The spectral representation of two-point boundary value problems for third order linear evolution PDEs

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Abstract

We use a spectral transform method to study general boundary value problems for third order linear evolution PDEs with constant coefficients, posed on a finite space domain. We show how this method yields a simple characterization of the discrete spectrum of the associated spatial differential operator, and discuss the obstructions that arise when trying to represent the solution of such a problem as a series of exponential functions.

We first review the theory for second order two-point boundary value problems, and present an alternative way to derive the classical series representation, as well as an equivalent integral representation, which in general involves complex contours. We illustrate the advantages of the integral representation by studying in some detail the case when Robin-type boundary conditions are prescribed.

We then consider the third order case, and show that the integral representation is in general *not* equivalent to a discrete series representation, justifying a posteriori the failure of some of the classical approaches. We illustrate the third order case in detail using the example of the equation $q_t + q_{xxx} = 0$, for various types of boundary conditions. In contrast with the second order case, the qualitative properties of the spectrum of the associated spatial differential operator depend in this case not only on the equation but also on the type of boundary conditions. In particular, the solution appears to admit a series representation only when the prescribed boundary conditions couple the two endpoints of the interval.

1 Introduction

In this paper, we consider boundary value problems for third order linear evolution partial differential equations (PDEs) with constant coefficients. Specifically, we consider the following general problem:

$$q_t + \sigma q_{xxx} + b q_x = 0, \quad x \in [0, L], \ t > 0, \quad b \in \mathbb{R}, \ \sigma = \pm 1, \tag{1.1}$$

$$q(x, 0) = q_0(x), \quad x \in [0, L],$$

$$\alpha_0 q(0, t) + \alpha_1 q_x(0, t) + \alpha_2 q_{xx}(0, t) = h_1(t), \quad t > 0, \tag{1.2}$$

$$\beta_0 q(L, t) + \beta_1 q_x(L, t) + \beta_2 q_{xx}(L, t) = h_2(t), \quad t > 0, \tag{1.3}$$

$$\begin{cases} \gamma_0 q(L,t) + \gamma_1 q_x(L,t) + \gamma_2 q_{xx}(L,t) = h_3(t), & t > 0, \quad \text{if } \sigma = 1, \\ \gamma_0 q(0,t) + \gamma_1 q_x(0,t) + \gamma_2 q_{xx}(0,t) = h_3(t), & t > 0, \quad \text{if } \sigma = -1, \end{cases} \tag{1.4}$$

where L, α_i, β_i and γ_i are given real constants, $L > 0$, and $q_0(x)$, $h_{1(t)}$, $h_2(t)$ and $h_3(t)$ are prescribed functions, compatible at $x = 0$, $x = L$ and $t = 0$, sufficiently smooth, and such that when two boundary conditions are imposed at the same endpoint, they are linearly independent and compatible. Since in this paper we are not concerned with issues of regularity, we assume that all known functions are infinitely differentiable on their domain of definition, and we will not refer again to the functional class of the prescribed data. A rigorous analysis of such regularity issues, including the derivation of the precise functional properties of the solution, can be carried out as for problems posed on the half line, see [10]. We note that equation (1.1) captures the general case, as any other third order linear evolution equation (for which the Cauchy problem is well posed) can be reduced to this form by a simple change of variables. We assume that at least one of the α_j's, one of the β_j's and one of the γ_j's is nonzero. The problems above are all well posed on $[0, L]$, in the sense that they admit a unique (regular) solution [8]. We also consider the case of coupled boundary conditions, for example of conditions of the form

$$q_x(L,t) = \alpha q_x(0,t), \quad \alpha \in \mathbb{R}, \tag{1.5}$$

to replace one of the conditions given above. It can be shown, indeed it is well known in some cases [13], how some such coupled conditions, for example when (1.5) replaces (1.4) for equation (1.1), also yield a well posed problem.

The paper is divided into two main parts. In the first, we outline a new spectral approach, proposed by Fokas [4], which we use to derive the main results. We illustrate its use in the analysis of the second order case, to show how classical results are recovered and indicate the advantages it offers. In the second part of the paper, we consider the third order case (1.1), which is our main interest.

The discrete spectrum of the x-differential operator associated with a given boundary value problemis characterised as the set of zeros of a certain complex function, determined uniquely by the PDE and by the given boundary conditions. This characterisation defines the set of values λ such that $\omega(\lambda)$ is in the classical discrete spectrum, where ω is the symbol of the differential operator. We call this set the *effective* discrete spectrum of the PDE (see Definition 1.1). It coincides with the zeros of the determinant of a certain algebraic system of equations. This determinant has the form of a finite sum of exponentials. This implies that, although its precise zeros depend on the particular boundary conditions prescribed, they always lie, at least asymptotically, in a neighbourhood of specific rays in the complex plane, passing through the origin, and whose direction is only determined by the equation and by the kind of boundary conditions. For example, in the case of the simple third order PDE $q_t + q_{xxx} = 0$, if the prescribed boundary conditions are uncoupled these zeros lie always on three rays, while for coupled conditions the relevant rays are six. We will show that the location of these rays immediately implies, for the solution of the problem with coupled boundary conditions, the existence of a series representation, but that this is not the case when uncoupled boundary conditions are prescribed.
In what follows, $q = q(x,t)$, $(x,t) \in [0, L] \times [0, \infty)$ satisfies

$$q_t + \omega(-i\partial_x) q = 0, \tag{1.6}$$

where $\omega(k)$ is a polynomial of degree n (for simplicity, we assume this polynomial has n distinct roots), and $L > 0$ is constant.

Definition 1.1 *Suppose that, for a given boundary value problem for the PDE (1.6) the solution exists, it is unique, and it can be represented as a sum of exponential functions of the form*

$$q(x,t) = \sum_{m\in\mathbb{Z}} \mathrm{e}^{-\omega(k_m)t}\left[a_0(k_m)\mathrm{e}^{ik_m x} + \ldots + a_{n-1}(k_m)\mathrm{e}^{i\lambda_{n-1}(k_m)x}\right]\mathbf{S}(t,k_m,\ldots,\lambda_{n-1}(k_m)), \tag{1.7}$$

where k_m are countably many points in $\mathbb{C}$, $\{k,\lambda_1(k),\ \ldots,\ \lambda_{n-1}(k)\}$ is the set of the n roots of the polynomial (in λ) $\omega(k)-\omega(\lambda)=0$, $a_i(k)$, $i=0,\ldots,n-1$ are rational functions and $\mathbf{S}(t,k,\ldots,\lambda_{n-1}(k))$ is an entire function of the complex variable k, explicitly defined in terms of the given initial and boundary conditions.
We then call the set $\Sigma=\{k_m\}_{m\in\mathbb{Z}}$ the **effective discrete spectrum** *of the boundary value problem.*

The terminology "effective discrete spectrum" refers to the fact that when this spectrum is not empty, then the integral representation is readily seen to be equivalent to an infinite series of the form (1.7). In the process, a basis of eigenfunctions for the associated x-differential operator is *algorithmically constructed.* Moreover, this construction is independent of the explicit knowledge of the eigenvalues or any other functional-theoretic notion.

The paper is organised as follows. In section 2 we briefly review the method of Fokas, to be used in the sequel. In section 3, we use the new approach to rederive systematically the well-known series representation of the solution of second order boundary value problems as well as the alternative integral representation, and comment on the respective advantages. In section 4, we apply the Fokas method to the case of the equation $q_t + q_{xxx}$. Although this is the simplest possible example, it illustrates the general features of the results, and we then indicate how to generalise it to the general case. We discuss how for PDEs of the form (1.1), when the prescribed boundary conditions are uncoupled, we can use our approach to derive a series representation only if one of the boundary conditions couples the two ends of the interval, namely is of the form (1.5). Then we can *explicitly* compute a representation for the solution in the form of a discrete infinite series of exponential functions. This is consistent with the classical spectral theory of the associated x-differential operator, that guarantees the existence of a basis of eigenfunctions corresponding to a discrete spectrum (constructed explicitly e.g. in [13]).

2 The spectral method of Fokas

In the past few years, a method for solving boundary value problems for linear evolution PDEs was proposed by Fokas. This method provides a rigorous characterisation of the boundary value problems which admit a unique regular solution, and are in this sense well posed, and yields an *explicit integral representation* of the solution of any such problem, in the form

$$q(x,t) = \int_\Gamma \mathrm{e}^{ikx-\omega(k)t}R(k)dk, \tag{2.1}$$

where k is complex, and Γ is a contour in the complex k plane. The function $R(k)$, called the *spectral function*, is explicitly determined in terms of the given initial and boundary conditions. This representation is equivalent to the classical Fourier spectral representation, whenever the latter can be derived.

The integral spectral representation thus obtained was originally introduced as a first step towards solving the case of *nonlinear integrable* evolution PDEs, and it is more general than our brief discussion suggests. We outline below the main results we use in the present context.

2.1 The general integral representation

In a series of works [4, 6, 8, 10, 12], the following results were obtained:

(a) Well posed boundary value problems
On $\mathcal{D}$, a well posed boundary value problem for a PDE of order n is determined if, in addition to the initial condition $q(x,0)=q_0(x)$, n boundary conditions are prescribed. If these n conditions do not couple the two ends of the interval, then N of them should be given at $x=0$ and $n-N$ at $x=L$, where $N=n/2$ if n is even, and $N=(n\pm 1)/2$ if n is odd. The sign in the latter equality is determined by the sign of the highest order x-derivative. For example, $N=1$ for the PDE $q_t+q_{xxx}=0$, while $N=2$ for $q_t-q_{xxx}=0$ (see conditions (1.4)).

(b) The formal integral representation
The solution of such a boundary value problem admits the representation

$$q(x,t)=\frac{1}{2\pi}\left\{\int_{-\infty}^{\infty}\mathrm{e}^{ikx-\omega(k)t}\hat{q}_0(k)dk-\int_{\partial D^+}\mathrm{e}^{ikx-\omega(k)t}\tilde{f}(t,k)dk-\int_{\partial D^-}\mathrm{e}^{ik(x-L)-\omega(k)t}\tilde{g}(t,k)dk\right\}. \tag{2.2}$$

where

$$D=\{k\in\mathbb{C}:\ \mathrm{Re}\,\omega(k)\le 0\},\quad D^{\pm}=D\cap\mathbb{C}^{\pm}, \tag{2.3}$$

$$\hat{q}_0(k) = \int_0^L \mathrm{e}^{-ikx}q(x,0)dx, \tag{2.4}$$

$$\tilde{f}(t,k)=\int_0^t \mathrm{e}^{\omega(k)s}X(0,s,k)ds,\qquad \tilde{g}(t,k)=\int_0^t \mathrm{e}^{\omega(k)s}X(L,s,k)ds, \tag{2.5}$$

$$X(x,t,k) = i\frac{\omega(k)-\omega(-i\partial_x)}{k-(-i\partial_x)}q(x,t). \tag{2.6}$$

These definitions show that $\tilde{f}(t,k)$ and $\tilde{g}(t,k)$ are each a linear combination of the n *spectral functions* defined respectively by

$$\tilde{f}_j(t,k)=\int_0^t \mathrm{e}^{\omega(k)s}(\partial_x^j q)(0,s)ds,\quad\text{and}\quad \tilde{g}_j(t,k)=\int_0^t \mathrm{e}^{\omega(k)s}(\partial_x^j q)(L,s)ds,\quad j=0,..,n-1. \tag{2.7}$$

The rigorous derivation of this integral representation is based on the formulation of a Riemann-Hilbert problem in the complex plane.

Remark 2.1 The spectral functions depend on the variable t, but, using analyticity, and replacing in their definition $\int_0^t$ by $\int_0^T$, for any $T > t$, the solution representation (2.2) does not change. Hence this is effectively a representation in the general form (2.1).

(c) The global relation

The global relation is the following algebraic relation between the functions defined by the expressions (2.4) and (2.5):

$$\tilde{f}(t,k) - \mathrm{e}^{-ikL}\tilde{g}(t,k) = \hat{q}_0(k) - \mathrm{e}^{\omega(k)t}\hat{q}(t,k), \quad \text{where } \hat{q}(t,k) = \int_0^L \mathrm{e}^{-ikx} q(x,t)dx. \tag{2.8}$$

The important result of [9] implies that the analysis of this single equation yields an explicit representation for the spectral functions $\tilde{f}(t,k)$ and $\tilde{g}(t,k)$, in terms only of $q_0(x)$ and of the given boundary conditions.

(d) The analysis of the global relation and the algebraic system characterising the spectral functions

The functions $\tilde{f}_j$ and $\tilde{g}_j$ are entire functions of k, bounded as $k \to \infty$ for $k \in D$, which are invariant under any transformation in the complex k plane which leaves $\omega(k)$ invariant, i.e. under all transformations

$$\lambda(k) : \mathbb{C} \to \mathbb{C} \quad \text{such that } \omega(k) = \omega(\lambda(k)). \tag{2.9}$$

These transformations are determined by the roots of the polynomial $\omega(k) - \omega(\lambda) = 0$, $\lambda \in \mathbb{C}$. We denote the $n-1$ roots different from $k = \lambda_0(k)$ by $\lambda_m(k)$, $m = 1, \ldots, n-1$. These roots, and the behaviour of the corresponding transformations, are completely characterised in [10].

When the global relation (2.8) is supplemented with the $n-1$ equations obtained by evaluating it at the $n-1$ roots $\lambda_j(k)$, $j \geq 1$, of (2.9), the result is a system of n equations, involving the $2n$ spectral functions $\tilde{f}_j(t,k)$ and $\tilde{g}_j(t,k)$, of which n are known via the prescribed boundary conditions. Hence this is a system for the n unknown spectral function:

$$\begin{aligned}
&\tilde{f}(t,k) - \mathrm{e}^{-ikL}\tilde{g}(t,k) = \hat{q}_0(k) - \mathrm{e}^{\omega(k)t}\hat{q}(t,k),\\
&\tilde{f}(t,\lambda_1(k)) - \mathrm{e}^{-i\lambda_1(k)L}\tilde{g}(t,\lambda_1(k)) = \hat{q}_0(\lambda_1(k)) - \mathrm{e}^{\omega(k)t}\hat{q}(t,\lambda_1(k))\\
&\ldots\\
&\tilde{f}(t,\lambda_{n-1}(k)) - \mathrm{e}^{-i\lambda_{n-1}(k)L}\tilde{g}(t,\lambda_{n-1}(k)) = \hat{q}_0(\lambda_{n-1}(k)) - \mathrm{e}^{\omega(k)t}\hat{q}(t,\lambda_{n-1}(k)).
\end{aligned} \tag{2.10}$$

This system involves also the *unknown* auxiliary function $\hat{q}(t,k)$, defined in (2.8). The central issue is the characterisation of those boundary value problems for which the solution of this system is a set of n functions with the correct boundedness and analyticity properties, and *not* depending on $\hat{q}(t,k)$.

(e) The effective integral representation

The solution of the system (2.10) involves linear combinations of the entire functions $\hat{q}(t,k)$, $\hat{q}(t,\lambda_1)$,… $\hat{q}(t,\lambda_{n-1})$ multiplied by the factor $\mathrm{e}^{\omega(k)t}/\Delta(k)$, where Δ denotes the determinant of the system. After multiplication by the factor $\mathrm{e}^{ikx-\omega(k)t}$ (or $\mathrm{e}^{ik(x-L)-\omega(k)t}$) these terms

are bounded as $k \to \infty$ in D^+ (respectively D^-). Hence if the determinant of (2.10) has no zeros in D, Jordan's lemma implies that the integral around $\partial D^{\pm}$ of these terms vanishes, and it does not contribute to the solution representation.
If the determinant of (2.10) has zeros in D, then it can be shown that they are necessarily on the boundary of D. In this case, the explicit computation of the principal value contribution due to these poles can be computed explicitly by solving the system (2.10) to obtain $\Delta(k)\tilde{f}_i(k) = H(t,k) + (\mathit{terms\ in}\ \hat{q}_t)$, where $H(t,k)$ is known. Hence if $\Delta(k_0) = 0$, $k_0 \in D^+$, noting that the function $\tilde{f}_i$ is by definition bounded in D, we obtain

$$(\mathit{terms\ in}\ \hat{q})(t,k_0) = H(t,k_0).$$

A similar arguments can be applied to $\tilde{g}_i$ and D^-. This simple observation, which we will show in detail for the examples studied below, is instrumental in the derivation of our main results.
In summary, even if $\hat{q}(k,t)$ is not known, its contribution to the integral representation of the solution can either be neglected or computed explicitly.

3 Second order problems

For PDEs of second order, using separation of variables it is always possible to solve boundary value problems on the finite interval using (generalised) Fourier series. In this section we illustrate the method of Fokas, based on a transform which is simultaneously a transform in x and in t, for second order boundary value problems of the form

$$\begin{aligned} q_t &= aq_{xx}, \quad x \in [0,L], \ t > 0, \quad Re(a) \geq 0, && (3.1)\\ q(x,0) &= q_0(x), \quad x \in [0,L], && (3.2)\\ q(0,t) &= \alpha q_x(0,t) + h_1(t), \quad q(L,t) = \beta q_x(L,t) + h_2(t), \quad t > 0 \ \alpha,\ \beta \in \mathbb{C}, && (3.3) \end{aligned}$$

and we prove the following result.

Proposition 3.1 *Consider the PDE (3.1) where $q = q(x,t)$. Suppose the initial condition (3.2) and the boundary conditions (3.3) are sufficiently regular and compatible.*
The solution of this problem admits the integral representation (2.2), with

$$\omega(k) = ak^2, \quad D^{\pm} = \{k = k_1 + ik_2 \in \mathbb{C} : \ \mathrm{Re}\,(a)(k_1^2 - k_2^2) \leq 2\,\mathrm{Im}(a)k_1k_2\} \cap \mathbb{C}^{\pm},$$

where the spectral functions $\tilde{f}(t,k)$ and $\tilde{g}(t,k)$ are explicitly determined in terms of the prescribed initial and boundary condition.
This integral representation always admits an alternative formulation as an infinite series of exponential functions over the effective discrete spectrum Σ.

Note that *any* second order boundary value problem can be put in this form by a simple change of variable. The assumption $\mathrm{Re}(a) \geq 0$ eliminates any PDE for which the Cauchy problem is ill posed.

3.1 Proof of proposition 3.1

The dispersion relation of this equation is $\omega(k) = ak^2$, the function (2.6) is $X(x,t,k) = a(q_x(x,t) + ikq(x,t))$, and the global relation is given by

$$a\left[\tilde{f}_1(k) + ik\tilde{f}_0(k) - \mathrm{e}^{-ikL}(\tilde{g}_1(k) + ik\tilde{g}_0(k))\right] = \hat{q}_0(k) - \mathrm{e}^{ak^2t}\hat{q}(t,k). \tag{3.4}$$

The boundary conditions and the definition (2.7) of $\tilde{f}_j$, $\tilde{g}_j$ yield

$$\tilde{f}_0(t,k) = \tilde{h}_1(t) + \alpha\tilde{f}_1(t,k), \quad \tilde{g}_0(t,k) = \tilde{h}_2(t) + \beta\tilde{g}_1(t,k), \quad \tilde{h}_j(t,k) = \int_0^t \mathrm{e}^{ak^2s}h_j(s)ds, \tag{3.5}$$

therefore the global relation (3.4) becomes

$$a\left[(1+ik\alpha)\tilde{f}_1 - \mathrm{e}^{-ikL}(1+ik\beta)\tilde{g}_1\right] = N(t,k) - \mathrm{e}^{ak^2t}\hat{q}(t,k), \tag{3.6}$$

where $N(t,k)$ denotes the *known* function

$$N(t,k) = \hat{q}_0(k) - ika\tilde{h}_1(t,k) + \mathrm{e}^{-ikL}ika\tilde{h}_2(t,k). \tag{3.7}$$

The two roots of the equation $\omega(k) = \omega(\lambda)$ are in this case $\lambda_0 = k$ and $\lambda_1 = -k$. The evaluation of (3.6) at $-k$ yields the additional equation

$$a\left[(1-ik\alpha)\tilde{f}_1 - \mathrm{e}^{ikL}(1-ik\beta)\tilde{g}_1\right] = N(t,-k) - \mathrm{e}^{ak^2t}\hat{q}(t,-k). \tag{3.8}$$

A derivation of the series representation using the global relation
Solving the system (3.6)-(3.8) for $\tilde{f}_1$ and $\tilde{g}_1$ we obtain

$$\begin{aligned}
a\Delta(k)\tilde{f}_1 &= \mathrm{e}^{ikL}(1-ik\beta)N(t,k) - \mathrm{e}^{-ikL}(1+ik\beta)N(t,-k) && (3.9)\\
&\quad -\mathrm{e}^{ak^2t}\left[\mathrm{e}^{ikL}(1-ik\beta)\hat{q}(t,k) - \mathrm{e}^{-ikL}(1+ik\beta)\hat{q}(t,-k)\right]\\
a\Delta(k)\tilde{g}_1 &= (1-ik\alpha)N(t,k) - (1+ik\alpha)N(t,-k) && (3.10)\\
&\quad -\mathrm{e}^{ak^2t}\left[(1-ik\alpha)\hat{q}(t,k) - (1+ik\alpha)\hat{q}(t,-k)\right],
\end{aligned}$$

where

$$\Delta(k) = \mathrm{e}^{ikL}(1+ik\alpha)(1-ik\beta) - \mathrm{e}^{-ikL}(1-ik\alpha)(1+ik\beta). \tag{3.11}$$

If k_0 is such that $\Delta(k_0) = 0$, we obtain explicitly the terms involving $\hat{q}(t,k)$ as

$$\mathrm{e}^{ik_0L}(1-ik_0\beta)\hat{q}(t,k_0) - \mathrm{e}^{-ik_0L}(1+ik_0\beta)\hat{q}(t,-k_0) =$$

$$\mathrm{e}^{-ak_0^2t}\left[\mathrm{e}^{ik_0L}(1-ik_0\beta)N(t,k_0) - \mathrm{e}^{-ik_0L}(1+ik_0\beta)N(t,-k_0)\right].$$

It is therefore important to study the zeros of the determinant function $\Delta(k)$. It is not difficult to check directly that this function has infinitely many real zeros $\{k_m\}$. Since $\tilde{f}_1(k)$ is bounded for all $k \in \mathbb{R}$, using the definition of $\hat{q}(t,k)$, and the fact that the k_m's are zeros of (3.11), we obtain

$$\int_0^L \left[(1-ik_m\alpha)\mathrm{e}^{-ik_mx} - (1+ik_m\alpha)\mathrm{e}^{ik_mx}\right]q(x,t)dx =$$

$$\mathrm{e}^{-ak_m^2 t}\left[(1-ik_m\alpha)N(t,k_m)-(1+ik_m\alpha)N(t,-k_m)\right]. \tag{3.12}$$

This suggests the following eigenfunction expansion for $q(x,t)$:

$$q(x,t)=\sum_{m\in\mathbb{Z}} a_m(t)\mathrm{e}^{-ak_m^2 t}\left[(1+ik_m\alpha)\mathrm{e}^{ik_m x}-(1-ik_m\alpha)\mathrm{e}^{-ik_m x}\right]. \tag{3.13}$$

Indeed, a simple direct computation shows that the functions

$$e_m(x)=(1+ik_m\alpha)\mathrm{e}^{ik_m x}-(1-ik_m\alpha)\mathrm{e}^{-ik_m x}$$

are *orthogonal eigenfunctions* of the linear differential operator $D = i\partial^2/\partial x^2$, satisfying the homogeneous boundary conditions $e_m(0)+\alpha e_m'(0)=0, \quad e_m(L)+\beta e_m'(L)=0$. Orthogonality, relation (3.12), and the fact that all series converge uniformly in $(0,L)$ (this is guaranteed by classical spectral analysis), yield

$$a_m(t)=\frac{1}{c(m,L)}\left[(1+ik_m\alpha)N(t,-k_m)-(1-ik_m\alpha)N(t,k_m)\right], \quad c(m,L) \text{ constant}. \tag{3.14}$$

Note that the expression (3.13) does *not* converge uniformly as $x\to 0$ or $x\to L$. Hence it is not immediate to prove that the function $q(x,t)$ satisfies the given boundary conditions. However, we show below that this series is equivalent to the general integral expression, which indeed satisfies the boundary conditions [8].

The present derivation has the advantage of being constructive, and a direct computation then shows that the eigenfunctions constructed satisfy the desired properties. However the complexity of these computations and the results obtained are comparable to those of the classical eigenfunction expansion approach. In particular, the zeros k_m cannot be found explicitly in general. However, our method yields an alternative integral representation which does *not* depend on the knownledge of the zeros k_m and is therefore completely specified. We now derive this alternative representation.

The integral representation

For this equation, the domains $D^\pm$ are given by $D^\pm=\{k\in\mathbb{C}^\pm : Re(ak^2)\le 0\}$. Using the boundary conditions, the general integral representation (2.2) becomes

$$\begin{aligned} q(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{ikx-ak^2t}\hat{q}_0(k)dk-\frac{a}{2\pi}\int_{\partial D+}\mathrm{e}^{ikx-ak^2t}\left[(1+ik\alpha)\tilde{f}_1(t,k)+ik\tilde{h}_1\right](t,k)dk \\ &- \frac{a}{2\pi}\int_{\partial D-}\mathrm{e}^{ik(x-L)-ak^2t}\left[(1+ik\beta)\tilde{g}_1+ik\tilde{h}_2\right](t,k)dk. \end{aligned} \tag{3.15}$$

where $\tilde{f}_1(t,k)$ and $\tilde{g}_1(t,k)$ are given by the expressions (3.9) and (3.10), and $\tilde{h}_j$ is defined by (3.5). In this expression all terms are explicitly known, except for the terms

$$\int_{\partial D+}\mathrm{e}^{ikx}(1+ik\alpha)\frac{\mathrm{e}^{ikL}(1-ik\beta)\hat{q}(t,k)-\mathrm{e}^{-ikL}(1+ik\beta)\hat{q}(t,-k)}{\Delta(k)}dk,$$

and the corresponding one along ∂D^-. It is easy to check that the integrands of these terms are bounded as $k\to\infty$ in D^+ and D^- respectively, except at the zeros of $\Delta(k)$, which are on $\mathbb{R}$. The real axis may be either exterior to D, or part of its boundary. Indeed, we can distiguish the following two cases:

Case (i): $\mathbf{a_1 \neq 0}$ In this case, setting $a = a_1 + ia_2$, we have

$$D = \{k_2 \geq \frac{k_1}{a_1}(\sqrt{a_1^2 + a_2^2} - a_2)\} \cup \{k_2 \leq -\frac{k_1}{a_1}(\sqrt{a_1^2 + a_2^2} + a_2)$$

and the real axis $k_2 = 0$ is not in D, except possibly for the point $k = 0$.
Case (ii): $\mathbf{a_1 = 0}$ In this case, $D = \{k_1 k_2 \geq 0\}$ if $a_2 > 0$, or $D = \{k_1 k_2 \leq 0\}$ if $a_2 < 0$. In both instances, the boundary of D contains the real axis.

When the real axis is exterior to D, the terms involving the unknown functions are analytic and bounded in D, and an application of Jordan's lemma implies that they do not contribute to the integral representation of the solution. Otherwise, using analyticity the contour ∂D can be indented to pass above (in D^+) or below (in D^-) each of these zeros, and then again the contribution of these terms vanishes.
The difference between the two cases (i) and (ii) above is also reflected in how the equivalence of the series and integral form of the solution representation is proved.
Equivalence of the series and integral representations
We now show how the explicit computation of the principal value contributions in the integral representation at these zeros of Δ reproduces the series representation.
Assume without loss of generality that $\alpha = 0$, and consider the term integrated along ∂D^- (analogous considerations apply to the integral on ∂D^+). This term is given by

$$\int_{\partial D^-} \mathrm{e}^{ik(x-L)-ak^2t} \left[\frac{N(t,k) - N(t,-k)}{\Delta(k)} + ika\tilde{h}_2(t,k)\right] dk.$$

Case (i) In this case, all terms appearing in this integral are analytic and bounded as $k \to \infty$ inside $\mathbb{C}^- \setminus D^-$, except at the zeros of $\Delta(k)$, which lie in this region. Therefore, deforming ∂D^- to $-R^+$, we pass through these zeros, which we denote by k_n, and compute the residue contribution at each one of them. This yields

$$\int_{\partial D^-} = -\int_{\mathbb{R}} + \sum_{k_n} \mathrm{e}^{ik_n(x-L)-ak_n^2t} \frac{N(t,k_n) - N(t,-k_n)}{\Delta'(k_n)}.$$

Case (ii) In this case, $\int_{\partial D^-} = -\oint_{\mathbb{R}}$, where we have indented the negative real line at each of the zeros of Δ by a circle of radius ε lying in C^-. The computation of the integral around these circles can be explicitly carried out in terms of the residues of the function at these points, which as before yields an extra term given by a series of the same form as the one in case (i).
In summary, we can deform the contour ∂D to the real axis and realise the integral representation entirely on $\mathbb{R}$, but in the process the representation acquires a series term due to the residues at the zeros of the function $\Delta(k)$. A straightforward computation then shows that the integral terms cancel out, and the only terms left is the infinite sum on the zeros. Indeed, the total sum of the integrals over $\mathbb{R}$ (up to a constant factor), setting $\lambda_1(k) = -k$ and $\Delta(k) = \mathrm{e}^{-ikL}(1 - \mathrm{e}^{2ikL})$, is given by

$$\int_{-\infty}^{\infty} \mathrm{e}^{ikx-ak^2t}\hat{q}_0(k)dk + \int_{-\infty}^{\infty} \mathrm{e}^{ikx-ak^2t} \left\{\frac{N(t,-k) - \mathrm{e}^{2ikL}N(t,k)}{1 - \mathrm{e}^{2ikL}} + ika\tilde{h}_1(k)\right\} dk$$

$$+\int_{-\infty}^{\infty} \mathrm{e}^{ikx-ak^2t}\left\{\frac{N(t,k)-N(t,-k)}{1-\mathrm{e}^{2ikL}}+ika\tilde{h}_2(k)\right\}dk.$$

Recalling the definition (3.7) of $N(t,k)$, this sum is easily seen to be zero, proving that the only contribution left in the solution representation is the series term.
QED

Remark 3.1 The same conclusion holds if the boundary conditions couple the ends of the interval. The proof is essentially identical to the one given above.

4 Third order evolution PDEs

We now consider third order boundary value problems of the general form (1.1). We show that, following the same approach as outlined in the previous section, the only case for which the contour deformation argument yields a series representation is the case that the prescribed boundary conditions couple the two ends of the interval $[0,L]$. Indeed, using the spectral approach of Fokas, we prove the following result:

Proposition 4.1 *Consider the boundary value problem (1.1)-(1.4).*
The solution of this problem admits the integral representation (2.2), with

$$\omega(k)=-\sigma ik^3-ibk,\quad D^{\pm}=\{k\in\mathbb{C}:\ \mathrm{Re}\omega(k)\le 0\}\cap\mathbb{C}^{\pm},$$

where the spectral functions $\tilde{f}(t,k)$ and $\tilde{g}(t,k)$ are explicitly determined in terms of the initial and boundary conditions prescribed. This representation is canonical, in the sense that the integration contours in (2.2) cannot be deformed to a different contour in the complex plane.

A consequence of the above results is that when the effective discrete spectrum Σ is outside D, we cannot rewrite the integral representation as a series by simply computing the residue contributions explicitly. However, this is possible if the given boundary conditions couple the two ends of the interval, as for periodic boundary conditions. In this case, the conclusions of the above proposition are essentially different. We will discuss this result at the end of this section.

4.1 The equation $q_t+q_{xxx}=0$

We start by analysing the example of the simplest possible such equation,

$$q_t+q_{xxx}=0,\quad 0<x<L,\ \ 0<t<T. \tag{4.1}$$

The dispersion relation of this equation is given by $\omega(k)=-ik^3$, so that the domain $D=D^+\cup D^-$ has three connected components, given by

$$D^+=\{k:\ \pi/3<\arg(k)<2\pi/3\},\quad D^-=D_1^-\cup D_2^-, \tag{4.2}$$

$$D_1^-=\{k:\ \pi<\arg(k)<4\pi/3\},\quad D_2^-=\{k:\ 5\pi/3<\arg(k)<2\pi\},$$

and $X(x,t,k) = k^2 q(x,t) - ikq_x(x,t) - q_{xx}(x,t)$. It follows that the spectral functions are given by

$$\tilde{f}(t,k) = \int_0^t \mathrm{e}^{-ik^3t}\left(k^2 q - ikq_x - q_{xx}\right)(0,t)dt = k^2\tilde{f}_0(t,k) - ik\tilde{f}_1(t,k) - \tilde{f}_2(t,k),$$

$$\tilde{g}(t,k) = \int_0^t \mathrm{e}^{-ik^3t}\left(k^2 q - ikq_x - q_{xx}\right)(L,t)dt = k^2\tilde{g}_0(t,k) - ik\tilde{g}_1(t,k) - \tilde{g}_2(t,k).$$

The global relation (with $\hat{q}(t,k)$ given in (2.8)) is

$$\tilde{f}(t,k) + \mathrm{e}^{-ikL}\tilde{g}(t,k) = \hat{q}_0(k) - \mathrm{e}^{-ik^3t}\hat{q}(t,k). \tag{4.3}$$

The representation of the solution of this problem is formally given by

$$\begin{aligned} q(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{ikx+ik^3t}\hat{q}_0(k)dk - \frac{1}{2\pi}\int_{\partial D+}\mathrm{e}^{ikx+ik^3t}\tilde{f}(t,k)dk \\ &- \frac{1}{2\pi}\int_{\partial D-}\mathrm{e}^{ik(x-L)+ik^3t}\tilde{g}(t,k)dk. \end{aligned} \tag{4.4}$$

The three roots of the equation $\omega(k) = \omega(\lambda)$ are

$$\lambda_0(k) = k, \quad \lambda_1(k) = \zeta k, \quad \lambda_2(k) = \zeta^2 k, \qquad \zeta = \mathrm{e}^{2\pi i/3}.$$

Hence the system involving the spectral functions is in this case

$$\begin{aligned} & k^2\tilde{f}_0 - ik\tilde{f}_1 - \tilde{f}_2 - \mathrm{e}^{-ikL}\left(k^2\tilde{g}_0 - ik\tilde{g}_1 - \tilde{g}_2\right) = \hat{q}_0(k) - \mathrm{e}^{-ik^3t}\hat{q}(t,k), \\ & \zeta^2k^2\tilde{f}_0 - i\zeta k\tilde{f}_1 - \tilde{f}_2 - \mathrm{e}^{-i\zeta kL}\left(\zeta^2k^2\tilde{g}_0 - i\zeta k\tilde{g}_1 - \tilde{g}_2\right) = \hat{q}_0(\zeta k) - \mathrm{e}^{-ik^3t}\hat{q}(t,\zeta k), \\ & \zeta k^2\tilde{f}_0 - i\zeta^2k\tilde{f}_1 - \tilde{f}_2 - \mathrm{e}^{-i\zeta^2kL}\left(\zeta k^2\tilde{g}_0 - i\zeta^2k\tilde{g}_1 - \tilde{g}_2\right) = \hat{q}_0(\zeta^2k) - \mathrm{e}^{-ik^3t}\hat{q}(t,\zeta^2k). \end{aligned} \tag{4.5}$$

“Dirichlet” boundary conditions

We now consider a specific boundary value problem for equation (4.1), defined when the prescribed boundary conditions are

$$q(0,t) = f_0(t), \quad q(L,t) = g_0(t), \quad q_x(L,t) = g_1(t), \tag{4.6}$$

which we improperly call “Dirichlet” boundary conditions.
The explicit representation of the solution is obtained by solving the system (4.5) with $\tilde{f}_0(k)$, $\tilde{g}_0(k)$ and $\tilde{g}_1(k)$ known. Setting

$$N(t,k) = \hat{q}_0(k) - k^2\left(\tilde{f}_0 - \mathrm{e}^{-ikL}\tilde{g}_0\right) + \mathrm{e}^{-ikL}ik\tilde{g}_1, \tag{4.7}$$

we obtain the system

$$\begin{aligned} & -ik\tilde{f}_1 - \tilde{f}_2 + \mathrm{e}^{-ikL}\tilde{g}_2 = N(t,k) - \mathrm{e}^{-ik^3T}\hat{q}(t,k), \\ & -i\zeta k\tilde{f}_1 - \tilde{f}_2 + \mathrm{e}^{-i\zeta kL}\tilde{g}_2 = N(t,\zeta k) - \mathrm{e}^{-ik^3t}\hat{q}(t,\zeta k), \\ & -i\zeta^2k\tilde{f}_1 - \tilde{f}_2 + \mathrm{e}^{-i\zeta^2kL}\tilde{g}_2 = N(t,\zeta^2k) - \mathrm{e}^{-ik^3t}\hat{q}(t,\zeta^2k), \end{aligned} \tag{4.8}$$

Taking as unknowns the functions $ik\tilde{f}_1$, $\tilde{f}_2$ and $\tilde{g}_2$, the determinant of this system is equal to $(\zeta - \zeta^2)\Delta(k)$, where

$$\Delta(k) = \left(\mathrm{e}^{-ikL} + \zeta \mathrm{e}^{-i\zeta kL} + \zeta^2 \mathrm{e}^{-i\zeta^2 kL}\right), \quad \zeta = \mathrm{e}^{2\pi i/3}, \tag{4.9}$$

If $\Delta(k)$ has zeros in D, the solutions of this syetms will be meromorphic functions in this domain. Hence we need to determine where the zeros of the function $\Delta(k)$ are located in the k complex plane. This is an entire function of order one, hence by the Hadamard factorization theorem it necessarily has infinitely many zeros. We now invoke Theorem 9, Ch. VI of [11], which implies that the zeros of this functions, which can accumulate only at infinity, must cluster along the semilines

$$L_1 = \{k : \arg(k) = 3\pi/2\}, \quad L_2 = \{k : \arg(k) = \pi/6\}, \quad L_3 = \{k : \arg(k) = 5\pi/6\}.$$

In this case, all zeros occur precisely on these semilines. A direct proof of this fact is given in the Appendix.

For the present discussion, the relevant fact is that the zeros *are all outside the domain D*. As a consequence, the unique solutions of the system (4.8) are entire functions of k, for $k \in D$, and the integral of the terms involving $\hat{q}(t, k)$ vanishes. In addition, the integration contour in the representation (4.4) is *canonical*:

Theorem 4.1 *Consider the boundary value problem for equation (4.1) with boundary conditions (4.6). The solution representation (4.4) for this problem can be written in the explicit form*

$$\begin{aligned} q(x,t) \;=\; & \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{ikx+ik^3t}\hat{q}_0(k)dk - \int_{\partial D^+} \mathrm{e}^{ikx+ik^3t}k^2\tilde{f}_0(t,k)dk \\ & - \int_{\partial D^-} \mathrm{e}^{ik(x-L)+ik^3t}(k^2\tilde{g}_0(t,k) - ik\tilde{g}_1(t,k))dk \\ + & \int_{\partial D^+} \mathrm{e}^{ikx+ik^3t}\frac{N(k)(\zeta^2\mathrm{e}^{-i\zeta^2kL} - \zeta\mathrm{e}^{-i\zeta kL}) + N(\zeta k)\zeta\mathrm{e}^{-ikL} + N(\zeta^2k)\zeta^2\mathrm{e}^{-ikL}}{\mathrm{e}^{-ikL} + \zeta\mathrm{e}^{-i\zeta kL} + \zeta^2\mathrm{e}^{-i\zeta^2kL}}dk \\ + & \int_{\partial D^-} \mathrm{e}^{ik(x-L)+ik^3t}\frac{N(t,k) + \zeta N(t,\zeta k) + \zeta^2N(t,\zeta^2k)}{\mathrm{e}^{-ikL} + \zeta\mathrm{e}^{-i\zeta kL} + \zeta^2\mathrm{e}^{-i\zeta^2kL}}dk, \end{aligned} \tag{4.10}$$

where $N(t, k)$ is defined by (4.7).
The countour ∂D for the integrals in (4.10) cannot be deformed to a different countour.

Proof: The proof is simply a computation, solving the system (4.8) and substituting the resulting expressions in the general expression (4.4). Since the zeros of the determinant are outside D, the terms containing $\hat{q}(t, k)$ are analytic and bounded in D^+ and D^- respectively, hence their integral around the boundary of D does not contibute to the representation.

To prove the last claim, we show that the integrand of the integral along ∂D^+ in the expression (4.4) is not bounded in the interior of either D^+ or $\mathbb{C}^+ \setminus D^+$. A similar argument applies to the integral along ∂D^-.

Clearly the contour cannot be deformed through D^+. Indeed, by definition of D, the exponential e^{ik^3t}, which is the only exponential involving t, has positive real part in D, hence it is not bounded as $k \to \infty$ inside this domain.

We now show the integrand in the fourth integral appearing in (4.4) is not bounded as $k \to \infty$ if $\pi/6 < \arg(k) < \pi/3$. As $k \to \infty$ inside this wedge, the dominant behaviour of the denominator is $\mathrm{e}^{-ikL} + \zeta \mathrm{e}^{-i\zeta kL}$ (the remaining terms are exponentially small). Hence we need to estimate the behaviour of

$$\mathrm{e}^{k_2L} + \zeta \mathrm{e}^{-k_2L/2+\sqrt{3}k_1L/2} = \mathrm{e}^{k_2L}(1 + \zeta \mathrm{e}^{-3k_2L/2+\sqrt{3}k_1L/2}).$$

Setting $k = k_1 + ik_2$, since in this wedge, $k_1 < \sqrt{3}k_2$, we have $\mathrm{e}^{-3k_2L/2+\sqrt{3}k_1L/2} < \mathrm{e}^{(-3/2+1/2)k_2L} = \mathrm{e}^{-k_2L}$ and asymptotically

$$\frac{1}{\mathrm{e}^{k_2L}(1 + \zeta \mathrm{e}^{-3k_2L/2+\sqrt{3}k_1L/2})} > \frac{1}{\mathrm{e}^{k_2L}(1 + \zeta \mathrm{e}^{-k_2L/2})} \sim \frac{1}{\mathrm{e}^{k_2L}}.$$

It follows that the dominant term in the denominator is e^{-ikL}. Therefore the integrand contains at least the term $\mathrm{e}^{ikx}N(\zeta k)$ which is *not* bounded in k, and we cannot deform the contour of integration.

QED

The fact that we cannot deform the contour of integration to the lines where the poles of the integrand lie implies in particular that we cannot include the contribution of the zeros of $\Delta(k)$ in the representation by a residue computation, indicating that the effective discrete spectrum of this boundary value problem is empty. See also Remark 4.1 below.

4.2 General uncoupled boundary conditions

We have analysed in detail the case that the given conditions are $q(0,t)$, $q(L,t)$ and $q_x(L,t)$. In general, the following lemma is valid. The proof is a straightforward computation.

Lemma 4.1 *The determinant $\Delta(k)$ of the system obtained from system (4.5) when any one of the functions $\tilde{f}_j$ and any two of the functions $\tilde{g}_j$ are known, has (up to linear terms in ζ and ζ^2) one of the following three forms:*

$$(1)\ \Delta(k) \sim \mathrm{e}^{ikL}\left(\mathrm{e}^{-ikL} + \zeta \mathrm{e}^{-i\zeta kL} + \zeta^2 \mathrm{e}^{-i\zeta^2 kL}\right), \quad (2)\ \Delta(k) \sim \mathrm{e}^{ikL}\left(\mathrm{e}^{-ikL} + \mathrm{e}^{-i\zeta kL} + \mathrm{e}^{-i\zeta^2 kL}\right),$$

$$(3)\ \Delta(k) \sim \mathrm{e}^{ikL}\left(\mathrm{e}^{-ikL} + \zeta^2 \mathrm{e}^{-i\zeta kL} + \zeta \mathrm{e}^{-i\zeta^2 kL}\right).$$

Note that the domain D depends only on the equation (as it is defined by the dispersion relation). Hence we only need to show that the zeros of the function $\Delta(k)$ lie outside of D in all three cases. Indeed, the zeros of these three functions, although they do not coincide, lie on the same lines L_i, $i = 1, 2, 3$. A direct proof is given in the Appendix.

4.3 Coupled boundary conditions

If the given boundary conditions are *periodic*, it is easy to see that the system for the unknown boundary conditions has determinant

$$\Delta(k) = 3(\zeta^2 - \zeta)(1 - \mathrm{e}^{-ikL})(1 - \mathrm{e}^{-i\zeta kL})(1 - \mathrm{e}^{-i\zeta^2 kL}).$$

The zeros of this function are clearly on the boundary of D, and in this case it is possible to obtain directly from the system the series representation of the solution.

This conclusion is more general. As an example, we assume now that the prescribed boundary conditions are

$$q(0,t) = q(L,t) = 0, \quad q_x(L,t) = \alpha q_x(0,t), \ \ \alpha \in \mathbb{R}.$$

For $\alpha = 0$, these are the "Dirichlet" conditions we have already studied. However, we now assume $\alpha \neq 0$.

This boundary value problem was considered in the context of control theory, with $|\alpha| < 1$, in [13], where it was shown that the associated linear ordinary differential operator admits a complete basis of orthogonal eigenfunctions. Therefore the solution of the boundary value problem can be expressed as a series in this eigenfunction basis.

We now consider this problem using our approach, and confirm that in this case this method yields the series representation as well as an equivalent integral representation for the solution of the boundary value problem. In addition, analysing the solvability of the system (4.11) below, it is not difficult to show, as in [12], that indeed this boundary value problem is well posed for every α. We do not therefore assume that $|\alpha| < 1$.

The substitution of the given conditions in the global relation (4.3), and its evaluation at k, ζk and $\zeta^2 k$ (where $\zeta = 2\pi i/3$) yields the system

$$\begin{aligned} &ik(\alpha \mathrm{e}^{-ikL} - 1)\tilde{f}_1 - \tilde{f}_2 + \mathrm{e}^{-ikL}\tilde{g}_2 = \hat{q}_0(k) - \mathrm{e}^{-ik^3t}\hat{q}(t,k), \\ &i\zeta k(\alpha \mathrm{e}^{-i\zeta kL} - 1)\tilde{f}_1 - \tilde{f}_2 + \mathrm{e}^{-i\zeta kL}\tilde{g}_2 = \hat{q}_0(\zeta k) - \mathrm{e}^{-ik^3t}\hat{q}(t,\zeta k), \\ &i\zeta^2 k(\alpha \mathrm{e}^{-i\zeta^2 kL} - 1)\tilde{f}_1 - \tilde{f}_2 + \mathrm{e}^{-i\zeta^2 kL}\tilde{g}_2 = \hat{q}_0(\zeta^2 k) - \mathrm{e}^{-ik^3t}\hat{q}(t,\zeta^2 k), \end{aligned} \tag{4.11}$$

with determinant

$$\Delta(k,\alpha) = (\zeta - \zeta^2)\left\{\mathrm{e}^{-ikL} + \zeta\mathrm{e}^{-i\zeta kL} + \zeta^2\mathrm{e}^{-i\zeta^2 kL} + \alpha(\mathrm{e}^{ikL} + \zeta\mathrm{e}^{i\zeta kL} + \zeta^2\mathrm{e}^{i\zeta^2 kL})\right\}.$$

To determine where the zeros of this function are located in the complex k plane, we now appeal to the cited theorem of [11]. This theorem states that the location of the zeros depend only on the exponents appearing in each exponential, and with a simple geometric construction identifies a number of rays, issuing from the origin, in whose neighbourhood the zeros must lie. In the present case, following this construction we find the rays

$$\arg(k) = 0, \ \ \arg(k) = \frac{\pi}{3}, \ \ \arg(k) = \frac{2\pi}{3}, \ \ \arg(k) = \pi, \ \ \arg(k) = \frac{4\pi}{3}, \ \ \arg(k) = \frac{5\pi}{3} \tag{4.12}$$

which coincide with the boundary of the domain D where $\mathrm{Re}\,\omega(k) \leq 0$. Although the theorem does not allow us to assert that the zeros are precisely on these rays, they will cluster along them asymptotically This result is sufficient for our purposes, as it immediately implies that these zeros lie, at least asymptotically, on ∂D.

We now solve the system (4.11) for one of the unknowns, e.g. $\tilde{f}_1(t,k)$. This yields

$$\begin{aligned} ik\Delta(k,\alpha)\tilde{f}_1(t,k) &= \hat{q}_0(k)(\mathrm{e}^{-i\zeta kL} - \mathrm{e}^{-i\zeta^2 kL}) + \hat{q}_0(\zeta k)(\mathrm{e}^{-i\zeta^2 kL} - \mathrm{e}^{-ikL}) + \hat{q}_0(\zeta^2 k)(\mathrm{e}^{-ikL} - \mathrm{e}^{-i\zeta kL}) \\ &-\mathrm{e}^{-ik^3t}\left[\hat{q}(t,k)(\mathrm{e}^{-i\zeta kL} - \mathrm{e}^{-i\zeta^2 kL}) + \hat{q}(t,\zeta k)(\mathrm{e}^{-i\zeta^2 kL} - \mathrm{e}^{-ikL}) + \hat{q}(t,\zeta^2 k)(\mathrm{e}^{-ikL} - \mathrm{e}^{-i\zeta kL})\right]. \end{aligned}$$

Consider e.g. the zeros $k_n \in \mathbb{R}^+$ of $\Delta(k,\alpha)$ (the argument for zeros on other rays is analogous). Evaluating this expression at these zeros we obtain

$$\hat{q}(t,k_n)(\mathrm{e}^{-i\zeta k_n L} - \mathrm{e}^{-i\zeta^2 k_n L}) + \hat{q}(t,\zeta k_n)(\mathrm{e}^{-i\zeta^2 k_n L} - \mathrm{e}^{-ik_n L}) + \hat{q}(t,\zeta^2 k_n)(\mathrm{e}^{-ik_n L} - \mathrm{e}^{-i\zeta k_n L}) =$$

$$\mathrm{e}^{ik_n^3 t}\left[\hat{q}_0(k_n)(\mathrm{e}^{-i\zeta k_n L} - \mathrm{e}^{-i\zeta^2 k_n L}) + \hat{q}_0(\zeta k_n)(\mathrm{e}^{-i\zeta^2 k_n L} - \mathrm{e}^{-ik_n L}) + \hat{q}_0(\zeta^2 k_n)(\mathrm{e}^{-ik_n L} - \mathrm{e}^{-i\zeta k_n L})\right]. \tag{4.13}$$

Up to constant multiple, the above expression is bounded as $k_n \to \infty$, where k_n are the zeros of Δ. Assuming that the right hand side of expression (4.3) can be written as the L^2 inner product (q_0, e_n), for some basis functions $e_n(x)$, and using the fact that $\overline{\zeta} = \zeta^2$ and that the k_n's are real, we find

$$e_n(x) = \mathrm{e}^{ik_n x}(\mathrm{e}^{i\zeta^2 k_n L} - \mathrm{e}^{i\zeta k_n L}) + \mathrm{e}^{i\zeta k_n x}(\mathrm{e}^{ik_n L} - \mathrm{e}^{i\zeta^2 k_n L}) + \mathrm{e}^{i\zeta^2 k_n x}(\mathrm{e}^{i\zeta k_n L} - \mathrm{e}^{ik_n L}). \tag{4.14}$$

Indeed, it is easy to verify that e_n's are eigenfunctions of the operator $D = \frac{\partial}{\partial x^3}$ and verify the boundary conditions $e_n(0) = e_n(1) = e_n'(1) - \alpha e_n'(0) = 0$ (the last condition follows from $\Delta(-k_n) = \overline{\Delta(k_n)}$). One can show directly, though this is not straightforward, that some appropriate multiples of these eigenfunctions, as well as the analogous eigenfunctions corresponding to negative real zeros, provide a complete orthogonal basis, see [13]. However in the present setting we can deduce this fact from the equivalence of the integral representation with the series obtained by computing the residues at the poles of the integrand, i.e. the points k_n, which are on the boundary of the domain of integration, D.

Remark 4.1 It is natural to try to derive a series representation for the solution $q(x,t)$ of the Dirichlet boundary value problem by directly evaluating the solution of the system (4.8) at the zeros of the determinant $\Delta(k)$, as done above for the case of coupled boundary conditions. However, in this case this approach is problematic, as the spectral relation obtained cannot be directly inverted. This is consistent with the fact that in this case the integral representation cannot be deformed to include the residue contributions. We know of no direct proof that a series representation exists in this case, see also Remark 3.1 in [13].

4.4 General third order dispersive case

We now indicate how the result obtained for the special case considered in the previous sections can be used to yield a result for the general third order PDE (1.1), for which the dispersion relation is of the form $\omega(k) = \pm ik^3 - ibk$, with $b \in \mathbb{R}$.

The main idea used in this generalisation is that, since the integrand in the integral representation of the solution of these problems is an entire function of k, the integration contour can be deformed inside any disc of radius $R < \infty$. This implies in particular that the only properties that affect the solutions are those valid asymptotically as $k \to \infty$.

It is proved in [10] that the domain $D = \{\pm k^2(3k_1^2 - k_2^2 \pm b \leq 0\}$, tends, as $k \to \infty$ inside it, to the simple picture obtained when considering the equation $q_t + q_{xxx} = 0$: as $k \to \infty$, D has three connected components, and these components are obtained from each other by rotations of $\pm 2\pi/3$. Therefore, it is easy to generalise the results proved above to the proof of proposition 4.1, which we omit.

5 Conclusions

We presented an alternative appproach to derive the representation of the solution of any well posed boundary value problem. This approach always yields *algorithmically* an integral representation involving complex contours of integration. This representation is explicit, as the integrand can be computed in terms of the dispersion relation and the given initial and boundary data. In the course of the derivation, we obtain the *global relation* among all boundary values of the solution, which plays a central role in the analysis, and it is indeed the fundamental new ingredient in the present approach.
For second order problems, this approach can be used also to derive the series representation through an algorithmic construction of the eigenfunctions of the associated linear ordinary differential x-operator. We stress however that the equivalent integral representation has some advantages over the series one, namely it does not depend explicitly on the eigenvalues (which cannot always be computed exactly) and has better convergence properties at the boundary points. It is also more convenient for studying the asymptotic behaviour of the solution for large times.
For third order equations, we have shown that a series representation can be derived in this manner only if the poles of the spectral functions are inside a certain domain, identified only by the equation, and thsi is only the case if the boundary conditions couple the two ends of the interval. The main tool in this analysis is the possibility of locating the zeros of the function which characterises the spectrum of the x-operator, exploiting general results in the theory of entire functions.

We remark that the original motivation for the present study was the analysis of integrable equations such as NLS and KdV on a finite interval, with boundary conditions that are not periodic. In case of periodic boundary conditions, these equations admit special solutions, given in terms of theta functions for NLS, and called “n-gap solitons”; “cnoidal waves” for KdV. These solutions are periodic, and their linear limit is precisely the trigonometric series solution. Hence these structures are ”linearisable”. This is to be contrasted with the case of soliton solutions, which do not have a linear counterpart: solitons are an intrinsically nonlinear phenomenon.
The interest was then to study whether these equations, when posed on a finite interval, could support special, linear or nonlinear, structures. Solitons are generically destroyed by the interaction with the boundary, and when they survive it, the mechanism to recover them analytically (in correspondence with the singularities of the associated Riemann-Hilbert problem) is well understood. The question of interest is whether other special, linearisable solutions can exist. The results presented here seem to indicate that, generically, they do not: indeed, in the linear limit, the KdV does not have the discrete spectrum that would result from the linearisations of such structures. Some specific boundary value problems for the NLS may on the other hand admit n-gap solitons. For recent results on this, see [7]. Recent developments in the analysis of evolution integrable nonlinear equations in one space dimension [2] indicate that it will be possible to use the analysis of the linear problem, presented here, to analyse the corresponding nonlinear boundary value problems, and to establish rigorously the results conjectured above.

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Appendix - The zeros of $\mathbf{F_0(z) = e^z + e^{\zeta z} + e^{\zeta^2 z}}$ and of its z-derivatives

Consider the function $F_0(z)$, where$\zeta = e^{2\pi i/3}$. The derivatives of this function with respect to z yield the two functions

$$F_1(z) = e^z + \zeta e^{\zeta z} + \zeta^2 e^{\zeta^2 z}, \quad F_2(z) = e^z + \zeta^2 e^{\zeta z} + \zeta e^{\zeta^2 z}.$$

By the change of variable $z = -ikL$, these three functions exhaust the possible determinants for any boundary value problem posed for the studied in section 4, for the case of uncoupled boundary conditions.
The functions $F_i(z)$ are entire functions of order one. This implies, via Hadamard's factorization and results in [11], that they have infinitely many zeros in the complex plane. These zeros can only accumulate at infinity, and are clustered along the negative real axis $L_1 = \{z : \arg(z) = -\pi\}$, and, by symmetry, on the two lines L_2, L_3 obtained by rotating the negative real axis by $2\pi/3$ and $-2\pi/3$. However, in this case the symmetry of the problem allows us to prove a stronger result.

Proposition 7.1 *The zeros of the function $F_\mu(z)$, $\mu = 0, 1, 2$, lie on the lines lines*

$$\mathcal{L}_1 = \{z : \arg(z) = -\pi\}, \quad \mathcal{L}_2 = \{z : \arg(z) = -\pi/3\}, \quad \mathcal{L}_3 = \{z : \arg(z) = \pi/3\}.$$

Proof: By symmetry, it is enough to restrict attention to one of these lines, say $\mathcal{L}_1$. Indeed, it will be enough to show that there cannot be zeros above this line, as a simple symmetry argument will then prove the same result for the region below the line. Hence, we show that, for $\mu = 0, 1, 2$, (1) the function $F_\mu(z)$ has infinitely many zeros on the real negative axis, (2) the function $F_\mu(z)$ has no zeros on the real positive axis, and (3) $F_\mu(z)$ has no zeros in the region $\{x < 0,\ 0 < y < \infty\}$.
Let $z = x + iy$. Since $\zeta = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $\zeta^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$ a direct computation yields

$$\begin{aligned} F(z) &= e^{-x/2} e^{-iy/2} \tilde{F}(z), & (7.1)\\ \tilde{F}(z) &= e^{3x/2} e^{i\,3y/2} + 2\cosh(\frac{\sqrt{3}y}{2})\cos(\frac{\sqrt{3}x}{2} + \mu\frac{2\pi}{3}) - 2i\sinh(\frac{\sqrt{3}y}{2})\sin(\frac{\sqrt{3}x}{2} + \mu\frac{2\pi}{3}). \end{aligned}$$

Separating real and imaginary parts, the condition $\tilde{F}(z) = 0$ yields the system of two equations

$$e^{3x/2}\cos(\frac{3y}{2}) \quad + \quad 2\cosh(\frac{\sqrt{3}y}{2})\cos(\frac{\sqrt{3}x}{2} + \mu\frac{2\pi}{3}) = 0 \qquad (7.2)$$

$$e^{3x/2}\sin(\frac{3y}{2}) \quad - \quad 2\sinh(\frac{\sqrt{3}y}{2})\sin(\frac{\sqrt{3}x}{2}+\mu\frac{2\pi}{3}) = 0. \tag{7.3}$$

Case (a): $y = 0$
We give the proof for $\mu = 1$. The other cases are entirely analogous.
In this case, equation (7.3) is an identity, and for the real part (7.2) we obtain

$$e^{3x/2} + 2\cos(\frac{\sqrt{3}x}{2} + \frac{2\pi}{3}) = 0.$$

$\mathbf{x \le 0}$ In this case it is easy to see that (7.2) has infinitely many solutions. The root at $x = 0$ is a double root (as can be seen by a simple Taylor expansion argument) and all the others are simple roots.
$\mathbf{x > 0}$ In this case, the equation can only be solved if $\cos(\frac{\sqrt{3}x}{2} + \frac{2\pi}{3})$ is negative, so if $2n\pi - \frac{\pi}{6} \le \frac{\sqrt{3}x}{2} \le \frac{5\pi}{6} + 2n\pi$, $n \in \mathbb{Z}$. The right hand side of this inequality must be a positive quantity, implying $n > -\frac{5}{12}$. In addition, if $n \ge 1$, then $x > 11\pi/\sqrt{3} > 11$ implying $e^{3x/2} >> 2$, yielding again that there cannot be roots of (7.2). Hence only the case $n = 0$ can yield a solution if $x > 0$, and any positive roots must lie in the interval $0 < x < \frac{5x}{3\sqrt{3}}$. However in this interval the derivative with respect to x of the function in (1.1b) is strictly increasing. Hence there cannot be any real positive roots of (7.2).
Case (b): $0 < y < \infty$
Consider equations (7.2) and (7.3). Divide the first by $2\cosh(\frac{\sqrt{3}y}{2})$ and the second by $2\sinh(\frac{\sqrt{3}y}{2})$, which are both different from zero if $y > 0$.
Squaring and adding the two resulting equations yields

$$e^{3x}f(y) = 1, \quad f(y) = \left(\frac{\cos^2(\frac{3y}{2})}{4\cosh^2(\frac{\sqrt{3}y}{2})} + \frac{\sin^2(\frac{3y}{2})}{4\sinh^2(\frac{\sqrt{3}y}{2})}\right).$$

For $x = 0$, $y > 0$, we want to show that the remaining term is always < 1, so that there cannot be any zeros in this region. To show that it is valid for all positive y, it is enough to show that the function $f(y)$ is monotonically decreasing for $y \ge 0$. Since $f(0) = 1$, this will show our claim. This is a simple but tedious analysis of the conditions that guarantee that $\frac{df}{dy} < 0$.
For $x < 0$, the exponential term $e^{3x} < 1$. Combined with the previous analysis, this shows that there cannot be any zeros if $y > 0$.
QED

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