu_0)$ and $P_-(\mu^+ - \mu_0)$ and adding the resulting expressions (taking into account that $P_+\mu^- = I = P_-\mu^+$), we obtain

$$\begin{aligned}\mu^0 = I + P_+ \int_{\mathbb{R}} \mu^0(x,y,t,l)\mathrm{e}^{il(xJ+tA+yI)}T^-(l,\lambda)dl\, \mathrm{e}^{-i\lambda(xJ+tA+yI)} \\ +P_- \int_{\mathbb{R}} \mu^0(x,y,t,l)\mathrm{e}^{il(xJ+tA+yI)}T^+(l,\lambda)dl\, \mathrm{e}^{-i\lambda(xJ+tA+yI)}. \quad (3.23)\end{aligned}$$

This is the desired integral equation for μ_0 in terms of $T^\pm(l,\lambda)$. The unique solvability of this equation for μ^0 can be established as in [10].
Using the two expressions (3.22), we also obtain the jump equation

$$(\mu^+ - \mu^-)(x,y,t,\lambda) = \left\{\int_{\mathbb{R}} \mu^0(x,y,t,l)\mathrm{e}^{il(xJ+tA+yI)} \left[T^-(l,\lambda) - T^+\left(l,\lambda)\right]dl\right\} \mathrm{e}^{-i\lambda(xJ+tA+yI)}. \quad (3.24)$$

Using equation (3.24) we can obtain a formal representation of the solution $\mu(x,y,t,\lambda)$ in terms of the eigenfunction μ^0 by solving the Riemann-Hilbert problem defined by this jump equation and by the asymptotic requirement $\mu \sim I$, $|\lambda| \to \infty$.
The solution of this Riemann-Hilbert problem is given by

$$\mu(x,y,t,\lambda) = I + \frac{1}{2\pi i}\int_{\mathbb{R}} \left[\int_{\mathbb{R}} \mu^0(x,y,t,l)\mathrm{e}^{il(xJ+tA+yI)} \left[T^-(l,k) - T^+\left(l,k)\right]\mathrm{e}^{-ik(xJ+tA+yI)}dl\right] \frac{dk}{k-\lambda}. \quad (3.25)$$

The solution $Q(x,y,t)$ of the boundary value problem can be computed from the knowledge of any solution μ of the spectral problem (3.1) that satisfies the asymptotic requirement $\lim_{|\lambda|\to\infty} \mu(x,y,t,\lambda) = I$. Indeed, let

$$\mu = I + \frac{\mu^*}{\lambda} + O\left(\frac{1}{\lambda^2}\right). \quad (3.26)$$

Since μ satisfies equation (3.1(a)), we have

$$\frac{(\mu^*)_x}{\lambda} = i\lambda[J, \frac{(\mu^*)}{\lambda}] + Q + \frac{Q\mu^*}{\lambda} + \frac{J(\mu^*)_y}{\lambda} + O\left(\frac{1}{\lambda^2}\right).$$

hence, equating powers of λ, we find that $Q + i[J,\mu^*] = 0$. On the other hand, the asymptotic behaviour of μ for large λ implies that

$$\lambda(\mu - I) = \mu^* + O\left(\frac{1}{\lambda}\right) \Rightarrow \quad \mu^* = \lim_{|\lambda|\to\infty} \lambda(\mu - I).$$

It follows that $Q(x,y,t) = -i\lim_{|\lambda|\to\infty} \lambda[J,\mu]$.
In particular, in view of (3.26), this holds for $\mu = \mu_0$. Hence

$$Q(x,y,t) = -i \lim_{|\lambda|\to\infty} \lambda[J,\mu_0]. \quad (3.27)$$

Alternatively, we can obtain an expression for the matrix $Q(x,y,t)$ by computing the large λ asymptotics of (3.25). This yields for Q the expression

$$Q(x,y,t) = \frac{\hat{J}}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \mu^0(x,y,t,l)\mathrm{e}^{il(xJ+tA+yI)}\left(T^-(l,k)-T^+(l,k)\right)\mathrm{e}^{-ik(xJ+tA+yI)}dl\,dk. \tag{3.28}$$

which is the right hand side of (3.3).
QED

Remark 3.2. The problem can also be solved without recourse to the auxiliary eigenfunction μ_0. Indeed, define $F(l,\lambda)$ as the solution of the integral equation:

$$F(l,\lambda) - \int_{\mathbb{R}} T^-(l,m)F(m,\lambda)dm = T^-(l,\lambda) - T^+(l,\lambda). \tag{3.29}$$

Then simple algebraic manipulations yield

$$(\mu^+ - \mu^-)(x,y,t,\lambda) = \int_{\mathbb{R}} \mu^-(x,y,t,l)\mathrm{e}^{il(xJ+tA+yI)}F(l,\lambda)\mathrm{e}^{-i\lambda(xJ+tA+yI)}dl. \tag{3.30}$$

From this by taking the projection P_- we obtain a linear integral equation for μ^-.
Equation (3.30) defines a nonlocal RH problem for the function μ in terms of the functions $F(l,\lambda)$. This means that μ is given only indirectly in terms of the spectral functions $T^{\pm}$. From the expression for μ analogous to (3.25) we can obtain a representation of $Q(x,y,t)$ in terms of $\mu^-(x,y,t,\lambda)$ as above. The advantage of defining the eigenfunction μ_0 is that it yields a Riemann-Hilbert defined directly, via the jump given by (3.24), in terms of the spectral functions $T^{\pm}$. The rigorous analysis of this Riemann-Hilbert is thereby significantly simplified.

4 Conclusions

In this paper we have brought together the results and techniques investigated in a series of earlier works for the analysis of first-order $N \times N$ hyperbolic systems. Namely, we have considered initial-boundary value problems of the N-wave interaction equations posed in semi-infinite domains.
We have extended the analysis given in the existing literature for the Cauchy problem to the case of an initial-boundary value problem posed for these equations. The main step for this extension is the analysis of both ODEs in the associated Lax pair simultaneously. This idea, due to Fokas, has already been used to extend the inverse scattering formalism to a methodology for solving boundary value problems for integrable PDEs, see for example the recent monograph [7]. In the cases analysed here, all boundary values needed to represent the solution are prescribed as boundary conditions, hence are known. However, the construction of the eigenfunctions of the spectral problem must be essentially modified to account for the time dependence.
The novelty of our presentation consists also in the analysis side by side of the one- and two-space dimension case. This illustrates the formal similarities between the two cases, while highlighting the essential technical difference, namely the fact that the associated spectral problem gives rise to a local, respectively non-local, Riemann-Hilbert problem.

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