to appear in *Inverse Problems*

Boundary value problems for third order linear PDEs in time dependent domains

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October 14, 2007

Abstract

We present the extension of a methodology to solve moving boundary value problems from the second order case to the case of the third order linear evolution PDE $q_t+q_{xxx} = 0$. This extension is the crucial step needed to generalise this methodology to PDEs of arbitrary order.

The methodology is based on the derivation of inversion formulas for a class of integral transforms that generalise the Fourier transform, and on the analysis of the global relation associated with the PDE. The study of this relation and its inversion using the appropriate generalised transform are the main elements of the proof of our results.

1 Introduction

In this work, we solve a Dirichlet-type boundary value problem for a third order evolution PDE posed on a time-dependent domain of the form

$$\mathbf{D} = \{(x,t): \quad 0 < t < T, \quad l(t) < x < \infty\}. \tag{1.1}$$

Here, and in what follows, T is a positive constant and $l(t)$ is a given function satisfying the following conditions:

$$l(t) \in \mathbf{C}^2[0,T], \quad l''(t) > 0, \quad l(0) = 0. \tag{1.2}$$

The convexity of the function $l(t)$ is not a necessary condition, and the case that $l''(t) < 0$ can be analysed in an analogous way. However the proof given requires that $l'(t)$ be monotone. It is possible to treat the case of slightly more general functions, see remark 2.4.
To illustrate the solution methodology we solve in detail a specific third order problem. The methodology we use is more general, though, and can be applied to any well posed moving boundary value problem for linear evolution PDEs of arbitrary order.

Our motivation and objective is the development of a general method for solving boundary value problems of this type. However, their study led to the consideration of integral

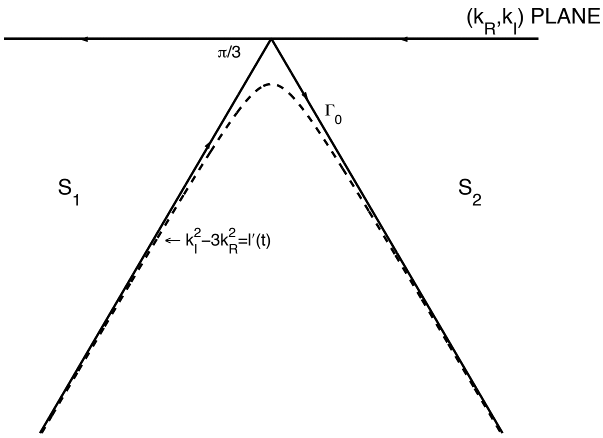

Figure 1: The curve Γ_0 (solid) and the curve $k_I(k_I^2 - 3k_R^2 - l'(t)) = 0$ (dashed) asymptotic to it. Both curves include the real line $k_I = 0$

transforms that generalise in a natural way the Fourier transform. The derivation of these integral transforms is interesting in its own right, and we present it in the first part of this paper without reference to PDE problems. Namely, we start by constructing the inversion formula for the integral transform defined by

$$F(k) = \int_0^T \mathrm{e}^{-ik^3 s - ikl(s)} f(s) ds, \quad k \in \mathbb{C}. \tag{1.3}$$

where $f(t)$ is an arbitrary, sufficiently smooth function. We show that if $F(k)$ is defined by equation (1.3), then $f(t)$ satisfies a Volterra integral equation [16] of the form

$$f(t) = \frac{3}{5\pi} \int_\Gamma \mathrm{e}^{ik^3 t + ikl(t)} k^2 F(k) dk + \frac{3}{5\pi} \int_0^t K(s,t) f(s) ds, \tag{1.4}$$

where the kernel $K(s,t)$ is defined explicitly (see equation (2.3)) and Γ is any curve in the lower half k complex plane, asymptotic to the contour

$$\Gamma_0 = \{k = k_R + ik_I \in \mathbb{C} : \; k_I(k_I^2 - 3k_R^2) = 0\} \tag{1.5}$$

traced counterclockwise. This contour is comprised of the real line and of the two half lines $\{k : arg(k) = \frac{4\pi}{3}\}$ and $\{k : arg(k) = \frac{5\pi}{3}\}$, see figure 1.
To derive the transform pair (1.3)-(1.4), we follow the approach of [7]. In this work, Fokas and Gelfand introduced an alternative method to obtain a number of classical integral transforms, including the Fourier transform, using the spectral analysis of appropriate eigenvalue ODEs (see also the review [12]). Indeed, our derivation of the inversion formula (1.4) is based on the spectral analysis of the ODE

$$\mu_t - i(k^3 + kl'(t))\mu = k^2 f(t), \quad t > 0, \;\; k \in \mathbb{C}, \tag{1.6}$$

and on the solution of an associated d-bar problem.
In the second part of the paper, we apply the Fokas transform method [3, 4], combined with the integral transform derived in the first part, to solve boundary value problems for a specific third-order evolution PDEs in a time-dependent domain $\mathbf{D}$ of the form (1.1).
The problem can be stated rather generally, for PDEs of the form $q_t + Tq = 0$ with T a linear differential operator with constant coefficients. For concreteness, we study here only the simplest third order PDE, as its solution contains all the features of the general case. Namely, we consider the following boundary value problem: find the function $q(x,t)$ that satisfies

$$\begin{aligned} &q_t + q_{xxx} = 0, \quad (x,t) \in \mathbf{D}, \\ &q(x,0) = q_0(x), \quad 0 < x < \infty; \\ &q(l(t),t) = f_0(t), \quad 0 < t < T. \end{aligned} \tag{1.7}$$

We assume that the initial and boundary conditions $q_0(x)$ and $f_0(t)$ are sufficiently smooth and that they are compatible at $x = t = 0$. We also assume that $q_0(x)$ decays as $x \to \infty$.
Moving boundary value problems of this kind were discussed in [8, 9], and the study of second order examples has been completed recently, see [6, 10]. The main difficulty for the solution of such problems is the characterisation of the *generalised Dirichlet to Neumann map*, namely the characterisation of the unknown boundary values in terms of the given initial and boundary conditions. For example, in [10] we solved the problem for the second order PDE $iq_t + q_{xx} = 0$, when the initial condition $q(x,0) = q_0(x)$ and one of the two boundary values, namely $q(l(t),t)$ is prescribed. Then we show that the other boundary value $q_x(l(t),t)$ is characterised as the unique solution of a certain linear Volterra integral equation.
In the present paper, we follow the same general approach as in [10]. However, in this case we must characterise *two* boundary values, as only one is given as boundary condition at the moving boundary. This problem is more difficult, because a direct generalisation of the method used for second order problems leads to strongly singular kernels. The main new result in this paper is a regularisation procedure, whereby we derive a *system* of Volterra integral equations characterising the unknown boundary values $\{q_x(l(t),t),\ q_{xx}(l(t),t)\}$ in terms of the given functions $\{q_0(x), f_0(t)\}$. This system is given by

$$\begin{cases} \frac{5\pi}{3} q_x(l(t),t) = & \int_0^t K(s,t) q_x(l(s),s) ds + i \int_0^t A_1(s,t) q_{xx}(l(s),s) ds + G_1(t) \\ \frac{5\pi}{3} q_{xx}(l(t),t) = & \int_0^t K(s,t) q_{xx}(l(s),s) ds - \frac{2}{1-\alpha^2} \mathrm{Re} \int_0^t A_2(s,t) q_x(l(s),s) ds + G_2(t) \end{cases}$$

where $K(s,t)$ is the kernel appearing in equation (1.4), $G_1(t)$ and $G_2(t)$ are known functions of the given initial and boundary conditions, $A_1(s,t)$, $A_2(s,t)$ are well defined integral kernels, see equation (3.6), and $\alpha = \mathrm{e}^{2\pi i/3}$.
We obtain our characterisation under the assumption of the existence of a unique solution. However this process can be justified a posteriori *without* this assumption, see Theorem 1.1 of [9].

The paper is divided in two main parts. In the first, we prove the inversion theorem for the transform pair (1.3)-(1.4). In the second, we use this transform pair to characterise the unknown boundary values of the solution of the boundary value problem (1.7).

In the conclusions, we briefly discuss how the particular result presented here can be generalised for solving moving boundary value problems for evolution PDEs of any order.
For comparison we also discuss, in the appendix, the solution of the analogous fixed boundary value problem (correspoding to $l(t) = 0$).

2 A generalised Fourier transform pair

In this section we assume that $f(t)$ is a given, sufficiently smooth function. The regularity needed for our results to be justified is as much as guarantees that the definition (1.3) of $F(k)$ makes sense, and also the validity and unique solvability of the d-bar formula (2.13). This holds under fairly mild assumptions, see [1].
Our main result in this section is the following.

Proposition 2.1 *Let $F(k)$ be defined in terms of a given, arbitrary function $f(t)$ by*

$$F(k) = \int_0^T \mathrm{e}^{-ik^3s-ikl(s)}f(s)ds, \quad k \in \mathbb{C}, \tag{2.1}$$

where $l(t)$ is a given function satisfying assumptions (1.2) and $f(t)$ is assumed to be sufficiently smooth (for example, $f(t) \in \mathbf{C}^1[0,T]$). Then $f(t)$ satisfies the following Volterra integral equation

$$\frac{5\pi}{3}f(t) = \int_{\partial\Omega_2^-} \mathrm{e}^{ik^3t+ikl(t)}F(k)k^2dk + \int_0^t K(s,t)f(s)ds, \tag{2.2}$$

where

$$K(s,t) = 2\mathrm{Re}\int_0^\infty \left\{ \left(1 + \frac{3i\lambda}{\sqrt{l'(s)+3\lambda^2}}\right) B(\lambda + i\sqrt{l'(s)+3\lambda^2}, s, t) \right\} d\lambda, \tag{2.3}$$

with

$$B(k,s,t) = k^2E(k,s,t), \quad E(k,s,t) = \mathrm{e}^{ik^3(t-s)+ik(l(t)-l(s))}, \tag{2.4}$$

and $\Omega_2^- = \Omega_2 \cup \mathbb{C}^-$ with Ω_2 the domain defined in (2.10) below, and the orientation of the boundary is counterclockwise, see figure 2.

Remark 2.1 In the limit $l(t) = 0$, $\Omega_2 \cap \mathbb{C}^+ = \Omega_2^+ = \{arg(k) \in [\pi/3, 2\pi/3]\}$ and

$$K(s,t) \to 2\mathrm{Re}\int_0^\infty \mathrm{e}^{i((1+i\sqrt{3})\lambda)^3(t-s)}(1+i\sqrt{3})^3\lambda)^2d\lambda = \int_{\partial\Omega_2^+} \mathrm{e}^{ik^3(t-s)}k^2dk.$$

This is the correct limit, see Appendix for details.

Remark 2.2 Using analyticity consideration, it is possible to show that the integral along the contour $\partial\Omega_2^-$ can also be computed along the contour composed of the real line and of the two semilines $\{k : arg(k) = \frac{4\pi}{3}\}$ and $\{k : arg(k) = \frac{5\pi}{3}\}$, or indeed any other contour asymptotic to this, see figure 1.
We discuss this reduction and its implications in the next section.

Remark 2.3 Since

$$|E(k,s,t)| = |e^{ik^3(t-s)+ik(l(t)-l(s))}| = e^{-k_I(s-t)[k_I^2-3k_R^2-l'(\tau)]}$$

we find that for $k = \lambda + i\sqrt{l'(s)+3\lambda^2}$, $\lambda > 0$, we have

$$|e^{ik^3(t-s)+ik(l(t)-l(s))}| = e^{-\sqrt{l'(s)+3\lambda^2}(s-t)[l'(s)-l'(\tau)]}$$

and the real part of the exponent is always negative as $s < t$ and $l'(s) < l'(\tau) < l'(t)$. Since $B(k,s,t) = k^2 E(k,s,t)$, the above computation shows that the integrand in the definition of the kernel $K(s,t)$ is exponentially decreasing, as a function of λ, as $\lambda \to \infty$.

Remark 2.4 The proof below, as well as remark 2.3, depend on the fact that we assume $l'(t)$ to be increasing on $[0,T]$. An entirely analogous argument holds if $l'(t)$ is decreasing. Since the PDE describes an evolution process, by subdividing $[0,T]$ into subintervals where $l'(t)$ is monotonic we can extend our result to the case of a $\mathbf{C^2}$ function, provided $l''(t)$ changes sign only finitely many times in $[0,T]$.

The spectral analysis

To prove this proposition, we consider the ODE (1.6) and we perform its spectral analysis. This means that we seek a solution μ of this ODE that is bounded, as a function of k, for *all* values of the complex parameter k on the Riemann sphere. Here $\mu = \mu(t,k)$, where k is an auxiliary complex parameter.

The idea of this derivation is due to Fokas and Gelfand, who showed how the usual Fourier transform pair, as applied the smooth function $f(t)$, can be derived using the spectral analysis of the ODE $\mu_t - ik\mu = f(t)$ and formulating an appropriate Riemann-Hilbert problem, see [7].

We seek a solution $\mu(t,k)$, of the ODE (1.6) , bounded for all $k = k_R + ik_I \in \mathbb{C}$. With this aim, we consider the three particular solutions of (1.6) given by

$$\begin{aligned}
\mu_1(t,k) &= \int_0^t e^{-ik^3(s-t)-ik(l(s)-l(t))} k^2 f(s) ds, && (2.5)\\
\mu_2(t,k) &= \int_T^t e^{-ik^3(s-t)-ik(l(s)-l(t))} k^2 f(s) ds, && (2.6)\\
\mu_3(t,k,\bar{k}) &= \int_{S(k)}^t e^{-ik^3(s-t)-ik(l(s)-l(t))} k^2 f(s) ds, && (2.7)
\end{aligned}$$

where the function $S(k,\bar{k})$ is defined on the domain Ω_3 given by (2.11) as

$$S : \Omega_3 \to [0,T], \quad S(k,\bar{k}) = t \Longleftrightarrow l'(t) = k_I^2 - 3k_R^2. \tag{2.8}$$

The real part of the exponent involved in these definition is

$$\mathrm{Re}(-ik^3(s-t) - ik(l(s)-l(t))) = k_I(s-t)[3k_R^2 - k_I^2 + l'(\tau)], \tau \in [s,t].$$

The functions μ_j's are bounded where the real part of the exponent is less than or equal to zero. Using the convexity of the function $l(t)$, this implies that they are bounded in the

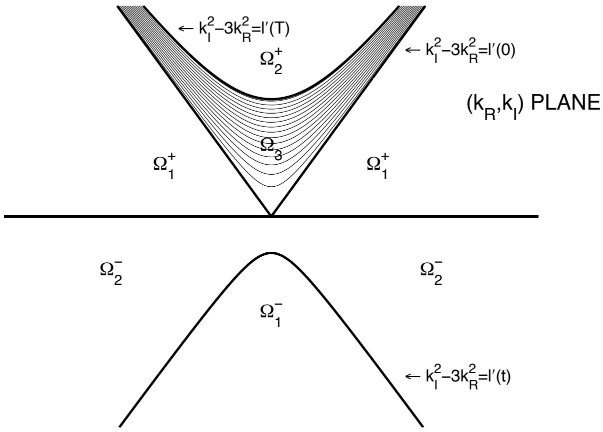

Figure 2: The domains Ω_1, Ω_2 and Ω_3 for the ODE $\mu_t - i(k^3 + kl'(t))\mu = k^2 f(t)$ in the k-plane (in this figure, we took $l'(0) = 0$, hence $l'(t) > 0$)

following domains:

$$\mu_1 \quad \Rightarrow \quad \Omega_1 = \{k: \ k_I \geq 0, \ k_I^2 - 3k_R^2 \leq l'(0)\} \cup \{k: \ k_I \leq 0, \ k_I^2 - 3k_R^2 \geq l'(t)\} \quad (2.9)$$

$$\mu_2 \quad \Rightarrow \quad \Omega_2 = \{k: \ k_I \geq 0, \ k_I^2 - 3k_R^2 \geq l'(T)\} \cup \{k: \ k_I \leq 0, \ k_I^2 - 3k_R^2 \leq l'(t)\} \quad (2.10)$$

$$\mu_3 \quad \Rightarrow \quad \Omega_3 = \{k: \ k_I \geq 0, \ l'(0) \leq k_I^2 - 3k_R^2 \leq l'(T)\} \quad (2.11)$$

These domains are depicted in figure (2) (where we assume for simplicity that $l'(0) = 0$.)
The two functions $\mu_1(t,k)$ and $\mu_2(t,k)$ are analytic functions of k inside the respective domains. However, the function $\mu_3(t,k,\bar{k})$, in contrast with the functions $\mu_1(t,k)$ and $\mu_2(t,k)$, depends on both k and $\bar{k}$, hence this function is *not analytic.*
Integration by parts of the equations defining the μ_j's implies the asymptotic behaviour

$$\mu_j = O\left(\frac{1}{k}\right), \quad k \in \ \Omega_j, \quad k \to \infty, \quad j = 1, 2. \quad (2.12)$$

The three equation (2.9)-(2.11) define a function $\mu(t,k,\bar{k})$ in terms of $f(t)$ which is bounded (as a function of k) on the entire Riemann sphere. Therefore, it is possible to find an alternative representation for this function using the Cauchy-Green (also known as d-bar) formula [1],

$$\mu(t,k,\bar{k}) = \frac{1}{2\pi i}\int_{\gamma(t)} \frac{J(\mu)}{\lambda - k} d\lambda + \frac{1}{2\pi i}\int\int_{\Omega(t)} \frac{\partial \mu(t,\lambda,\bar{\lambda})}{\partial \bar{\lambda}} \frac{d\lambda \wedge d\bar{\lambda}}{\lambda - k}, \quad 0 < t < T, \ k \in \mathbb{C}, \quad (2.13)$$

where $d\lambda \wedge d\bar{\lambda} = -2i d\lambda_R d\lambda_I$, $\gamma(t)$ denotes the contour along which the function μ has a jump discontinuity $J(\mu)$ and $\Omega(t)$ is the domain where $\partial\mu/\partial\bar{k} \neq 0$. To find this alternative representation, we need to compute the jump of the function μ along the contour where its representation is not unique, because two of the function $\mu_j(x,t,k)$ are defined. Note that μ_3 coincide with μ_1 on the hyperbola $\{k_I > 0,\ k_I^2 - 3k_R^2 = l'(0)\}$ and with μ_2 on the hyperbola $\{k_I > 0,\ k_I^2 - 3k_R^2 = l'(T)\}$. This is a consequence of the definition of $S(k, \bar{k})$. Hence the only jump discontinuity is given by

$$\mu_1 - \mu_2 = \int_0^T E(k, s, t) k^2 f(s) ds, \quad k \in \partial\Omega_2^-. \tag{2.14}$$

The counterclockwise direction of integration is depicted in figure 2. Equation (2.7) implies

$$\frac{\partial \mu_3(t, k, \bar{k})}{\partial \bar{k}} = -\frac{\partial S(k, \bar{k})}{\partial \bar{k}} k^2 \{ \mathrm{e}^{-ik^3(s-t) - k(l(s) - l(t))} k^2 f(s) \}_{s = S(k, \bar{k})}. \tag{2.15}$$

Hence, using equations (2.14) and (2.15) in equation (2.13), we find the following alternative representation of μ:

$$\begin{aligned} \mu(t, k, \bar{k}) &= \frac{1}{2\pi i} \int_{\partial\Omega_2^-} \int_0^T E(\lambda, s, t) \lambda^2 f(s) ds \frac{d\lambda}{\lambda - k} \\ &\quad - \frac{1}{2\pi i} \int\int_{\Omega_3} \frac{\partial S(\lambda, \bar{\lambda})}{\partial \bar{\lambda}} E(\lambda, S(\lambda, \bar{\lambda}), t) f(S(\lambda, \bar{\lambda})) \lambda^2 \frac{d\lambda \wedge d\bar{\lambda}}{\lambda - k}. \end{aligned} \tag{2.16}$$

Substituting the right hand side of this equation in the ODE (1.6) we find the expression

$$\begin{aligned} k^2 f(t) &= \mu_t - i(k^3 + kl'(t))\mu = \\ &= \frac{1}{2\pi} \Bigg\{ \int_{\partial\Omega_2^-} \left[(\lambda^2 + l'(t)) + \lambda k + k^2 \right] E(\lambda, s, t) \lambda^2 f(s) ds\, d\lambda \\ &\quad - \int\int_{\Omega_3} \left[(\lambda^2 + l'(t)) + \lambda k + k^2 \right] \frac{\partial S(\lambda, \bar{\lambda})}{\partial \bar{\lambda}} E(\lambda, S(\lambda, \bar{\lambda}), t) f(S(\lambda, \bar{\lambda})) \lambda^2 d\lambda \wedge d\bar{\lambda} \Bigg\}. \end{aligned}$$

Equating the coefficients of the powers of k on both sides of this expression we finally arrive at the following representations for the function $f(t)$:

$$f(t) = \left\{ \frac{1}{2\pi} \int_{\partial\Omega_2^-} \int_0^T E(k, s, t) f(s) ds\, k^2 dk - \frac{1}{2\pi} \int\int_{\Omega_3} \frac{\partial S(k, \bar{k})}{\partial \bar{k}} E(k, S(k, \bar{k}), t) f(S(k, \bar{k})) k^2\, dk \wedge d\bar{k} \right\}. \tag{2.17}$$

This formal representation of the function $f(t)$ was first derived in [9]. We now use it to prove our results.

Proof of proposition 2.1

We split the domain Ω_3 as $\Omega_3 = \Omega_3^{(1)} \cup \Omega_3^{(2)}$, where

$$\Omega_3^{(1)} = \{k :\ k_I \geq 0,\ l'(0) \leq k_I^2 - 3k_R^2 \leq l'(t)\}, \tag{2.18}$$

$$\Omega_3^{(2)} = \{k :\ k_I \geq 0,\ l'(t) \leq k_I^2 - 3k_R^2 \leq l'(T)\}. \tag{2.19}$$

and we appeal to Green's theorem to write the double integral over $\Omega_3^{(2)}$ as a simple contour integral. The boundary of the domain $\Omega_3^{(2)}$ is the union of the two hyperbolas

$$\mathcal{H}_T = \{k : k_I > 0,\ \ k_I^2 - 3k_R^2 = l'(T)\}, \tag{2.20}$$

$$\mathcal{H}_t = \{k : k_I > 0,\ \ k_I^2 - 3k_R^2 = l'(t)\}. \tag{2.21}$$

For $k \in \mathcal{H}_T$, $S(k,\bar{k}) = T$, while for $k \in \mathcal{H}_t$, $S(k,\bar{k}) = t$. Hence using the definition of μ_3,

$$\int\int_{\Omega_3^{(2)}} \frac{\partial \mu_3(k,\bar{k})}{\partial \bar{k}} dk \wedge d\bar{k} = -\int_{\mathcal{H}_T}\int_T^t E(k,s,t) f(s) ds k^2 dk.$$

Hence we obtain

$$2\pi f(t) = \int_{\partial\Omega_2^-}\int_0^T E(k,s,t)f(s)dsk^2dk - \int_{\mathcal{H}_T}\int_T^t E(k,s,t)f(s)dsk^2dk$$
$$+2i\int\int_{\Omega_3^{(1)}} \frac{\partial S(k,\bar{k})}{\partial \bar{k}} E(k,S(k,\bar{k}),t)f(S(k,\bar{k}))k^2 dk_R dk_I.$$

The second term can be computed explicitly. Indeed, this term is

$$\int_{\mathcal{H}_T}\int_T^t \mathrm{e}^{ik^3(t-s)+ik(l(t)-l(s))}k^2 f(s)dsdk =$$

$$\int_{\mathcal{H}_T}\left[\int_T^t \mathrm{e}^{ik^3(t-s)+ik(l(t)-l(s))}k^2 f(s)ds - \frac{ik^2 f(t)}{k^3+kl'(t)}\right]dk + \int_{\mathcal{H}_T}\frac{ik^2 f(t)}{k^3+kl'(t)}dk. \qquad (2.22)$$

The first integral vanishes by analyticity, as the exponential involved decays in the region Ω_2^+ bounded by $\mathcal{H}_T$, and is of order $O\left(\frac{1}{k^2}\right)$ as $k \to \infty$ in this region, while the second integral is given by

$$\int_{\mathcal{H}_T}\frac{ik^2 f(t)}{k^3+kl'(t)}dk = -f(t)\lim_{R\to\infty}\int_{\pi/3}^{2\pi/3}\frac{\mathrm{e}^{2i\theta}d\theta}{\mathrm{e}^{2i\theta}+\frac{l'(t)}{R^2}} = -\frac{\pi}{3}f(t). \qquad (2.23)$$

This yields

$$\frac{5\pi}{3}f(t) = \int_{\partial\Omega_2^-}\int_0^T E(k,s,t)f(s)dsk^2dk + 2i\int\int_{\Omega_3^{(1)}}\frac{\partial S(k,\bar{k})}{\partial\bar{k}}E(k,S(k,\bar{k}),t)f(S(k,\bar{k}))k^2dk_Rdk_I. \qquad (2.24)$$

We now compute the remaining double integral explicitly. We start by writing it as

$$2i\int\int_{\Omega_3^{(1)}}\frac{\partial S(k,\bar{k})}{\partial\bar{k}}E(k,S(k,\bar{k}),t)f(S(k,\bar{k}))k^2dk_Rdk_I$$

$$= \int\int_{\Omega_3^{(1)}}\left(i\frac{\partial S}{\partial k_R} - \frac{\partial S}{\partial k_I}\right)E(k_R+ik_I, S(k_R,k_I),t)f(S)(k_R+ik_I)^2dk_Rdk_I.$$

and we make the change of variables

$$(k_R,k_I)\to(\lambda,\sigma), \quad \lambda = k_R, \quad \sigma = k_I^2 - 3k_R^2.$$

Then $\Omega_3^{(1)} = \{(\lambda,\sigma) : -\infty \le \lambda < \infty,\ l'(0) \le \sigma \le l'(t),\}$, and $S = S(\sigma)$. The Jacobian of this change of variable is

$$J = \frac{1}{2k_I} = \frac{1}{2\sqrt{\sigma+3\lambda^2}}.$$

and we obtain

$$2i\int\int_{\Omega_3^{(1)}}\frac{\partial S(k,\bar{k})}{\partial\bar{k}}E(k,S(k,\bar{k}),t)f(S)k^2dk_Rdk_I=$$

$$\int_{l'(0)}^{l'(t)}\int_{-\infty}^{\infty}(1+\frac{3i\lambda}{\sqrt{\sigma+3\lambda^2}})\frac{dS(\sigma)}{d\sigma}B(\lambda+i\sqrt{\sigma+3\lambda^2},S(\sigma),t)f(S(\sigma))d\lambda\,d\sigma$$

where $B(k,s,t)$ is defined by (2.4).
Changing variable once more to (λ,s) where $s=S(\sigma)$, so that $\sigma=l'(s)$, the above expression can be written as

$$\int_0^t\int_0^\infty\left[\left(1+\frac{3i\lambda}{\sqrt{l'(s)+3\lambda^2}}\right)B(\lambda+i\sqrt{l'(s)+3\lambda^2},s,t)\right.$$

$$\left.+\left(1-\frac{3i\lambda}{\sqrt{l'(s)+3\lambda^2}}\right)B(-\lambda+i\sqrt{l'(s)+3\lambda^2},s,t)\right]d\lambda\,f(s)ds. \qquad (2.25)$$

The term inside square brackets is equal to the kernel $K(s,t)$ defined by equation (2.3). Indeed,

$$\left[\left(1+\frac{3i\lambda}{\sqrt{l'(s)+3\lambda^2}}\right)B(\lambda+i\sqrt{l'(s)+3\lambda^2},s,t)+\left(1-\frac{3i\lambda}{\sqrt{l'(s)+3\lambda^2}}\right)B(-\lambda+i\sqrt{l'(s)+3\lambda^2},s,t)\right]=$$

$$=2\mathrm{Re}\left[\left(1+\frac{3i\lambda}{\sqrt{l'(s)+3\lambda^2}}\right)B(\lambda+i\sqrt{l'(s)+3\lambda^2},s,t)\right].$$

We now show that this kernel does not have a strong singularity at $s=t$. Let

$$\Lambda=\lambda+i\sqrt{l'(s)+3\lambda^2}.$$

We observe that

$$\begin{aligned}\left(1+\frac{3i\lambda}{\sqrt{l'(s)+3\lambda^2}}\right)B(\Lambda,s,t) &= \frac{1}{3i(s-t)}\frac{d\mathrm{e}^{i\Lambda^3(s-t)}}{d\lambda}\mathrm{e}^{-i\Lambda(l(s)-l(t))}\\ \left(1-\frac{3i\lambda}{\sqrt{l'(s)+3\lambda^2}}\right)B(-\overline{\Lambda},s,t) &= -\frac{1}{3i(s-t)}\frac{d\mathrm{e}^{(i\overline{\Lambda})^3(s-t)}}{d\lambda}\mathrm{e}^{i\overline{\Lambda}(l(s)-l(t))}\end{aligned}$$

Hence integrating by parts the sum of these terms with respect to λ, $0<\lambda<\infty$, we obtain that the contribution of $1/(s-t)$ at $s=t$ cancels.

Finally, using the definition (2.4) of $B(k,s,t)$ we obtain

$$\frac{5\pi}{3}f(t)=\int_{\partial\Omega_2^-}\int_0^T E(k,s,t)f(s)dsk^2dk+\int_0^t K(s,t)f(s)ds. \qquad (2.26)$$

Recalling the definition (2.1) of $F(k)$, this yields equation (2.2).
QED

2.1 Reductions

In order to apply the transform just derived to the solution of a boundary value problem, we now make two reductions.

First reduction - deformation of the integration contour

We use the fact that the integrand of the integral appearing in the expression (2.26), equal to $I(k) = \int_0^T E(k,s,t)f(s)ds$, is analytic in any bounded subdomain of the complex k plane. In particular, using Cauchy's theorem we find that, for every $R > 0$,

$$\int_{\partial(\Omega_2^- \cap B_R)} I(k)dk = 0, \quad B_R = \{k : |k| \leq R\},$$

This observation is used implicitly to give the following meaning to the integral of this function along the unbounded contour $\partial\Omega_2^-$:

$$\int_{\partial\Omega_2^-} I(k)dk = \lim_{R\to\infty} \int_{\{k\in\Omega_2^- : |k|=R\}} I(k)dk.$$

Now let S_1, S_2 denote the sectors given by

$$S_1 = \{k : \ \pi \leq arg(k) \leq 4\pi/3\}, \quad S_2 = \{k : \ 5\pi/3 \leq arg(k) \leq 2\pi\}. \tag{2.27}$$

These sectors are shown on figure 1. Note that $\partial(S_1 \cup S_2) = \Gamma_0$, where Γ_0 is the curve given by (1.5).
Since the curve defined by $3k_R^2 - k_I^2 - l'(t) = 0$ is asymptotic to the rays bounding the two sectors S_1 and S_2 (which are characterised by $3k_R^2 - k_I^2 = 0$), using the definition of Ω_2^-, it can easily by verified that $\{\Omega_2 \setminus (S_1 \cup S_2)\} \cap B_R \to_{R\to\infty} \emptyset$, hence we have

$$\lim_{R\to\infty} \int_{\{k\in\Omega_2^- : |k|=R\}} I(k)dk = \lim_{R\to\infty} \int_{\{k\in S_1\cup S_2 : |k|=R\}} I(k)dk$$

so that

$$\int_{\partial\Omega_2^-} I(k)dk = \int_{\Gamma_0} I(k)dk.$$

The same argument shows that Γ_0 can be deformed to any curve with the same asymptotic as $k \to \infty$, $k \in C^-$.

Second reduction - integration around any simply connected component of Ω_2^-

It is useful to write the final expression (2.26) in an equivalent form.
Using the asymptotic reduction shown above, we write (2.26) as an integral over $\Gamma_0 = \partial(S_1 \cup S_2)$.
The change of variable $k \to \lambda = -\bar{k}$ maps S_1 to S_2. Note that

$$E(\lambda, s, t)\lambda^2 = \overline{E(k, s, t,)k^2}.$$

Using the definition of $F(k)$, the first integral in (2.2) is given by

$$\int_{\partial(S_1\cup S_2)}\int_0^T E(k,s,t)f(s)ds\,k^2dk$$

hence, using the map $k\to -\bar{k}$, and assuming that $f(s)$ is real-valued, this integral can be written as follows:

$$\int_{\partial(S_1\cup S_2)} E(k,0,t)k^2F(k)dk = \left[\int_{\partial S_1}+\int_{\partial S_2}\right]\int_0^T E(k,s,t)f(s)ds\,k^2dk$$

$$= \int_{\partial S_1}\int_0^T E(k,s,t)f(s)ds\,k^2dk + \overline{\int_{\partial S_1}\int_0^T E(k,s,t)f(s)ds\,k^2dk}$$

$$= 2\mathrm{Re}\int_{\partial S_1}\int_0^T E(k,s,t)f(s)ds\,k^2dk$$

and similarly

$$\int_{\partial(S_1\cup S_2)} E(k,0,t)k^2F(k)dk = 2\mathrm{Re}\int_{\partial S_2}\int_0^T E(k,s,t)f(s)ds\,k^2dk$$

In summary, we have the following corollary of our main result.

Corollary 2.1 *Let $F(k)$, $f(t)$ be as in the statement of proposition 2.1. Then the inversion formula (2.2) can be written in the following two equivalent forms:*

$$\frac{5\pi}{3}f(t) = 2\mathrm{Re}\int_{\partial S_1} E(k,0,t)k^2F(k)dk + \int_0^t K(s,t)f(s)ds, \tag{2.28}$$

$$\frac{5\pi}{3}f(t) = 2\mathrm{Re}\int_{\partial S_2} E(k,0,t)k^2F(k)dk + \int_0^t K(s,t)f(s)ds. \tag{2.29}$$

Remark 2.5 It is also possible to use rotations by $2\pi/3$, mapping S_1 onto S_2, to write

$$\frac{5\pi}{3}f(t) = \int_{\partial S_1}\int_0^T [E(k,s,t)+E(\mathrm{e}^{2\pi i/3}k,s,t)]f(s)ds\,k^2dk,$$

and similarly for an expression along ∂S_2 only. These expressions can be used as well as (2.28) or (2.29) to derive the final result in the next section. We prefer the form (2.28), (or (2.29)) as it shows explicitly that the result of the inversion is a real function of t.
However, it is the use of rotations (rather than of the symmetry with respect to the imaginary axis) that generalises tothe case of higher order problems.

3 The Dirichlet to Neumann map for the linear KdV equation $q_t + q_{xxx} = 0$

In this section we give the characterisation of the unknown boundary values in terms of the prescribed initial and boundary conditions for the boundary value problem described in the statement of our main proposition below.

Proposition 3.1 *Consider the following boundary value problem for the linear KdV equation*

$$\begin{cases} q_t + q_{xxx} = 0, & x > l(t), \ \ 0 < t < T, \\ q(x,0) = q_0(x), & x > l(t), \\ q(l(t),t) = f_0(t), & 0 < t < T \end{cases} \tag{3.1}$$

where $l(t)$ satisfies (1.2), $q_0(x)$ is a smooth function of x, decaying as $x \to \infty$, $f_0(t)$ is a smooth function of t, and $f_0(0) = q_0(0)$.
The generalised Dirichlet to Neumann map for this problem is given by the system of linear Volterra integral equations

$$\frac{5\pi}{3} q_x(l(t),t) = \int_0^t K(s,t) q_x(l(s),s)ds + i \int_0^t A_1(s,t) q_{xx}(l(s),s)ds + G_1(t) \tag{3.2}$$

$$\frac{5\pi}{3} q_{xx}(l(t),t) = \int_0^t K(s,t) q_{xx}(l(s),s)ds - \frac{2}{1-\alpha^2}\mathrm{Re}\int_0^t A_2(s,t) q_x(l(s),s)ds + G_2(t) \tag{3.3}$$

where $K(s,t)$ is given by (2.3), $\Omega_2^- = \Omega_2 \cup \mathbb{C}^-$ is given by (2.10), the known functions $G_1(t)$ and $G_2(t)$ are defined by

$$G_1(t) = \int_{\partial\Omega_2^-} \mathrm{e}^{ik^3t + ikl(t)} \left[\int_0^\infty \mathrm{e}^{-ikx}(q_0)_x(x)dx - \int_0^t \mathrm{e}^{-ik^3s - ikl(s)} f_0'(s)ds \right] dk, \tag{3.4}$$

$$G_2(t) = \frac{2i}{1-\alpha^2}\mathrm{Re}\left[\int_{\partial S_1} - \alpha^2 \int_{\partial S_2} \right] \mathrm{e}^{ik^3t + ikl(t)} \left[\int_0^t \mathrm{e}^{-ik^3s - ikl(s)} f_0'(s)ds - k\int_0^\infty \mathrm{e}^{-ikx}(q_0)'(x)dx \right] dk, \tag{3.5}$$

and

$$A_1(s,t) = \int_{\mathbb{R}} \mathrm{e}^{ik^3(t-s) + ik(l(t)-l(s))} k dk, \quad A_2(s,t) = \int_{\partial S_1} \mathrm{e}^{ik^3(t-s)} \left[\mathrm{e}^{ik(l(t)-l(s))} - \mathrm{e}^{i\alpha k(l(t)-l(s))} \right] ik^3 dk \tag{3.6}$$

where the domain S_1 is defined in (2.27) and

$$\alpha = \mathrm{e}^{2\pi i/3} \quad \textit{is the rotation by } \frac{2\pi}{3}. \tag{3.7}$$

3.1 The global relation

We start by deriving the global relation between the initial and boundary values of the solution of (3.1). The PDE can be written in divergence form as

$$\left[\mathrm{e}^{-ikx - ik^3t} q(x,t)\right]_t - \left[\mathrm{e}^{-ikx - ik^3t}(k^2 q - ikq_x(x,t) - q_{xx}(x,t)\right]_x = 0. \quad k \in \mathbb{C}. \tag{3.8}$$

Hence an application of Green's theorem in the domain **D** given by (1.1) yields the following expression for the global relation:

$$\int_0^T \mathrm{e}^{-ik^3 s - ikl(s)} \left[(k^2 + l'(s)) q(l(s), s) - ikq_x(l(s), s) - q_{xx}(l(s), s) \right] ds$$
$$= \int_0^\infty \mathrm{e}^{-ikx} q_0(x) dx - \mathrm{e}^{-ik^3 T} \int_{l(T)}^\infty \mathrm{e}^{-ik\xi} q(\xi, T) d\xi. \tag{3.9}$$

The use of the global relation in Ω_2^-

We multiply (3.9) by $k\mathrm{e}^{ik^3 t + ikl(t)}$ and by $k^2 \mathrm{e}^{ik^3 t + ikl(t)}$, respectively, to get

$$\begin{aligned}
\int_0^T \mathrm{e}^{ik^3(t-s) + ik(l(t)-l(s))} ik^2 q_x(l(s), s) ds &= \int_0^T \mathrm{e}^{ik^3(t-s)+ik(l(t)-l(s))} (k^3 + kl'(s)) q(l(s), s) ds \\
&- \int_0^T \mathrm{e}^{ik^3(t-s)+ik(l(t)-l(s))} k q_{xx}(l(s), s) ds \qquad (3.10) \\
-\mathrm{e}^{ik^3 t + ikl(t)} \int_0^\infty k \mathrm{e}^{-ikx} q_0(x) dx &+ k\mathrm{e}^{ik^3(t-T)+ik(l(t)-l(T))} \int_{l(T)}^\infty \mathrm{e}^{ik(l(T)-\xi)} q(\xi, T) d\xi, \\
\int_0^T \mathrm{e}^{ik^3(t-s)+ik(l(t)-l(s))} k^2 q_{xx}(l(s), s) ds &= \int_0^T \mathrm{e}^{ik^3(t-s)+ik(l(t)-l(s))} k(k^3 + kl'(s)) q(l(s), s) ds \\
&- \int_0^T \mathrm{e}^{ik^3(t-s)+ik(l(t)-l(s))} ik^3 q_x(l(s), s) ds \qquad (3.11) \\
-\mathrm{e}^{ik^3 t + ikl(t)} \int_0^\infty k^2 \mathrm{e}^{-ikx} q_0(x) dx &+ k^2 \mathrm{e}^{ik^3(t-T)+ik(l(t)-l(T))} \int_{l(T)}^\infty \mathrm{e}^{ik(l(T)-\xi)} q(\xi, T) d\xi.
\end{aligned}$$

We note:

(a) Integration by parts of the terms involving the given function $q_0(x)$ (using the assumption that $q_0(x) \to_{x\to\infty} 0$) yields

$$k \int_0^\infty \mathrm{e}^{-ikx} q_0(x) dx = -iq_0(0) - i \int_0^\infty \mathrm{e}^{-ikx} (q_0)_x(x) dx,$$

(b) Integration by parts of the terms involving the given function $q(l(t), t) = f_0(t)$ yields

$$\begin{aligned}
\int_0^T \mathrm{e}^{-ik^3 s - ikl(s)} (k^3 + kl'(s)) f_0(s) ds &= i\mathrm{e}^{-ik^3 T - ikl(T)} f_0(T) - if_0(0) - i \int_0^T \mathrm{e}^{-ik^3 s - ikl(s)} f_0'(s) ds \\
\int_0^T \mathrm{e}^{-ik^3 s - ikl(s)} k(k^3 + kl'(s)) f_0(s) ds &= ik\mathrm{e}^{-ik^3 T - ikl(T)} f_0(T) - ikf_0(0) - i \int_0^T k\mathrm{e}^{-ik^3 s - ikl(s)} f_0'(s) ds
\end{aligned}$$

(c) The integrals containing the unknown function $q(\xi, T)$ are analytic and bounded in the domain $\partial\Omega_2^-$. Indeed, the term $k^2 \mathrm{e}^{ik^3(t-T)+ik(l(t)-l(T))}$ is analytic and is of order $O\left(\frac{1}{k}\right)$ for $k \to \infty$ in this domain, while the integral with respect to ξ is analytic, and of order $O\left(\frac{1}{k}\right)$ as $k \to \infty$, for $\mathrm{Im}(k) \leq 0$.
Similarly, the terms containing $f_0(T)$ and $q_x(l(T), T)$ are analytic and bounded in the domain $\partial\Omega_2^-$.

3.2 Proof of the proposition

We now use the two equation (3.10) and (3.11) obtained from the global relation to prove our result.

Reduction of (3.10) to a Volterra linear integral equation

We first integrate equation (3.10) along $\partial\Omega_2^-$. The terms listed in **(c)** give a vanishing contribution, hence, recalling the definition of $E(k,s,t)$, we obtain

$$\int_{\partial\Omega_2^-}\int_0^T E(k,s,t) i q_x(l(s),s) ds\, k^2 dk = -if_0(0)\int_{\partial\Omega_2^-} E(k,0,t)dk$$
$$-i\int_{\partial\Omega_2^-}\int_0^T E(k,s,t) f_0'(s) ds\, dk - \int_{\partial\Omega_2^-}\int_0^T E(k,s,t) q_{xx}(l(s),s) ds\, kdk$$
$$+iq_0(0)\int_{\partial\Omega_2^-} E(k,0,t)dk + i\int_{\partial\Omega_2^-} E(k,0,t)\int_0^\infty \mathrm{e}^{-ikx} q_0'(x) dx\, dk$$

Since $q_0(0) = f_0(0)$, the terms involving these two values cancel.
Our aim is now to write the term involving $q_{xx}(l(s),s)$ as a Volterra integral. By splitting the inner integral, we can write

$$\int_{\partial\Omega_2^-}\int_0^T E(k,s,t) q_{xx}(l(s),s) ds\, kdk = \int_{\partial\Omega_2^-}\left[\int_0^t + \int_t^T\right] E(k,s,t) q_{xx}(l(s),s) ds\, kdk. \quad (3.12)$$

The behaviour as $k \to \infty$ of the integrand in the two integrals on the right hand side is $\mathrm{O}\left(\frac{1}{k^2}\right)$, and the exponential $E(k,s,t)$ with $s > t$ is bounded in Ω_2. It follows from Cauchy's theorem that the second integral on the right hand side vanishes. Similarly,

$$\int_{\partial\Omega_2^-}\int_0^T E(k,s,t) f_0'(s) ds\, dk = \int_{\partial\Omega_2^-}\int_0^t E(k,s,t) f_0'(s) ds\, dk.$$

Since $E(k,s,t)$ with $s < t$ is bounded in Ω_1^-, and $\partial\Omega_1^- = -\mathbb{R} \cup \partial\Omega_2^-$, we can write

$$\int_{\partial\Omega_2^-}\int_0^t E(k,s,t) q_{xx}(l(s),s) ds\, kdk = \int_{-\infty}^\infty\int_0^t E(k,s,t) q_{xx}(l(s),s) ds\, kdk.$$

The kernel

$$A_1(s,t) = \int_{\mathbb{R}} E(k,s,t) kdk$$

has an integrable singularity at $s = t$. This singularity is of order $1/(t-s)^{2/3}$ (as can be seen by changing variable to $\lambda = (t-s)^{1/3}k$). Hence we can invert the order of integration, and write the term (3.12) as

$$\int_{\partial\Omega_2^-}\int_0^t \mathrm{e}^{ik^3(t-)+ik(l(t)-l(s))} q_{xx}(l(s),s) ds\, kdk = \int_0^t A_1(s,t) q_{xx}(l(s),s) ds. \quad (3.13)$$

In summary, equation (3.10) becomes

$$\int_{\partial\Omega_2^-}\int_0^T e^{ik^3(t-)+ik(l(t)-l(s))} i q_x(l(s),s)ds\, k^2 dk = -\int_0^t A_1(s,t) q_{xx}(l(s),s)ds$$
$$+i\int_{\partial\Omega_2^-} e^{ik^3t+ikl(t)}\left[\int_0^\infty e^{-ikx}(q_0)_x(x)dx - \int_0^t e^{-ik^3s-ikl(s)} f_0'(s)ds\right]dk. \qquad (3.14)$$

Reduction and regularisation of (3.11) to a Volterra integral equation
We now turn to equation (3.11). In this case we have the term

$$\int_0^T e^{-ik^3s-ikl(s)} ik^3 q_x(l(s),s)ds,$$

which does not have sufficient decay as $k\to\infty$ to ensure that we can convert it to a well-defined Volterra integral term.Indeed, if $t=T$, integration by parts shows that this term is $O(1)$ as $k\to\infty$. To overcome this difficulty, we consider the two domains S_1 and S_2 and integrate equation (3.11) along ∂S_1 and along ∂S_2. This gives

$$\int_{\partial S_1}\int_0^T e^{ik^3(t-s)+ik(l(t)-l(s))} q_{xx}(l(s),s)ds\, k^2 dk =$$
$$-i\int_{\partial S_1} e^{ik^3t+ikl(t)}\left[\int_0^t e^{-ik^3s-ikl(s)} f_0(s)' ds - k\int_0^\infty e^{-ikx}(q_0)_x(x)dx\right]dk$$
$$-\int_{\partial S_1}\int_0^T e^{ik^3(t-s)+ik(l(t)-l(s))} ik^3 q_x(l(s),s)ds. \qquad (3.15)$$

and

$$\int_{\partial S_2}\int_0^T e^{ik^3(t-s)+ik(l(t)-l(s))} q_{xx}(l(s),s)ds\, k^2 dk =$$
$$-i\int_{\partial S_2} e^{ik^3t+ikl(t)}\left[\int_0^t e^{-ik^3s-ikl(s)} f_0(s)' ds - k\int_0^\infty e^{-ikx}(q_0)_x(x)dx\right]dk$$
$$-\int_{\partial S_2}\int_0^T e^{ik^3(t-s)+ik(l(t)-l(s))} ik^3 q_x(l(s),s)ds. \qquad (3.16)$$

Note that a rotation of $2\pi/3$ maps S_1 to S_2. Hence for any function $H(k)$ defined on S_2, we have

$$\int_{\partial S_2} H(k)dk = \alpha\int_{\partial S_1} H(\alpha k)dk, \quad \alpha = e^{2\pi i/3}.$$

Hence we can write equation (3.16) as

$$\alpha^2\int_{\partial S_2}\int_0^T e^{ik^3(t-s)+ik(l(t)-l(s))} q_{xx}(l(s),s)ds\, k^2 dk =$$
$$-i\alpha^2\int_{\partial S_2} e^{ik^3t+ikl(t)}\left[\int_0^t e^{-ik^3s-ikl(s)} f_0(s)' ds - k\int_0^\infty e^{-ikx}(q_0)_x(x)dx\right]dk$$
$$-\int_{\partial S_1}\int_0^T e^{ik^3(t-s)+i\alpha k(l(t)-l(s))} ik^3 q_x(l(s),s)ds, \qquad (3.17)$$

where we used $\alpha^3 = 1$. The difference between equation (3.15) and (3.17) yields

$$\int_{\partial S_1}\int_0^T \mathrm{e}^{ik^3(t-s)+ik(l(t)-l(s))} q_{xx}(l(s),s)ds\, k^2 dk - \alpha^2 \int_{\partial S_2}\int_0^T \mathrm{e}^{ik^3(t-)+ik(l(t)-l(s))} q_{xx}(l(s),s)ds\, k^2 dk =$$

$$i\left[\alpha^2\int_{\partial S_2} - \int_{\partial S_1}\right] \mathrm{e}^{ik^3t+ikl(t)} \left[\int_0^t \mathrm{e}^{-ik^3s-ikl(s)} f_0(s)' ds - k\int_0^\infty \mathrm{e}^{-ikx}(q_0)_x(x)dx\right] dk$$

$$-\int_{\partial S_1}\int_0^T \mathrm{e}^{ik^3(t-s)}\left[\mathrm{e}^{ik(l(t)-l(s))} - \mathrm{e}^{i\alpha k(l(t)-l(s))}\right] ik^3 q_x(l(s),s)ds\, dk. \tag{3.18}$$

we now analyse the last integral. Our aim is to write this as a Volterra integral term. Hence we split the t- integral as

$$\int_0^T = \int_0^t + \int_t^T .$$

In order to use Cauchy's theorem to conclude that the second integral does not contribute to the k-integral, we need to check its behaviour as $k \to \infty$. Integration by parts yields

$$\int_t^T \mathrm{e}^{ik^3(t-s)}\left[\mathrm{e}^{ik(l(t)-l(s))} - \mathrm{e}^{i\alpha k(l(t)-l(s))}\right] ik^3 q_x(l(s),s)ds =$$

$$-\mathrm{e}^{ik^3(t-T)}\left[\mathrm{e}^{ik(l(t)-l(T))} - \mathrm{e}^{i\alpha k(l(t)-l(T))}\right] q_x(l(T),T) + \int_t^T \mathrm{e}^{ik^3(t-s)}[\ldots]\, q_x(l(s),s)ds$$

The exponential $\mathrm{e}^{ik^3(t-T)}$ is analytic and bounded for $k \in S_1$ while $\mathrm{e}^{ik(l(t)-l(T))}$ is analytic and bounded for $k \in \mathbb{C}^-$. Since both k and αk are in $\mathbb{C}^-$, overall this integral decays at infinity in S_1, hence the integral along the boundary of this domain vanishes. It follows that the last integral in (3.18) is

$$\int_{\partial S_1}\int_0^t \mathrm{e}^{ik^3(t-s)}\left[\mathrm{e}^{ik(l(t)-l(s))} - \mathrm{e}^{i\alpha k(l(t)-l(s))}\right] ik^3 q_x(l(s),s)ds\, dk.$$

In order to invert the order of integration, we verify that the integral kernel

$$A_2(s,t) = \int_{\partial S_1} \mathrm{e}^{ik^3(t-s)}\left[\mathrm{e}^{ik(l(t)-l(s))} - \mathrm{e}^{i\alpha k(l(t)-l(s))}\right] ik^3 dk$$

is well defined. Changing variable to $\lambda = k(t-s)^{1/3}$ we find that

$$A_2(s,t) = \int_{\partial S_1} \mathrm{e}^{i\lambda^3}\left[\mathrm{e}^{i\lambda\frac{l(t)-l(s)}{(t-s)^{1/3}}} - \mathrm{e}^{i\alpha\lambda\frac{l(t)-l(s)}{(t-s)^{1/3}}}\right]\frac{i\lambda^3 d\lambda}{t-s}.$$

Hence this kernel is regular at $s = t$, and we can write

$$\int_{\partial S_1}\int_0^T \mathrm{e}^{ik^3(t-s)+ik(l(t)-l(s))} q_{xx}(l(s),s)ds\, k^2 dk - \alpha^2 \int_{\partial S_2}\int_0^T \mathrm{e}^{ik^3(t-s)+ik(l(t)-l(s))} q_{xx}(l(s),s)ds\, k^2 dk =$$

$$i\left[\alpha^2\int_{\partial S_2} - \int_{\partial S_1}\right] \mathrm{e}^{ik^3t+ikl(t)} \left[\int_0^t \mathrm{e}^{-ik^3s-ikl(s)} f_0(s)' ds - k\int_0^\infty \mathrm{e}^{-ikx}(q_0)_x(x)dx\right] dk$$

$$-\int_0^t A_2(s,t)q_x(l(s),s)ds. \tag{3.19}$$

Equation (2.26) with $f(t) = iq_x(l(t),t)$ yields

$$\int_{\partial\Omega_2^-}\int_0^T E(k,s,t)iq_x(l(s),s)ds\,k^2dk = \frac{5\pi}{3}iq_x(l(t),t) - \int_0^t K(s,t)iq_x(l(s),s)ds,$$

while equations (2.28)-(2.29) with $f(t) = q_{xx}(l(t),t)$ give

$$2\,\mathrm{Re}\int_{\partial S_1}\int_0^T E(k,s,t)q_{xx}(l(s),s)ds\,k^2dk = \frac{5\pi}{3}q_{xx}(l(t),t) - \int_0^t K(s,t)q_{xx}(l(s),s)ds,$$

$$2\,\mathrm{Re}\int_{\partial S_2}\int_0^T E(k,s,t)q_{xx}(l(s),s)ds\,k^2dk = \frac{5\pi}{3}q_{xx}(l(t),t) - \int_0^t K(s,t)q_{xx}(l(s),s)ds.$$

Substituting these expression in the system (3.14)-(3.19) we finally obtain a system of Volterra integral equations for the two unknown boundary values $q_x(l(t),t))$ and $q_{xx}(l(t),t)$:

$$\begin{aligned}\frac{5\pi}{3}iq_x(l(t),t) &= \int_0^t K(s,t)iq_x(l(s),s)ds - \int_0^t A_1(s,t)q_{xx}(l(s),s)ds \\ &+ \left[i\int_{\partial\Omega_2^-} \mathrm{e}^{ik^3t+ikl(t)}\left[\int_0^\infty \mathrm{e}^{-ikx}(q_0)_x(x)dx - \int_0^t \mathrm{e}^{-ik^3s-ikl(s)}f_0'(s)ds\right]dk\right],\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}\frac{5\pi}{3}q_{xx}(l(t),t) &= \int_0^t K(s,t)q_{xx}(l(s),s)ds - \frac{2}{1-\alpha^2}\mathrm{Re}\int_0^t A_2(s,t)q_x(l(s),s)ds \\ &- \frac{2i\alpha^2}{1-\alpha^2}\mathrm{Re}\int_{\partial S_2}\mathrm{e}^{ik^3t+ikl(t)}\left[\int_0^t \mathrm{e}^{-ik^3s-ikl(s)}f_0(s)'ds - k\int_0^\infty \mathrm{e}^{-ikx}(q_0)_x(x)dx\right]dk \\ &+ \frac{2i}{1-\alpha^2}\mathrm{Re}\int_{\partial S_1}\mathrm{e}^{ik^3t+ikl(t)}\left[\int_0^t \mathrm{e}^{-ik^3s-ikl(s)}f_0(s)'ds - k\int_0^\infty \mathrm{e}^{-ikx}(q_0)_x(x)dx\right]dk.\end{aligned} \tag{3.21}$$

These expressions coincide with (3.2)-(3.3).
QED

4 Conclusions

In the recent paper [10], we presented the inversion formula for the integral transform defined by

$$F(k) = \int_0^T \mathrm{e}^{ik^2s-ikl(s)}f(s)ds.$$

This formula was used to give an expression for the Dirichlet to Neumann map associated with the linear Schrödinger equation posed on a time-dependent domain. This map is a

Volterra integral equation, and the main result of this earlier work was the transformation of the solution obtained from the solution of a d-bar problem, hence containing double complex integrals, into a Volterra integral equation for the unknown boundary data.

The results presented in this paper extend this analysis to the case of an equation of third order. This problem is considerably more challenging than the one considered earlier, and it results in a system of Volterra integral equations characterising the two unknown boundary values. We start by deriving the generalised Fourier transform associated with the third order dispersion relation. This derivation is similar to the derivation of the analogous transform associated to the second order problem. However, to obtain the system characterising the unknown boundary values, it is not sufficient to apply this transform, as in this case the straightforward procedure leads to strongly singular integrals. The singular part of the expression must be regularised before a well-defined system is obtained. This regularisation is the main new element in the present work.

The same approach can be used to generalise the analysis presented here to solve moving boundry value problems for linear evolution PDEs of any order. This is not done here, but it is straightforward. Indeed, the important property leading to the regularisation is the fact that the domain Ω_2^- can always be deformed to the domain given by the union of $(n \pm 1)/2$ sectors of angle $2\pi/n$ (where n is the order of the x differential operator in the PDE) that intersect only at the origin. Since a boundary value problem for such a PDE can be shown to be well posed precisely when $(n \mp 1)/2$ conditions are prescribed (the choice of sign depending on the sign of the highest order term in the PDE), the number of sectors implies that we can derive as many integral equations as there are unknown boundary values. This is based also on the fact that the inversion theorem can be given in terms of an integral along the boundary of any one of the sectors (as we do in the present case, see 2.1).

This approach is important also because it generalises to the case of nonlinear intgerable evolution PDEs in one space dimension. For example, the PDE studied here is essentially the linear part of the KdV equation

$$q_t + q_x + q_{xxx} + qq_x = 0.$$

Combining the recent solution of the boundary value problem posed on the half line for this linear equation [5] with the ideas of the present paper it should be possible to characterise the solution of moving boundary value problems on domains of the kind considered here for the nonlinear integrable case.

The expression for the solution derived here is very convenient for numerical evaluation. Indeed, the analyticity properties of the kernels involved in equations (3.2)-(3.3) can be exploited to deform the contour of integration to contours along which the decay of the integrand is exponential. A similar computation, based on the numerical method given in [2], is presented in [6]. The details of this analysis, and its numerical implementation, are presently being developed and will be presented in a subsequent paper.

5 Acknowledgements

The work leading to the present development was done in collaboration with Thanasis Fokas, to whom I am indebted for continuing mathematical inspiration and encouragement.
This work has been partially funded by EPSRC through grant EP/E022960/1.

Appendix: the case $l(t) = 0$

In this section we sketch the analysis of the case $l(t) = 0$, and derive the associated integral transform. In this case, the integral transform pair can be derived from the usual Fourier transform pair by a change of variable.
However, we show here a direct derivation, that does not depend on the knowledge of the Fourier transform. This is the derivation that can be generalised to the case of a moving boundary ($l(t) \neq 0$).

Proposition 5.1 *Let $f(t)$ be an arbitrary smooth function defined on $[0, T]$, and let the function $F(k)$ be defined in terms of $f(t)$ by*

$$F(k) = \int_0^T e^{-ik^3 s} f(s) ds. \tag{5.1}$$

Then the function $f(t)$ satisfies

$$f(t) = \frac{1}{2\pi} \int_{\partial\Omega_2} e^{ik^3 t} F(k) k^2 dk, \quad 0 < t < T, \tag{5.2}$$

where

$$\partial\Omega_2 = \{k = k_R + ik_I \in \mathbb{C} : \ k_I(k_I^2 - 3k_R^2) = 0\} \tag{5.3}$$

Sketch of the proof: The proof of this proposition follows the lines of the spectral analysis given in section 2, but since $l(t) = 0$ in this case, the solution is obtained through a Riemann-Hilbert, rather than a d-bar problem.
Namely, the two particular solutions of the associated ODE $\mu_t - ik^3\mu = f(t)$ given by

$$\mu^1(t,k) = \int_0^t e^{ik^3(t-s)} f(s) ds, \quad \mu^2(t,k) = -\int_t^\infty e^{ik^3(t-s)} f(s) ds$$

are analytic in $\Omega_1 = \{k : \ 0 \leq 3arg(k) \leq \pi\}$ and $\Omega_2 = \{k : \ \pi \leq 3arg(k) \leq 2\pi\}$ respectively. Hence these two solutions cover the entire complex k plane, and determine a RIemann-Hilbert problem whose unique solution is given by

$$\mu(t,k) = \frac{1}{2\pi i} \int_{\partial\Omega_2} \frac{e^{i\lambda^3 t} F(\lambda)}{\lambda - k} d\lambda,$$

where $F(\lambda)$ is given by (5.1). This yields expression (5.2) for $f(t)$.
QED

We now rewrite the inverse transform (5.2) in a form where it becomes evident that this is the limit, for $l(t) = 0$, of expression (2.2). We first split Ω_2 is the component lying in the upper and lower complex k plane, respectively, as

$$\Omega_2 = \Omega_2^+ \cup \Omega_2^-.$$

Note that $\Omega_2^- = S_1 \cup S_2$ where S_1, S_2 are given by (2.27).

Now we write

$$f(t) = \frac{1}{2\pi}\int_{\partial\Omega_2^-} e^{ik^3t}F(k)k^2dk + \frac{1}{2\pi}\int_{\partial\Omega_2^+} e^{ik^3t}F(k)k^2dk.$$

The first integral on the right hand side of this expression coincides with the first term in expression (2.2), when $l(t) = 0$. Hence we concentrate on the second integral, and we split the time integral in the definition of $F(k)$ and note that (see (2.23)

$$\int_{\partial\Omega_2^+}\int_t^T e^{ik^3(t-s)}f(s)ds\, k^2dk = \frac{\pi}{3}f(t).$$

This yields

$$\frac{5\pi}{3}f(t) = \int_{\partial\Omega_2^-} e^{ik^3t}F(k)k^2dk + \int_0^t \left[\int_{\partial\Omega_2^+} e^{ik^3(t-s)}k^2dk\right] f(s)ds. \qquad (5.4)$$

The integral kernel in the last integral on the right hand side of formula (5.4) is equal to the limit as $l(t) \to 0$ of the kernel (2.3).

References

[1] M.J Ablowitz and A.S. Fokas, *Introduction and applications of complex variables*, Cambridge University Press, 2nd ed. 2004.

[2] P. Baratella, and A.P. Orsi, A new approach to the numerical solution of weakly singular Volterra integral equations, *J. Comp. Appl. Anal.* **163**, 2004, 401-418.

[3] A.S. Fokas, A unified transform method for solving linear and certain nonlinear PDE's, *Proc. Royal Soc. Series A*, **453** 1997, 1411–1443.

[4] A.S. Fokas, A new transform method for evolution PDEs, *IMA J. Appl. Math*, **67** 2002, 559-590.

[5] A.S. Fokas, A generalised Dirichlet to Neumann map for certain nonlinear evolution PDEs, *Comm. Pure Appl. Math.* **LVIII** 2005, 639-670

[6] A.S. Fokas and S. De Lillo, The Dirichlet to Neumann map for the heat equation on a moving boundary,Inv. Prob. (in press)

[7] A.S. Fokas and I.M. Gelfand, Integrability of linear and nonlinear evolution equations and the associated nonlinear Fourier transform, *Lett. Math. Phys.* (1994) **32**, 189-210.

[8] A.S. Fokas and B. Pelloni, A method of solving moving boundary value problems for linear evolution equations, *Phys. Rev. Lett.* **84** 2000, 4785-4789.

[9] A.S. Fokas and B. Pelloni, Integrable evolution equations in time-dependent domains, *Inv. Prob.* **17** 2001, 919-935.

[10] A.S. Fokas and B. Pelloni, The generalised Dirichlet to Neumann map for moving initial-boundary value problems, J. Math. Phys. published online, 9 January 2007 Vol.48, No.1.

[11] A.S. Fokas and L.Y. Sung, Initial boundary value problems for linear evolution equations on the half line, (submitted).

[12] A.S. Fokas and L.Y. Sung, Generalized Fourier transforms, their nonlinearization and the imaging of the brain, *Notices AMS* (2005) **52**, 1178-1192.

[13] Y. Katznelson, *An introduction to harmonic analysis*, Dover, 1976.

[14] B. Pelloni. Well-posed boundary value problems for linear evolution equations on a finite interval, *Math. Proc. Camb. Phil. Soc.* (2004), **136**, 361-382.

[15] B. Pelloni. The spectral representation of two-point boundary value problems for linear PDEs, *Proc. Royal Soc. Lond. A* **461** 2005, 2965-2984.

[16] V. Volterra, *Theory of functionals and of integral and integro-differential equations* Dover, 1959.