# Spectral analysis of the elliptic sine-Gordon equation in the quarter plane

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We study the elliptic sine-Gordon equation in the quarter plane by a spectral transform approach. Namely, we determine the Riemann-Hilbert problem associated with well-posed boundary value problems in this domain, and use it to derive a formal representation of the solution. Our analysis is based on a generalisation of the usual inverse scattering transform recently introduced by Fokas for the study of linear elliptic problems.

## 1 Introduction

The elliptic sine-Gordon equation is the nonlinear partial differential equation

$$q_{xx} + q_{yy} = \sin q, \quad q = q(x, y). \tag{1.1}$$

This equation appears e.g. in the theory of Josephson effects, superconductors, spin waves in ferromagnets, see [10].
From a strictly mathematical point of view, the elliptic sine-Gordon equation is an example of *integrable* PDE, at least in the sense of admitting a Lax pair formulation.This means that, in principle, the tools developed in connection with the so-called inverse scattering transform can be used for its analysis. Indeed, the inverse scattering analysis for this integrable equation has been considered in [3] for a problem posed on $\mathbb{R}^2$ with prescribed periodic behaviour at infinity, and in [10] for the problem on a half plane $y \geq 0$, with all boundary values prescribed, but constrained by a certain nonlinear relation. Due to the limitations of the inverse scattering approach when dealing with boundary value problems, in these works the solution is not constructed effectively. Special exact solutions for the problem posed in the whole of $\mathbb{R}^2$ were found in the 80's by various authors (see the references in [10]), but the more realistic case of boundary value problems for this equation is still essentially open.
In the present work, we use the generalisation of the inverse scattering transform method to boundary value problems due to Fokas [4]. This is based on the analysis of both ODEs in the Lax pair simultaneously, and on the analysis of a relation among all boundary values, called by Fokas the *global relation*. The global relation gives an algorithmic way to derive a representation of the solution under the assumption that the given boundary conditions are *admissible*, in the sense that they satisfy the global relation identically. We note that the result of [10] is a particular case when admissible boundary conditions are assumed to hold, hence it is contained in our general formualtion.
In this work we present the general approach, and characterise the solution under the assumption that the boundary conditions are admissible. To illustrate the methodology, we consider the particular case when the problem is posed in the positive quarter plane, $x \geq 0$, $y \geq 0$. This method allows also the explicit analysis of the concrete boundary value problems obtained when the boundary conditions are *linearisable*, see

[7, 8]. The analysis of such cases, and of the more difficult case of general boundary conditions, is in progress.

## 2 Notation and main result

### Notation

In what follows, we adopt the following notations.

- For any $2 \times 2$ matrix $M$, we denote
$$\widehat{\sigma_3} M = [\sigma_3, M], \qquad \mathrm{e}^{\widehat{\sigma_3}} M = \mathrm{e}^{\sigma_3} M \mathrm{e}^{-\sigma_3}.$$
- The matrices $\sigma_i$ are the usual Pauli matrices:
$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
- The parameter $\lambda \in \mathbb{C}$ denotes the *spectral parameter*, and we denote
$$w_1(\lambda) = \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right), \quad w_2(\lambda) = \frac{1}{2i}\left(\lambda - \frac{1}{\lambda}\right)$$
- We write
$$\lambda \in (i) \quad \text{or} \quad \lambda \in (ij), \quad i, j = 1, 2, 3, 4,$$
to denote that $\lambda$ is in the $i$-th quadrant of the $\lambda$ complex plane, or the union of the $i$-th and $j$-th quadrants, respectively.
- For a $2 \times 2$ matrix $M(\lambda)$ the notation $\lambda \in ((i), (j))$ means that we consider $\lambda \in (i)$ for the first column vector, and $\lambda \in (j)$ for the second column vector.

We can now give the definition of admissible boundary conditions.

**Definition 2.1** *Let $d_1(y)$ $u_1(y)$, $d_2(x)$, $u_2(x)$ defined on $[0, \infty)$ be smooth functions, that decay at infinity and are compatible at the corner $(x, y) = (0, 0)$*[1]*.*
*Define the column vectors $(a_0(\lambda), b_0(\lambda))^\tau$, $(a_1(\lambda), b_1(\lambda))^\tau$ by*

$$\begin{pmatrix} a_0(\lambda) \\ b_0(\lambda) \end{pmatrix} = \psi_3(0, \lambda), \ \lambda \in (12), \qquad \begin{pmatrix} a_1(\lambda) \\ b_1(\lambda) \end{pmatrix} = \psi_1(0, \lambda), \ \lambda \in (14) \tag{2.2}$$

*where $\psi_1$, $\psi_3$ are the unique solutions of*

$$\psi_3(x, \lambda)_x + w_2(\lambda) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \psi_3(x, \lambda) = Q_0(x, 0, \lambda) \psi_3(x, \lambda),$$

$$\psi_1(y, \lambda)_y + w_1(\lambda) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \psi_1(y, \lambda) = i Q_0(0, y, -\lambda) \psi_1(y, \lambda),$$

$$\lim_{x \to \infty} \psi_3(x, \lambda) \to (1, 0)^\tau, \qquad \lim_{y \to \infty} \psi_1(y, \lambda) \to (1, 0)^\tau,$$

*where*

$$Q_0(x, 0, \lambda) = \frac{i d_2'(x) + u_2(x)}{2} \sigma_1 - \frac{i}{4\lambda} (\sin d_2(x)) \sigma_2 + \frac{i}{4\lambda} (1 - \cos d_2(x)) \sigma_3, \tag{2.3}$$

[1]The given boundary conditions are called compatible if $d_1(0) = d_2(0)$, $d_1'(0) = u_2(0)$, $d_2'(0) = u_1(0)$.

$$Q_0(0,y,\lambda)=\frac{iu_1(y)+d_1'(y)}{2}\sigma_1-\frac{i}{4\lambda}(\sin d_1(y))\sigma_2+\frac{i}{4\lambda}(1-\cos d_1(y))\sigma_3, \qquad (2.4)$$

*where $\sigma_i$, $i=1,2,3$ are the usual Pauli matrices.*
*The set of functions $\{d_1(y),\, u_1(y),\, d_2(x),\, u_2(x\}$ is called* **admissible** *if*

$$a_0=a_1, \qquad b_0=b_1, \quad \lambda\in(1).$$

We can now state our main theorem.

**Proposition 2.1** *Consider the Dirichlet boundary vlaue problem for the elliptic sine-Gordon equation in the quarter plane*

$$q_{xx}+q_{yy}=\sin q, \quad q=q(x,y), \quad x\geq 0,\ y\geq 0, \qquad (2.5)$$

$$q(0,y)=d_1(y), \qquad q(x,0)=d_2(x), \qquad d_1(0)=d_2(0), \qquad (2.6)$$

*where $d_1$, $d_2$ are in the Schwarz class.*
*Suppose that there exist two functions $u_1(y)$, $u_2(x)$ such that the set $\{d_1,\, d_2\, u_1,\, u_2\}$ is admissible in the sense of definition (2.1). Assume also that $a(\lambda)\neq 0$, $\lambda\in(1)$.*
*Consider the complex variable $z=x+iy$, $x\geq 0$, $y\geq 0$, and define $\Psi(z,\bar z,\lambda)$ as the solution of the following Riemann-Hilbert problem (see figure 1):*

$$\Psi_-(z,\bar z,\lambda)=\Psi_+(z,\bar z,\lambda)J(z,\bar z,\lambda), \quad \lambda\in\Gamma \qquad (2.7)$$

*where $\Gamma=\mathbb{R}\cup i\mathbb{R}$ and*

$$J(z,\bar z,\lambda)=J^{\alpha}(z,\bar z,\lambda), \text{ if } arg(\lambda)=\alpha, \quad \alpha=0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}, \qquad (2.8)$$

*with*

$$J^0=\begin{pmatrix} 1 & -\frac{\bar b_0}{a_0}e^{i\theta(z)} \\ 0 & 1\end{pmatrix}, \quad J^{\pi/2}=\begin{pmatrix} 1 & -\frac{\bar b_1}{a_0}e^{i\theta(z)} \\ 0 & 1\end{pmatrix}, \quad J^{\pi}=\begin{pmatrix} 1 & 0 \\ \frac{b_0}{\bar a_1}e^{-i\theta(z)} & 1\end{pmatrix},$$

*and*

$$J^{3\pi/2}=J_\pi^{-1}J_0J_{\pi/2} \qquad (2.9)$$

*where the functions $a_0(\lambda)$, $a_1(\lambda)$, $b_0(\lambda)$, $b_1(\lambda)$ be given by (2.2), $\bar b=\overline{b(\bar\lambda)}$ and*

$$\theta(z)=\frac{1}{2}\left[\lambda z-\frac{\bar z}{\lambda}\right]. \qquad (2.10)$$

*Then the function $q(x,y)$ defined by*

$$\cos q(z,\bar z)=1+\frac{2}{\pi}\frac{\partial}{\partial\bar z}\int_{\Gamma_-}(\Psi_+)_{12}\frac{b(\lambda)}{a(-\lambda)}e^{-\frac{i}{2}(\lambda z-\frac{\bar z}{\lambda})}d\lambda, \qquad (2.11)$$

*where $\Gamma_-=\Gamma\cap\mathbb{C}^-$, $b=b_0$ on $\mathbb{R}^-$, $b=b_1$ on $i\mathbb{R}^-$, $a(-\lambda)=a_0(-\lambda)=a_1(-\lambda)$ for $\lambda\in(3)$, satisfies $q_{xx}+q_{yy}=\sin q$, as well as the boundary conditions*

$$q(0,y)=d_1(y)\ \ q_x(0,y)=u_1(y) \quad q(x,0)=d_2(x),\ ;\ q_y(x,0)=u_2(x).$$

The proof of this result follows the lines of the proof of Theorem 3.1 of [7], once the appropriate Riemann-Hilbert problem is determined. In this article, we present the construction of the Riemann-Hilbert problem, and characterise the function $q(x,y)$ that satisfies (1.1)-(2.6) in terms of its solution.

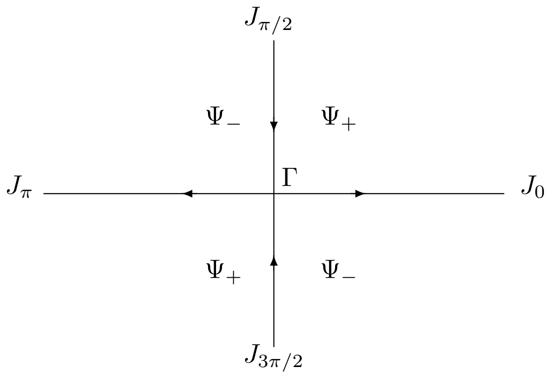

Figure 1: The Riemann-Hilbert problem

# 3 The linearised problem: modified Helmholtz equation

In the limit for small $q$, we find that equation (1.1) linearises to the modified Helmholtz equation

$$q_{xx} + q_{yy} = q, \quad q = q(x, y). \tag{3.1}$$

This equation has been extensively analysed by Fokas and his collaborator by spectral transform method proposed by Fokas [4, 6, 9]. In terms of the variables $z = x + iy$ and $\bar{z} = x - iy$, the equation can be written as

$$4q_{z\bar{z}} = q \Leftrightarrow \mu_{z\bar{z}} = \mu_{\bar{z}z},$$

where $\mu$ satisfies the "Lax pair"

$$\mu_z - \frac{i\lambda}{2}\mu = 2q_z, \qquad \mu_{\bar{z}} + \frac{i}{2\lambda}\mu = \frac{i}{\lambda}q. \tag{3.2}$$

Note that such $\mu$ satisfies $\mu \sim O\left(\frac{1}{\lambda}\right), \ |\lambda| \to \infty$.

### The Dirichlet problem in the quarter plane

Suppose the modified Helmholtz equation (3.1) is posed in the quarter plane $x \geq 0$, $y \geq 0$, with prescribed Dirichlet boundary conditions:

$$q(0, y) = d_1(y), \quad q(x, 0) = d_2(x), \tag{3.3}$$

appropriately smooth, decaying at infinity, and compatible at $x = y = 0$.
By performing the spectral analysis of the Lax pair (3.2), it is possible prove the following result [4].

**Proposition 3.1** *The solution of equation (3.1) in the quarter plane, given the Dirichlet conditions (3.3), admits the integral representation*

$$q(z, \bar{z}) = \frac{1}{4\pi i}\left[\int_{r_1} \mathrm{e}^{\frac{i}{2}(\lambda z - \frac{\bar{z}}{\lambda})}\hat{q}_1(\lambda)\frac{d\lambda}{\lambda} + \int_{r_2} \mathrm{e}^{\frac{i}{2}(\lambda z - \frac{\bar{z}}{\lambda})}\hat{q}_2(\lambda)\frac{d\lambda}{\lambda}\right] \tag{3.4}$$

*where $r_1$ denotes the positive imaginary axis, $r_2$ denotes the positive real axis, and the spectral functions $\hat{q}_j(\lambda)$ are defined by*

$$\hat{q}_1(\lambda) = (\lambda - \frac{1}{\lambda})D_1(\lambda) - \frac{i}{2}(\lambda + \frac{1}{\lambda})D_2(i\lambda), \quad \lambda \in r_1, \tag{3.5}$$

$$\hat{q}_2(\lambda) = i(\lambda + \frac{1}{\lambda})D_2(-i\lambda) - \frac{i}{2}(\lambda + \frac{1}{\lambda})D_2(i\lambda) - (\lambda - \frac{1}{\lambda})D_1(-\lambda), \quad \lambda \in r_2 \tag{3.6}$$

*with*

$$D_j(\lambda) = \int_0^\infty e^{\frac{1}{2}(\lambda+\frac{1}{\lambda})s} d_j(s)ds, \quad j = 1, 2. \tag{3.7}$$

The characterisation of the functions $\hat{q}_1$, $\hat{q}_2$ in terms of $d_1(y)$, $d_2(x)$ is the the *Dirichlet to Neumann map* for this problem. This characterisation is based on the analysis of the global relation, obtained as a result of the spectral analysis of the Lax pair, and on its invariance under the transformations $\lambda \to \pm 1/\lambda$. For this problem, the global relation is the following explicit algebraic constraint for $\hat{q}_1$, $\hat{q}_2$:

$$\hat{q}_1(\lambda) + \hat{q}_2(\lambda) = 0, \quad \pi \le \arg(\lambda) \le \frac{3\pi}{2}. \tag{3.8}$$

# 4 The nonlinear problem

**A Lax pair for the nonlinear problem**

We write the elliptic sine-Gordon equation (1.1) in terms of the variables $z = x + iy$, $\bar{z} = x - iy$ as

$$q_{z\bar{z}} = \frac{1}{4}\sin q.$$

This equation is the compatibility condition of the following pair of ODEs:

$$\Psi_z - \frac{i\lambda}{4}[\sigma_3, \Psi] = Q\Psi \tag{4.1}$$

$$\Psi_{\bar{z}} + \frac{i}{4\lambda}[\sigma_3, \Psi] = \frac{1}{\lambda}\tilde{Q}\Psi \tag{4.2}$$

where $\lambda \in \mathbb{C}$, $\Psi = \Psi(z, \bar{z}, \lambda)$ is a nonsingular $2 \times 2$ matrix, and the matrices $Q$ and $\tilde{Q}$ are given by

$$Q(z, \bar{z}) = \frac{iq_z(z, \bar{z})}{2}\sigma_1, \qquad \tilde{Q}(z, \bar{z}) = \frac{i}{4}(1 - \cos q(z, \bar{z}))\sigma_3 - \frac{i}{4}(\sin q(z, \bar{z}))\sigma_2. \tag{4.3}$$

This Lax pair, that reduces to (3.2) in the linear limit, is similar, but not identical, to the pair used in [3, 10].
The solutions of (4.1)-(4.2) satisfy

$$\Psi(z, \bar{z}, \lambda) = I + \frac{\psi_1(z, \bar{z})}{\lambda} + O\left(\frac{1}{\lambda^2}\right) \quad \text{for } |\lambda| \to \infty. \tag{4.4}$$

In terms of the variables $(x, y)$ (denoting $\Psi(x, y)$ by the same symbol as the function in terms of the variables $(z, \bar{z})$) we have

$$\Psi_x + \frac{w_2(\lambda)}{2}[\sigma_3, \Psi] = Q_0(x, y, \lambda)\Psi, \qquad \Psi_y + \frac{w_1(\lambda)}{2}[\sigma_3, \Psi] = iQ_0(x, y, -\lambda)\Psi, \tag{4.5}$$

with

$$Q_0(x, y, \lambda) = Q(x, y) + \frac{1}{\lambda}\tilde{Q}(x, y) = \frac{i(q_x - iq_y)}{4}\sigma_1 + \frac{i}{4\lambda}(1 - \cos q)\sigma_3 - \frac{i}{4\lambda}(\sin q)\sigma_2. \tag{4.6}$$

where $q$, $q_x$ and $q_y$ are considered as functions of $(x, y)$.
Note that this is a Lax pair with the same $\lambda$-dependence on the right hand side as the Lax pair for the hyperbolic sine-Gordon equation. In what follows we sometimes write $Q_0(\lambda)$ for $Q_0(x, y, \lambda)$.
The matrices (4.3) and (4.6) satisfy the following two properties:
**(1)**: $\mathrm{Tr}(Q) = \mathrm{Tr}\tilde{Q} = 0$
**(2a)**: $\det\left(\frac{1}{2}w_1(\lambda)\sigma_3 - iQ_0(-\lambda)\right)$ is a function of $\lambda$ only through $w_1(\lambda)$.
**(2b)**: $\det\left(\frac{1}{2}w_2(\lambda)\sigma_3 - Q_0(\lambda)\right)$ is a function of $\lambda$ only through $w_2(\lambda)$.
The latter property is crucial in the determination of linearisable boundary conditions.

## 4.1 Boundary value problems posed in the quarter plane

We consider the elliptic sine-Gordon equation for $(z, \bar{z})$ in a given convex polygon $\mathcal{P}$. The Lax pair (4.1)-4.2) is equivalent to the single equation

$$d\left(\mathrm{e}^{(-\frac{i\lambda}{4}z+\frac{i}{4\lambda}\bar{z})\widehat{\sigma_3}}\Psi(z,\bar{z},\lambda)\right) = W(z,\bar{z},\lambda), \tag{4.7}$$

where $W$ is an exact 1-form, defined by

$$W(z,\bar{z},\lambda) = \mathrm{e}^{(-\frac{i\lambda}{4}z+\frac{i}{4\lambda}\bar{z})\widehat{\sigma_3}}\left(Q\Psi dz + \frac{1}{\lambda}\tilde{Q}\Psi d\bar{z}\right). \tag{4.8}$$

From this formulation, we find immediately that the solutions of this equation have the form

$$\Psi(z,\bar{z},\lambda) = I + \int_{z^*}^{z} \mathrm{e}^{(\frac{i\lambda}{4}z-\frac{i}{4\lambda}\bar{z})\widehat{\sigma_3}}W(\zeta,\bar{\zeta},\lambda)$$

where $z*$ is an arbitrary point inside the polygon.
In addition, using the convexity of the polygon, we obtain that the form $W$ is closed, hence

$$\int_{\partial\mathcal{P}} W(z,\bar{z},\lambda) = 0. \tag{4.9}$$

This integral identity is the **global relation**.

We now specialise to the case that the boundary value problem is posed in the quarter plane $x > 0$, $y > 0$, hence $\mathcal{P} = \mathcal{I}$ is the first quadrant of the complex $z$ plane.
In this case, the corner of the polygon are

$$z_1 = 0 + i\infty, \qquad z_2 = 0, \qquad z_3 = \infty + i0,$$

Motivated by the analysis of the linear problem, we consider $z_j$ to be the vertices of the polygon, and define the particular solutions obtained when $z^* = z_j$, $j = 1,\ldots,3$. Namely, we define

$$\Psi_j = I + \int_{z_j}^{z} \mathrm{e}^{\frac{i}{4}[\lambda(z-\zeta)-\frac{1}{\lambda}(\bar{z}-\bar{\zeta})]\widehat{\sigma_3}}\left(Q\Psi d\zeta + \frac{1}{\lambda}\tilde{Q}\Psi d\bar{\zeta}\right). \tag{4.10}$$

The function $\Psi_j$ is the unique solution of (4.8) such that $\Psi_j(z_j) = I$.

We now analyse where each solution is a bounded analytic function. The boundedness of the exponential terms involved in the definition of $\Psi_j$ clearly depends on the location of the parameter $\lambda$ in the complex plane. Note that

$$R_j = Re\left(\frac{i}{4}[\lambda(z-\zeta) - \frac{1}{\lambda}(\bar{z}-\bar{\zeta})]\right) = -\frac{1}{4}\left(1+\frac{1}{|\lambda|^2}\right) Im\ (\lambda(z-\zeta)).$$

hence we can compute explicitly the sector $\Sigma_j$ where the exponentials $\exp(R_j)$ are bounded (as $\lambda \to \infty$ and $\lambda \to 0$). We also define by $\overline{\Sigma_j}$ the "conjugate" sectors, where $-R_j$ is bounded. We find that the sectors $\Sigma_j$, $j = 1,2,3$ are given explicitly by

$$\begin{aligned}
\Sigma_1 &= \{\lambda \in \mathbb{CP}^1 :\ \arg(\lambda) \in [\frac{\pi}{2}, \frac{3\pi}{2}]\}, \quad \overline{\Sigma_1} = \{\lambda \in \mathbb{CP}^1 :\ \arg(\lambda) \in [\frac{3\pi}{2}, \frac{\pi}{2}]\} \\
\Sigma_2 &= \{\lambda \in \mathbb{CP}^1 :\ \arg(\lambda) \in [0, \frac{\pi}{2}]\}, \quad \overline{\Sigma_2} = \{\lambda \in \mathbb{CP}^1 :\ \arg(\lambda) \in [\pi, \frac{3\pi}{2}]\} \\
\Sigma_3 &= \{\lambda \in \mathbb{CP}^1 :\ \arg(\lambda) \in [\pi, 2\pi]\}, \quad \overline{\Sigma_3} = \{\lambda \in \mathbb{CP}^1 :\ \arg(\lambda) \in [0, \pi]\}
\end{aligned} \tag{4.11}$$

These sectors cover the complex plane. In addition, $\Sigma_1 \cap \Sigma_3 = (3)$ and $\overline{\Sigma_1} \cap \overline{\Sigma_3} = (1)$. Taking into account the definition of $\widehat{\sigma_3}$ and $\mathrm{e}^{A\widehat{\sigma_3}}$, we find that the $z$, $\zeta$ dependence of each solution $\Psi_j$ is of the form

$$\Psi_j = \begin{pmatrix} * & * \,\mathrm{e}^{i\theta(z-\zeta)} \\ * \,\mathrm{e}^{-i\theta(z-\zeta)} & * \end{pmatrix},$$

where $*$ stand for a function of $\lambda$ only, and $\theta(z)$ is defined by(2.10).
Hence the elements in the first column of $\Psi_j$ are bounded for $\lambda \in \overline{\Sigma_j}$, while the elements in the second column of $\Psi_j$ are bounded for $\lambda \in \Sigma_j$.
Using the notation $\Psi = (\Psi^{(\cdot)}, \Psi^{(\cdot)})$ for a $2 \times 2$ matrix and its component column vectors, we obtain

$$\Psi_1 = (\Psi_1^{(14)}, \Psi_1^{(23)}), \qquad \Psi_2 = (\Psi_2^{(3)}, \Psi_2^{(1)}), \qquad \Psi_3 = (\Psi_3^{(12)}, \Psi_3^{(34)}).$$

where the superscript indicates the region of boundedness of each column vector (as a function of $\lambda$).
On account of the trace property $tr(Q) = tr(\tilde{Q}) = 0$, any solution of this Lax pair has determinant equal to 1, hence it is invertible. Hence any two solutions of (4.1)-(4.2) are related by

$$\Psi^{-1}\tilde{\Psi} = \mathrm{e}^{(\frac{i\lambda}{4}z - \frac{i}{4\lambda}\bar{z})\widehat{\sigma_3}}\rho(\lambda).$$

Since, by its definition, $\Psi_2(0,0,\lambda) = I$, we can write

$$\Psi_3(z,\bar{z},\lambda) = \Psi_2(z,\bar{z},\lambda)\mathrm{e}^{(\frac{i\lambda}{4}z - \frac{i}{4\lambda}\bar{z})\widehat{\sigma_3}}\Psi_3(0,0,\lambda), \quad \lambda \in (\mathbb{R}^-, \mathbb{R}^+) \tag{4.12}$$

$$\Psi_1(z,\bar{z},\lambda) = \Psi_2(z,\bar{z},\lambda)\mathrm{e}^{(\frac{i\lambda}{4}z - \frac{i}{4\lambda}\bar{z})\widehat{\sigma_3}}\Psi_1(0,0,\lambda), \quad \lambda \in (i\mathbb{R}^-, i\mathbb{R}^+), \tag{4.13}$$

as well as

$$\Psi_3(z,\bar{z},\lambda) = \Psi_1(z,\bar{z},\lambda)\mathrm{e}^{(\frac{i\lambda}{4}z - \frac{i}{4\lambda}\bar{z})\hat{\sigma}_3}\Psi_1(0,0,\lambda)^{-1}\Psi_3(0,0,\lambda), \quad \lambda \in ((1),(3)). \tag{4.14}$$

We note in particular that (4.14) is valid for each column vector not just along half lines, but in a whole quadrant of the $\lambda$ plane.

### The global relation and the spectral functions

As in the general case (4.9), the integral of the exact differential form $W(z,\bar{z},\lambda)$ given by (4.8) along the boundary of $\mathcal{I}$ vanishes:

$$\int_{\partial\mathcal{I}} \mathrm{e}^{(-\frac{i\lambda}{4}z + \frac{i}{4\lambda}\bar{z})\widehat{\sigma_3}} \left( Q\Psi dz + \frac{1}{\lambda}\tilde{Q}\Psi d\bar{z} \right) = 0.$$

Computing this integral explicitly (under the assumption that $q$, $q_z$, $q_{\bar{z}}$ vanish for $|z| \to \infty$) we obtain

$$\int_0^\infty \mathrm{e}^{-\frac{i}{4}(\lambda - \frac{1}{\lambda})x\hat{\sigma}_3}(Q + \frac{1}{\lambda}\tilde{Q})\Psi(x,x,\lambda)dx - i\int_0^\infty \mathrm{e}^{\frac{1}{4}(\lambda + \frac{1}{\lambda})y\hat{\sigma}_3}(Q - \frac{1}{\lambda}\tilde{Q})\Psi(iy,-iy,\lambda)dy = 0. \tag{4.15}$$

The global relation (4.15) is valid in particular for $\Psi = \Psi_3$. In this case, by definition, the first integral on the left hand side is

$$\int_0^\infty \mathrm{e}^{-\frac{i}{4}(\lambda - \frac{1}{\lambda})x\hat{\sigma}_3}(Q + \frac{1}{\lambda}\tilde{Q})\Psi_3(x,x,\lambda)dx = I - \Psi_3(0,0,\lambda).$$

To compute the second integral, we notice that using (4.14), we have

$$i\int_0^\infty \mathrm{e}^{\frac{1}{4}(\lambda + \frac{1}{\lambda})y\hat{\sigma}_3}(Q - \frac{1}{\lambda}\tilde{Q})\Psi_3(iy,-iy,\lambda)dy$$

$$= i\int_0^\infty \mathrm{e}^{\frac{1}{4}(\lambda + \frac{1}{\lambda})y\hat{\sigma}_3}(Q - \frac{1}{\lambda}\tilde{Q})\Psi_1(iy,-iy,\lambda)dy\ \Psi_1(0,0,\lambda)^{-1}\Psi_3(0,0,\lambda) =$$

$$= (\Psi_1(0,0,\lambda) - I)\Psi_1(0,0,\lambda)^{-1}\Psi_3(0,0,\lambda) = \Psi_3(0,0,\lambda) - \Psi_1(0,0,\lambda)^{-1}\Psi_3(0,0,\lambda).$$

Hence in this case the condition (4.15) yields

$$\Psi_1(0,0,\lambda)^{-1}\Psi_3(0,0,\lambda) = I, \quad \lambda \in ((1),(3)).$$

and (4.14) then implies that the two eigenfunctions coincide in their common domain:

$$\Psi_1(z,\bar{z},\lambda) = \Psi_3(z,\bar{z},\lambda), \quad \lambda \in ((1),(3)). \tag{4.16}$$

We now give the following definition.

**Definition 4.1** *The spectral functions $S_0(\lambda)$, $S_1(\lambda)$ are the functions defined through the particular solutions $\Psi_3$, $\Psi_1$ by*

$$S_0(\lambda) = \Psi_3(0,0,\lambda), \qquad S_1 = \Psi_1(0,0,\lambda), \tag{4.17}$$

More specifically, the spectral functions $S_i$ are defined through the solutions of the integral equations

$$\Psi_3(x,0,\lambda) = I - \int_0^\infty e^{-\frac{i}{4}[\lambda-\frac{1}{\lambda}]\xi\hat{\sigma}_3} Q_0(\xi,0,\lambda)\Psi_3(\xi,0,\lambda)d\xi, \tag{4.18}$$

$$\Psi_1(0,y,\lambda) = I - \int_0^\infty e^{\frac{1}{4}[\lambda+\frac{1}{\lambda}]\eta\hat{\sigma}_3} iQ_0(0,\eta,-\lambda)\Psi_1(0,\eta,\lambda)d\eta. \tag{4.19}$$

where $Q_0$ is defined by (4.6). Then setting

$$\psi_3(x,\lambda) = \Psi_3(x,0,\lambda), \qquad \psi_1(y,\lambda) = \Psi_1(0,y,\lambda),$$

we have

$$S_0(\lambda) = \psi_3(0,\lambda), \qquad S_1(\lambda) = \psi_1(0,\lambda).$$

These functions are bounded as follows:

$$S_0(\lambda) = (S_0^{(12)}, S_0^{(34)}), \quad S_1(\lambda) = (S_1^{(14)}, S_1^{(23)}).$$

In terms of the spectral functions we can write the global relation as

$$S_0(\lambda) = S_1(\lambda), \qquad \lambda \in ((1),(3)). \tag{4.20}$$

This follows from the identity (4.16) of $\Psi_3$ and $\Psi_1$ in these sectors.

### Properties of the spectral functions

**(1):** The trace property $tr(Q) = tr(\tilde{Q}) = 0$ implies that

$$\det S_0 = \det S_1 = 1. \tag{4.21}$$

**(2):** Set

$$Q_0(z,\bar{z},\lambda) = Q(z,\bar{z}) + \frac{1}{\lambda}\tilde{Q}(z,\bar{z}) = \frac{iq_z}{2}\sigma_1 - \frac{i}{4\lambda}(\sin q)\sigma_2 + \frac{i}{4\lambda}(1-\cos q)\sigma_3.$$

For $q$ real-valued, $Q_0$ satisfy the symmetry properties

$$Q_0(\lambda)_{22} = Q_0(-\lambda)_{11} = \overline{Q_0(\bar{\lambda})_{11}}; \qquad Q_0(\lambda)_{12} = Q_0(-\lambda)_{21} = -\overline{Q_0(\bar{\lambda})_{21}}.$$

Hence any solution $\Psi$ of (4.1)-(4.2) satisfies the same symmetry properties, and we can assume $S_0$, $S_1$ to have the form

$$S_0(\lambda) = \begin{pmatrix} a_0(\lambda) & -\overline{b_0(\bar{\lambda})} \\ b_0(\lambda) & \overline{a_0(\bar{\lambda})} \end{pmatrix} = \begin{pmatrix} a_0(\lambda) & -b_0(-\lambda) \\ b_0(\lambda) & a_0(-\lambda) \end{pmatrix},$$

$$S_1(\lambda) = \begin{pmatrix} a_1(\lambda) & -\overline{b_1(\bar{\lambda})} \\ b_1(\lambda) & \overline{a_1(\bar{\lambda})} \end{pmatrix} = \begin{pmatrix} a_1(\lambda) & -b_1(-\lambda) \\ b_1(\lambda) & a_1(-\lambda) \end{pmatrix}.$$

The global relation (4.15) takes the form

$$a_0(\lambda) = a_1(\lambda), \qquad b_0(\lambda) = b_1(\lambda), \qquad \lambda \in (1). \tag{4.22}$$

Hence for $\lambda \in (1)$ we simply write $a(\lambda)$, $b(\lambda)$ without any subscript. Similarly, for $\lambda \in (3)$, we write $a(-\lambda)$, $b(-\lambda)$.

### The Riemann-Hilbert problem

The analysis of the boundedness of each of the eigenfunctions $\Psi_i$ shows that in each quadrant of the $\lambda$ plane there exist at least two among the column vectors defining the collection of matrix functions $\Psi_j$ that are bounded and analytic in that quadrant. In the first and third quadrants , there are two eigenfunctions that can be selected for the first or second column, respectively. Namely we can form the following matrices (see figure 4.1):

$$first\ quadrant: \quad (\left\{\begin{array}{c}\Psi_3^{(12)}\\ \Psi_1^{(14)}\end{array}\right\}, \Psi_2^{(1)}), \qquad second\ quadrant: \quad (\Psi_3^{(12)}, \Psi_1^{(23)}),$$

$$third\ quadrant: \quad (\Psi_2^{(3)}, \left\{\begin{array}{c}\Psi_3^{(34)}\\ \Psi_1^{(23)}\end{array}\right\}), \qquad fourth\ quadrant: \quad (\Psi_1^{(14)}, \Psi_3^{(34)}).$$

These matrices provide the solution of the direct problem. Note that in the quadrants where two eigenfunctions are bounded, they actually coincide. Explicitly, we have the following identities for column vectors:

$$\begin{aligned} \Psi_3^{(12)}(z, \bar{z}, \lambda) &= \Psi_1^{(14)}(z, \bar{z}, \lambda), \qquad Re(\lambda) \geq 0, Im(\lambda) \geq 0, \\ \Psi_3^{(34)}(z, \bar{z}, \lambda) &= \Psi_1^{(23)}(z, \bar{z}, \lambda), \qquad Re(\lambda) \leq 0, Im(\lambda) \leq 0. \end{aligned} \tag{4.23}$$

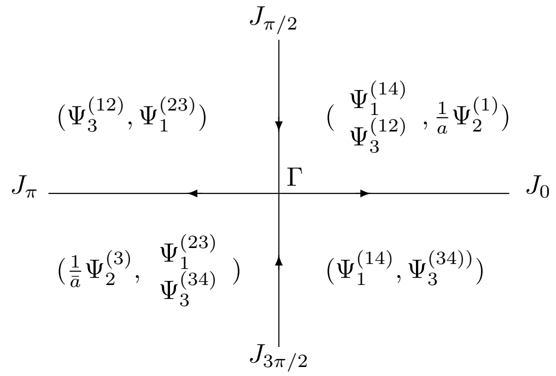


Figure 2: Bounded eigenfunctions and the Riemann-Hilbert problem

The equations (4.12)-(4.14) that relate the various analytic eigenfunctions can be rewritten in a form that determines uniquely a Riemann-Hilbert problem, with jumps on the real and imaginary axis. Because of the identity (4.23), the jumps on each of the four half lines only involve either the first or the second column vector.
Indeed, we find

$$\Psi_-(z, \bar{z}, \lambda) = \Psi_+(z, \bar{z}, \lambda) J(z, \bar{z}, \lambda), \tag{4.24}$$

where, using the notation $\bar{a} = \bar{a}(\bar{\lambda})$, $\bar{b} = \bar{b}(\bar{\lambda})$, the matrices $\Psi_\pm$ and $J$ are defined as follows:

$$\Psi_+ = \begin{cases} \left( \Psi_3^{(12)}, \frac{\Psi_2^{(1)}}{a(\lambda)} \right), & arg(\lambda) \in [0, \frac{\pi}{2}], \\ \left( \frac{\Psi_2^{(3)}}{\bar{a}(\bar{\lambda})}, \Psi_3^{(34)} \right), & arg(\lambda) \in [\pi, \frac{3\pi}{2}], \end{cases}$$

$$\Psi_- = \begin{cases} \left( \Psi_3^{(12)}, \Psi_1^{(2)} \right), & arg(\lambda) \in [\frac{\pi}{2}, \pi], \\ \left( \Psi_1^{(4)}, \Psi_3^{(34)} \right), & arg(\lambda) \in [\frac{3\pi}{2}, 2\pi], \end{cases}$$

$$J(z,\bar{z},\lambda) = J^{\alpha}(z,\bar{z},\lambda), \text{ if } arg(\lambda)=\alpha, \quad \alpha = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \tag{4.25}$$

with $J^{\alpha}$ given by (2.9). The normalisation is such that all the matrices $J^{\alpha}$ have unit determinant.

For simplicity, we assume that $a(\lambda) \neq 0$, hence that the solution does not have poles. Poles can be included by a standard procedure, see [4].

Rewriting the jump condition, we obtain

$$\Psi_+ - \Psi_- = \Psi_+ - \Psi_+ J - = \Psi_+(I - J) \Rightarrow \Psi_+ - \Psi_- = \Psi_+ \tilde{J} \tag{4.26}$$

where $\tilde{J}$=$I - J$. The relation (4.26) and the asymptotic condition (4.4) determine uniquely a Riemann-Hilbert problem. The solution of this RH problem is given by

$$\Psi(z,\bar{z},\lambda) = I + \frac{1}{2\pi i}\int_{\Gamma} \frac{\Psi_+(z,\bar{z},\lambda')\tilde{J}(z,\bar{z},\lambda')}{\lambda' - \lambda} d\lambda', \quad \lambda \in \Gamma\mathbb{R} \cup i\mathbb{R}. \tag{4.27}$$

This implies immediately

$$\psi_1 = -\frac{1}{2\pi i}\int_{\Gamma} \Psi_+(z,\bar{z},\lambda)\tilde{J}(z,\bar{z},\lambda)d\lambda, \quad (\psi_1)_{12} = \lim_{\lambda\to\infty}(\lambda\Psi_{12}). \tag{4.28}$$

**The characterisation of $q(z,\bar{z})$**

The last step in the construction is the derivation of an expression for $q$ in terms of the solution of the Riemann-Hilbert problem.

Using (4.4) in the first ODE in the Lax pair (4.1), we find that the $\lambda^0$ coefficient gives

$$-\frac{i}{4}[\sigma_3, \psi_1] = i\frac{q_z}{2}\sigma_1 \Rightarrow q_z = -(\psi_1)_{12} = -\lim_{\lambda\to\infty}(\lambda\Psi_{12}),$$

while using the second ODE (4.2), the (1,1) element of the $\lambda^{-1}$ coefficients yields

$$\cos q(z,\bar{z}) = 1 + 4i\frac{\partial(\psi_1)_{11}}{\partial\bar{z}}.$$

In view of the explicit expression of $\tilde{J}$, this gives for $\cos q(x,y)$ the expression 2.11. This concludes the proof of Proposition 2.1.

# 5 Conclusions

The formal procedure outlined in the present article represents the first step towards solving effectively boundary value problems for the elliptic sine-Gordon equation (1.1) posed in a convex polygon. The main feature of this procedure are (a) the simultaneous analysis of the two ODEs in the Lax pair and (b) the derivation and explicit use of the global relation. In contrast to the case of evolution PDEs such as the sine-Gordon equation $q_{tt} - q_{xx} = \sin q$, the global relation is valid in two-dimensionl region of the complex spectral plane, and this implies that the eigenfunctions of the spectral problem coincide in this region. The important consequence of this property of the spectral problem is that there are only two unknown spectral functions to be characterised, $a(\lambda)$ and $b(\lambda)$. However, again in contrast with the evolution case, when the initial conditions characterise fully one of the spectral functions, in the present case *no one spectral function is explicitly known.* It appears that in order to characterise the spectral data, it is necessary to determine an auxiliary Riemann-Hilbert problem for the functions $a(\lambda, b(\lambda)$, see also [2]. Preliminary investigations suggest that, for constant Dirichlet conditions, this can be done effectively, and the resulting Riemann-Hilbert problem can be solved explicitly. Hence we call these conditions *linearisable.* The solution in this case will be presented in work currently being drafted. For general boundary conditions, the determination of this auxiliary Riemann-Hilbert problem is more complicated, and the analysis is in progress.

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