A Galerkin Boundary Element Method for an Acoustic Scattering Problem, with Convergence Rate Independent of Frequency

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Abstract

In this paper we consider the problem of acoustic scattering in a half plane by a surface of piecewise constant impedance. To achieve good approximation at high frequencies with a relatively low number of degrees of freedom we propose a novel Galerkin boundary element method, using a graded mesh with smaller elements adjacent to discontinuities in impedance and a special set of basis functions so that, on each element, the approximation space contains polynomials (of degree \( \nu \)) multiplied by traces of plane waves on the boundary. The numerical scheme is stable and convergent, and the error in computing the total acoustic field is of order \( N^{-((\nu+1)/2)} \log^{1/2} \min(N, k) \), where the number of degrees of freedom is proportional to \( N \log \min(N, k) \) and \( k \) is the wavenumber. We focus here particularly on details regarding the implementation of this scheme.

1 Introduction

In this paper we consider the problem of acoustic scattering of an incident wave by a planar surface with spatially varying acoustical surface impedance. In the case in which there is no variation in the acoustical properties of the surface or the incident field in some fixed direction parallel to the surface,
the problem is effectively two-dimensional. Adopting Cartesian coordinates \(0x_1,x_2,x_3\), let this direction be that of the \(x_3\)-axis and the surface be the plane \(x_2 = 0\). Under the further assumption that the incident wave and scattered fields are time harmonic, the total acoustic field \(u^t \in C(U) \cap C^2(U)\) then satisfies the Helmholtz equation

\[ \Delta u^t + k^2 u^t = 0, \quad \text{in } U := \{(x_1,x_2) \in \mathbb{R}^2 : x_2 > 0\}, \]  

supplemented with the impedance boundary condition

\[ \frac{\partial u^t}{\partial x_2} + ik \beta u^t = f, \quad \text{on } \Gamma := \{(x_1,0) : x_1 \in \mathbb{R}\}, \]  

with \(f \equiv 0\), where \(k = \omega/c > 0\), with \(\omega = 2\pi \mu\), \(\mu\) is the frequency of the incident wave and \(c\) is the speed of sound in \(U\). The acoustic pressure at time \(t\), position \((x_1,x_2,x_3)\) is then given by \(\text{Re}(e^{-ikt}u^t(x))\), where \(x = (x_1,x_2) \in \overline{U}\). The function \(\beta\) in (2) is the relative ground surface admittance and typically in outdoor sound propagation depends on the frequency and the ground properties. We assume throughout that \(\beta\) is piecewise constant, and constant outside some finite interval \([a,b]\), so that for some real numbers \(a = t_0 < t_1 < \ldots < t_n = b\),

\[ \beta(x_1) = \begin{cases} \beta_j, & x_1 \in (t_{j-1},t_j], \\ \beta_c, & x_1 \in \mathbb{R}\backslash(t_0,t_n], \end{cases} \]

where we assume that for some \(\epsilon > 0\), \(\text{Re}\beta_c \geq \epsilon, \text{Re}\beta_j \geq \epsilon, |\beta_c| \leq \epsilon^{-1}, |\beta_j| \leq \epsilon^{-1}, j = 1,\ldots,n\). The condition \(\text{Re}(\beta) \geq 0\) ensures that the ground surface absorbs rather than emits energy.

For simplicity of exposition, we restrict our attention to the case of plane wave incidence, so that the incident field \(u^i\) is given by \(u^i(x) = e^{ik(x_1\sin \theta - x_2 \cos \theta)}\), where \(\theta \in (-\pi/2, \pi/2)\) is the angle of incidence. The reflected or scattered part of the wave field is \(u := u^i - u^t \in C(\overline{U}) \cap C^2(U)\), and this also satisfies (1) and (2) with \(f(x_1) := -\partial u^t/\partial x_2 - ik \beta u^t, x_1 \in \mathbb{R}\).

In this paper we are concerned with solving (1)-(2) numerically specifically in the case in which \(k\) may be large. This corresponds to the high frequency, low wavelength case (the wavelength is \(\lambda = c/\mu\)) and presents a number of numerical difficulties. Standard finite or boundary element schemes for scattering problems become prohibitively expensive as \(k \to \infty\), with a fixed number of elements
being required per wavelength in order to achieve reasonable accuracy. When the wavelength is
short compared to the size of the obstacle this leads to excessively large systems of equations. Much
recent work has focused on enriching the approximation space with plane wave solutions of (1),
in order to more efficiently represent the highly oscillatory solution. Although excellent numerical
results have been reported, until recently there has been little in the way of error analysis for this
approach, specifically with regard to the dependence of the error estimates on the wavenumber $k$.
For a full discussion and literature review regarding this approach we refer to [3].

Here we use similar ideas, and we provide an error estimate showing a convergence rate which is
independent of the wavenumber $k$. To achieve this, we begin by using ideas in the spirit of the
geometrical theory of diffraction (GTD) to identify and subtract off the leading order behaviour
(namely the incident and reflected rays) as $k \to \infty$. The remaining scattered wave (consisting of
the rays diffracted at impedance discontinuities) can then be expressed (on the boundary $\Gamma$) as the
product of the known oscillatory functions $e^{\pm ikx_1}$ and unknown non-oscillatory functions denoted
as $f_j^\pm$, which can be shown to decay quickly away from impedance discontinuities. We then use a
Galerkin boundary element method with a plane wave basis on a graded mesh, so as to obtain a
piecewise polynomial (order $\nu$) representation of the non-oscillatory functions $f_j^\pm$.

This approach was first used in [2], leading to an error bound depending only logarithmically on
the wavenumber $k$. This has been improved upon in [3], where stronger regularity results on
$f_j^\pm$ have been proved, leading to a slightly different approximation space. For this scheme it
was shown in [3] that the error in computing an approximation to the total wave field is of or-
der $N^{-(\nu+1)} \log^{1/2} |\min(N, k)|$ where the total number of degrees of freedom is proportional to
$(N \log |\min(N, k)|)$. Here, in §2 we briefly describe the Galerkin boundary element method approach
of [3], and we present the error estimates proved there. In §3 we discuss some of the issues regarding
the implementation of the method, and extend the presentation of [3] to include formulae for the
coefficients needed to set up the linear system. Finally in §4 we present some numerical results,
demonstrating that the scheme is stable and convergent even when certain assumptions on $\beta$, needed
to prove the error estimate in [3], are not satisfied.
2 The Galerkin method and error estimates

It is shown in [3] that the boundary value problem (1)–(2), accompanied by an appropriate radiation condition, can be reformulated as the second kind boundary integral equation

$$\phi = \psi_{\beta_c} + K_{\beta_c}^{\beta} \phi,$$  

(3)

where for $s \in R$, $\phi(s) := u^t((s/k,0))$, $\psi_{\beta_c}(s) := u_{\beta_c}^t((s/k,0))$ (with $u_{\beta_c}^t$ denoting the (known) total acoustic field in the case that $\beta \equiv \beta_c$), and

$$K_{\beta_c}^{\beta} \chi(s) := i \int_{R}^{k_b} G_{\beta_c}(((s - t)/k,0),(0,0))(\beta(t/k) - \beta_c)\chi(t) dt,$$

with $G_{\beta_c}(x,y)$ denoting the Green’s function for the above problem which satisfies the standard Sommerfeld radiation and boundedness conditions, and (2) with $\beta \equiv \beta_c$ and $f \equiv 0$. Our numerical scheme for solving (3) is based on a consideration of the contribution of the reflected and diffracted ray paths as predicted by the GTD. In particular, to leading order as $k \to \infty$, on the interval $(t_{j-1}, t_j)$ the GTD predicts that the total field $\phi \approx \psi_{\beta_j}$, the total field there would be if the whole boundary had the admittance $\beta_j$ of the interval $(t_{j-1}, t_j)$. Thus, for $s \notin \{t_j := kt_j, j = 0, \ldots, n\}$, the GTD predicts that $\phi(s) \to \Psi(s)$ as $k \to \infty$, where $\Psi(s) := \psi_{\beta_j}(s)$, $s \in (t_{j-1}, t_j]$, $j = 1, \ldots, n$, and $\Psi(s) := \psi_{\beta_c}(s)$, $s \in R \setminus \{t_0, t_n\}$. In our numerical scheme we compute the difference between $\phi$ and $\Psi$,

$$\Phi(s) := \phi(s) - \Psi(s) = \Psi_{\beta_c}(s) + K_{\beta_c}^{\beta} \phi(s), \quad s \in \mathbb{R}.$$  

(4)

where $\Psi_{\beta_c} \in L^\infty(R)$ is known and is given by $\Psi_{\beta_c} := \psi_{\beta_c} - \Psi + K_{\beta_c}^{\beta} \Psi$. Equation (4) will be the integral equation that we solve numerically, and we approximate its solution by $\Phi_N \in V_{\Omega, \nu}$, where

$$(\Phi_N, \chi) = (\Psi_{\beta_c}, \chi) + (K_{\beta_c}^{\beta} \Phi_N, \chi), \quad \text{for all } \chi \in V_{\Omega, \nu}.$$  

(5)

To determine the approximation space we begin by writing

$$\Phi(s) = e^{is} f^+(s - \tilde{t}_{j-1}) + e^{-is} f^-(\tilde{t}_j - s), \quad s \in (\tilde{t}_{j-1}, \tilde{t}_j], j = 1, \ldots, n.$$  

(6)

In GTD terms, the first term in (6) is an explicit summation of all the diffracted rays scattered at the discontinuities in impedance at $\tilde{t}_{j-1}$ which travel from left to right along $(\tilde{t}_{j-1}, \tilde{t}_j)$. Similarly, the
other term in (6) is the contribution to the diffracted field diffracted by the discontinuity at \( \hat{t}_j \). It is shown in [3] that the functions \( f_j^\pm \) are not oscillatory, and in particular that there exist constants \( c_m \) dependent only on \( m \) and \( \epsilon \) such that \(|f_j^\pm(m)(r)| \leq c_mr^{-3/2-m} \), for \( r > 1 \).

We then seek a piecewise polynomial approximation to \( f_j^\pm \) on a graded mesh. For \( j = 1, \ldots, n \) we define \( A_j := \min\{\alpha N^{\nu+1}, \hat{t}_j - \hat{t}_{j-1}\} \), where \( N = 2, 3, \ldots \), and \( \alpha \) is a constant (chosen experimentally), independent of \( N \) and \( k \). Assuming \( A_j > 1 \), the mesh \( \Lambda_{N,A_j} : 0 = y_0 < \ldots < y_{N+N_{A_j}} = A_j \) is a composition of two parts and consists of the points \( y_i := (i/N)^{1+2\nu/3}, i = 0, \ldots, N \), \( y_{N+j} := A_j^{j/N_{A_j}} \), \( j = 1, \ldots, N_{A_j} \), with \( N_{A_j} = [-\log A_j/q \log(1-1/N)] \) chosen to ensure a smooth transition between the two parts of the mesh. For \( j = 1, \ldots, n \) we then define the two meshes \( \Omega_j^+ := \hat{t}_j - \Lambda_{N,A_j} \), \( \Omega_j^- := \hat{t}_j + \Lambda_{N,A_j} \), and also \( \Pi_{l_j}^{\pm, \nu} := \{ \sigma : \sigma|_{y_{l_j}^\pm, y_{l_j}^\pm} \} \) is a polynomial of degree \( \leq \nu \), for \( j = 1, \ldots, N + N_{A_j} \), where \( y_{l_j}^\pm \) are the mesh points of \( \Omega_j^\pm \). Letting \( e_{\pm}(s) := e^{\pm \epsilon s}, s \in \mathbb{R} \), we then define \( V_{l_j}^{\pm, \nu} := \{ e_{\pm}(s) : \sigma \in \Pi_{l_j}^{\pm, \nu} \} \), and then our approximation space is \( V_{l_j}^{\nu} := \bigcup_{j=1,\ldots,n} \{ V_{l_j}^{\pm, \nu} \cup V_{l_j}^{\pm, \nu} \} \).

The number of degrees of freedom, i.e. \( \dim(V_{l_j}^{\nu}) \) is of order \( N \log(\min(N,k)) \). We rewrite (5) as

\[
\Phi_N - P_N K_\beta^\nu \Phi_N = P_N \Psi_{\beta_c}
\]  

(7)

where \( P_N : L_\infty(R) \cup L_2(R) \rightarrow V_{l_j}^{\nu} \) is the operator of orthogonal projection onto \( V_{l_j}^{\nu} \). Assuming

\[
|\beta_j - \beta_c| < \Re \beta_c, \quad j = 1, \ldots, n,
\]

(8)

it is shown in [3] that \((I - P_N K_\beta^{\nu})^{-1} : L_2(\mathbb{R}) \rightarrow L_2(\mathbb{R})\) exists, so that (7) is uniquely solvable, and \( \| (I - P_N K_\beta^{\nu})^{-1} \| \leq \Re \beta_c / (\Re \beta_c - \| \beta - \beta_c \|_\infty) \). Numerical results in §4 suggest stability and convergence of the scheme even if (8) does not hold. The following theorem is proved in [3].

**Theorem 2.1** Assuming (8), there exist constants \( C_1, C_2 > 0 \) dependent only on \( \nu \) and \( \epsilon \) such that

\[
\| \Phi - \Phi_N \|_{2(\mathbb{R},\mathbb{R})} \leq \frac{C_1 N^{7/2}(1 + \log^{1/2}(\min(\alpha N^{\nu+1}, k(b-a))))}{\cos^2 \theta (\Re \beta_c - \| \beta - \beta_c \|_\infty) N^{\nu+1}},
\]

\[
|u'(x) - u_N'(x)| \leq \frac{C_2 N^{7/2}(1 + \log^{1/2}(\min(\alpha N^{\nu+1}, k(b-a))))}{\cos^2 \theta (\Re \beta_c - \| \beta - \beta_c \|_\infty) N^{\nu+1}},
\]

where for \( x \in U \), \( u_N'(x) \) is the approximation to \( u' \) defined, for \( \phi_N := \Phi_N + \Psi \), by

\[
u'N(x) := u_{\beta_c}^N(x) + ik \int_0^1 G_{\beta_c}(x, (y_1, 0))(\beta(y_1) - \beta_c) \phi_N(ky_1) \, dy_1.
\]

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3 Implementation

We restrict our attention in this section to the case $\nu = 0$. The implementation of the scheme is similar for higher values of $\nu$, but the formulas for the coefficients of the linear system will be considerably more complicated. Writing $\Phi_N$ as a linear combination of the basis functions of $V_{\Omega,0}$, we have $\Phi_N(s) = \sum_{j=1}^{M_N} v_j \rho_j(s)$, where $M_N := \dim(V_{\Omega,0})$ and $\rho_j$ is the $j^{th}$ basis function, given by

$$\rho_j(s) := \frac{e^{i\alpha} \chi([s_k^+, s_k^-])}{(s_j^+ - s_{j-1}^-)^{1/2}}, \quad j = \tilde{j} + 2 \sum_{n=1}^{p-1} (N + N_{A_m}), \quad \tilde{j} = 1, \ldots, N + N_{A_p},$$

for $p = 1, \ldots, n$, where $s_k^+ \in \Omega^+_p$, $s_k^- \in \Omega^-_p$ for $l = 0, \ldots, N + N_{A_p}$, and $\chi_{[s_1, s_2]}$ denotes the characteristic function of the interval $[s_1, s_2]$. The scaling of $\rho_j$, $j = 1, \ldots, M_N$, ensures that if $k$ is large compared to $N$, such that the two meshes $\Omega^+_j$ and $\Omega^-_j$ do not overlap, then the basis functions $\rho_j$, $j = 1, \ldots, M_N$, form an orthonormal basis for $V_{\Omega,0}$ (this is not true for the Galerkin method of [2]).

Equation (5) then becomes the linear system

$$\sum_{j=1}^{M_N} v_j ((\rho_j, \rho_l) - (K_{\beta_c}^\alpha \rho_j, \rho_l)) = (\Psi_{\beta_c}^\alpha, \rho_l), \quad l = 1, \ldots, M_N. \quad (9)$$

To form (9) we need to determine the coefficients $(\rho_j, \rho_l)$, $(K_{\beta_c}^\alpha \rho_j, \rho_l))$ and $(\Psi_{\beta_c}^\alpha, \rho_l)$.

We evaluate $(\rho_j, \rho_l)$ analytically. For $j = 1, \ldots, M_N$, we have for some constants $\theta_j$, $c_j$ and $d_j$ that

$$\rho_j(s) = e^{i\theta_j} \chi_{[c_j, d_j]}(t)/(d_j - c_j)^{1/2}, \quad (10)$$

where $\theta_j = \pm 1$, and $a \leq c_j < d_j \leq b$. It is then straightforward to show that $(\rho_j, \rho_j) = 1, (\rho_j, \rho_l) = 0$ if $[c_j, d_j]$ and $[c_l, d_l]$ do not overlap (using the notation of (10)), and if $[c_j, d_j]$ and $[c_l, d_l]$ do overlap such that $j \neq l$ and max$(c_j, c_l) \leq \min(d_j, d_l)$ (in which case $\theta_j \neq \theta_l$) then

$$\rho_j, \rho_l) = e^{i(\theta_j - \theta_l) \min(d_j, d_l)} - e^{i(\theta_j - \theta_l) \max(c_j, c_l)} / (d_j - c_j)^{1/2}(d_l - c_l)^{1/2}(\theta_j - \theta_l).$$

To evaluate $(K_{\beta_c}^\alpha \rho_j, \rho_l)$, from [3] and using the representation (10) we have $(K_{\beta_c}^\alpha \rho_j, \rho_l) = i(\beta_j + \beta_c)(d_j - c_j)^{-1/2}(d_l - c_l)^{-1/2}(I_1/\pi + C_{\beta_c}^\alpha I_2)$, where

$$I_1 := \int_{c_l}^{d_l} \int_{c_j}^{d_j} F(r)e^{i[-r\theta_j + (\theta_j - \theta_l)e]} \, dr \, ds, \quad I_2 := \int_{c_l}^{d_l} \int_{c_j}^{d_j} e^{i[-r\theta_j + (\theta_j - \theta_l)e]} \, dt \, ds, \quad (11)$$

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\( \hat{\beta}_j \) denotes the value of \( \beta \) on \((c_j, d_j)\), \( F(r) := (r^{1/2}(r - 2i)^{1/2})/(r^2 - 2ir - \beta_c^2) \), \( \eta := (1 - \beta_c^2)^{1/2} \), with \( \text{Re}(\eta) \geq 0 \), and \( C_{\beta_c} := \beta_c/\eta \) if \( \text{Im}(\beta_c) < 0 \) and \( \text{Re}(\eta) > 1 \), \( C_{\beta_c} := \beta_c/2\eta \) if \( \text{Im}(\beta_c) < 0 \) and \( \text{Re}(\eta) = 1 \), and \( C_{\beta_c} := 0 \) otherwise.

If \( \rho_j \) and \( \rho_l \) are supported on the same interval, with \( c_j = c_l =: c \) and \( d_j = d_l =: d \), then

\[
I_2 = \frac{2i\eta(d - c)}{\eta^2 - 1} + \frac{1 - e^{i(\eta - 1)(d - c)}}{(\eta - 1)^2} + \frac{1 - e^{i(\eta + 1)(d - c)}}{(\eta + 1)^2},
\]

\[
I_1 = (d - c) \int_0^\infty \frac{2(i - r)F(r)}{r(2i - r)} \, dr + \int_0^\infty \frac{F(r)}{r^2} (e^{-r(d - c)} - 1) \, dr + \int_0^\infty \frac{F(r)}{(2i - r)^2} (e^{(2i - r)(d - c)} - 1) \, dr.
\]

If the supports of \( \rho_j \) and \( \rho_l \) do not overlap, then assuming without loss of generality that \( d_j \leq c_l \) (the case \( d_l \leq c_j \) is identical with \( c_j \leftrightarrow c_l, d_j \leftrightarrow d_l, \theta_j \leftrightarrow -\theta_l \)), we have

\[
I_2 = \left( \frac{e^{i(\theta_j - \eta)\theta_j} - e^{i(\theta_j - \eta)\theta_j}}{i(\theta_j - \eta)} \right) \left( \frac{e^{i(\eta - \theta_j)\theta_j} - e^{i(\eta - \theta_j)\theta_j}}{i(\eta - \theta_j)} \right),
\]

\[
I_1 = e^{i(\theta_j - \eta)d_j} \int_0^\infty \frac{F(r)(e^{i(\theta_j - \eta)r - r(d_j - d_l)} - e^{i(\theta_j - \eta)r - r(c_l - d_l)})}{i(1 - \theta_j - r)(i(\theta_j - 1) + r)} \, dr
\]

\[+ e^{i(\theta_l - \eta)c_l} \int_0^\infty \frac{F(r)(e^{i(\theta_j - \eta)r - r(c_l - c_j)} - e^{i(\theta_j - \theta_l)r - r(c_l - c_j)})}{i(1 - \theta_l - r)(i(\theta_j - 1) + r)} \, dr.
\]

If \( j \neq l \) but the supports of \( \rho_j \) and \( \rho_l \) overlap then \( I_2 \) can be computed analytically as above, and \( I_1 \) can be expressed as a combination of integrals of the forms above.

To evaluate \((\Psi_\beta^\beta, \rho_l)\) we again use (10) and write \((\Psi_\beta^\beta, \rho_l) = I_3 + i(I_4/\pi + C_{\beta_c} I_5)\), where

\[
I_3 := \int_{c_l}^{d_l} (\psi_{\beta_c}(s) - \Psi(s)) e^{-i\theta_1 s} \, ds,
\]

\[
I_4 := \int_{c_l}^{d_l} \int_{t_m}^{t_n} \int_0^\infty F(r)e^{i(\theta - \eta)r - \eta} \beta \left( \frac{t}{k} \right) - \beta_c \right) \Psi(t) e^{-i\theta_1 t} \, dr \, dt \, ds
\]

\[
I_5 := \int_{c_l}^{d_l} \int_{t_m}^{t_n} e^{i(\theta - \eta)t} \left( \beta \left( \frac{t}{k} \right) - \beta_c \right) \Psi(t) e^{-i\theta_1 t} \, dt \, ds.
\]

Without loss of generality we assume \([c_l, d_l] \in [t_{m-1}, t_m]\), for some \( m = 1, \ldots, n \), and then

\[
I_3 = \frac{R_{\beta_c}(\theta) - R_{\beta_c}(\theta)}{i(\sin \theta - \theta_l)}(e^{i(\sin \theta - \theta_l)d_l} - e^{i(\sin \theta - \theta_l)c_l}),
\]

where \( R_{\beta_c}(\theta) := (\cos \theta - \beta^*)(\cos \theta + \beta^*) \). To evaluate \( I_5 \), note that if \( C_{\beta_c} \neq 0 \) then \( \text{Im}(\beta_c) < 0 \) and \( \text{Re}(\eta) \geq 1 \), in which case \( \eta \notin [-1, 1] \), and thus

\[
I_5 = \left[ - \frac{e^{i(\theta - \eta)d_l} - e^{i(\theta - \eta)c_l}}{\eta - \theta_l} \right] \sum_{j=1}^{n-1} (\beta_j - \beta_c)(1 + R_{\beta_j}(\theta)) \left( \frac{e^{i(\sin \theta - \eta)\tilde{t}_j} - e^{i(\sin \theta - \eta)\tilde{t}_{j-1}}}{\sin \theta - \eta} \right)
\]

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\[
(\beta_m - \beta_c)(1 + R_{\beta_m}(\theta)) \left( -2\eta (e^{i(\sin \theta - \theta_i)}d_t - e^{i(\sin \theta - \theta_i)c_t}) \right) \\
+ \frac{e^{i(\sin \theta - \eta)\tilde{t}_{m-1}} (e^{i(\eta - \theta_i)d_t} - e^{i(\eta - \theta_i)c_t})}{(\sin \theta - \eta)(\eta - \theta_t)} + \frac{e^{i(\sin \theta + \eta)\tilde{t}_m} (e^{-i(\eta + \theta_i)d_t} - e^{-i(\eta + \theta_i)c_t})}{(\sin \theta + \eta)(\eta + \theta_t)} \\
+ \left( \frac{e^{-i(\eta + \theta_i)d_t} - e^{-i(\eta + \theta_i)c_t}}{\eta + \theta_t} \right) \sum_{j=m+1}^{n} (\beta_j - \beta_c)(1 + R_{\beta_j}(\theta)) \left( \frac{e^{i(\sin \theta + \eta)\tilde{t}_j} - e^{i(\sin \theta + \eta)\tilde{t}_j-1}}{\sin \theta + \eta} \right).
\]

To evaluate \( I_1 \) we note that \( \sin \theta \neq \pm 1 \), as \( \theta \in (-\pi/2, \pi/2) \), and thus, after some algebra,

\[
I_1 = \left\{ \begin{array}{l}
\sum_{j=1}^{m-1} (\beta_j - \beta_c)(1 + R_{\beta_j}(\theta)) \left( e^{i(\sin \theta - \eta)\tilde{t}_j} \int_{0}^{\infty} F(r) \left( e^{i(1-\theta_i)c_t} + r(\tilde{t}_{j-1} - c_t) - e^{i(1-\theta_i)d_t} + r(\tilde{t}_{j-1} - d_t) \right) dr \right) \\
+ e^{i(\sin \theta - \eta)\tilde{t}_j} \int_{0}^{\infty} F(r) \left( e^{i(1-\theta_i)c_t} + r(\tilde{t}_{j-1} - c_t) - e^{i(1-\theta_i)d_t} + r(\tilde{t}_{j-1} - d_t) \right) dr \\
\end{array} \right. \\
+ (\beta_m - \beta_c)(1 + R_{\beta_m}(\theta)) \left( e^{i(\sin \theta - \eta)\tilde{t}_m} \int_{0}^{\infty} F(r) \left( e^{i(1-\theta_i)c_t} + r(\tilde{t}_{m-1} - c_t) - e^{i(1-\theta_i)d_t} + r(\tilde{t}_{m-1} - d_t) \right) dr \right) \\
+ e^{i(\sin \theta - \eta)\tilde{t}_m} \int_{0}^{\infty} F(r) \left( e^{i(1-\theta_i)c_t} + r(\tilde{t}_{m-1} - c_t) - e^{i(1-\theta_i)d_t} + r(\tilde{t}_{m-1} - d_t) \right) dr \\
+ \sum_{j=m+1}^{n} (\beta_j - \beta_c)(1 + R_{\beta_j}(\theta)) \left( e^{i(\sin \theta + \eta)\tilde{t}_j} \int_{0}^{\infty} F(r) \left( e^{-i(1+\theta_i)c_t} + r(\tilde{t}_{j-1} - c_t) - e^{-i(1+\theta_i)d_t} + r(\tilde{t}_{j-1} - d_t) \right) dr \right) \\
+ e^{i(\sin \theta + \eta)\tilde{t}_j} \int_{0}^{\infty} F(r) \left( e^{-i(1+\theta_i)c_t} + r(\tilde{t}_{j-1} - c_t) - e^{-i(1+\theta_i)d_t} + r(\tilde{t}_{j-1} - d_t) \right) dr \\
\right\}
\]

The remaining integrals for \( I_1 \) and \( I_4 \) must be evaluated numerically. We note that the integrals to be evaluated are similar in difficulty to the integral representation for the Green’s function, ([2, equation (2.7)]), for which very efficient numerical schemes are proposed in [1]. In particular, we remark that the integrals are not oscillatory although the original integrands in (11) and (12) are highly oscillatory: the oscillating part of the integrands has been removed by the integrations which have been carried out analytically. Hence the coefficients do not become more difficult to evaluate as \( k \to \infty \). Difficulties in evaluation of the remaining integrals are the infinite range of integration, the branch point singularity in the integrand at \( r = 0 \), and a simple pole in the integrand at \( r = i(1 - \sqrt{1 - \beta_j^2}) \) which may lie on or close to the positive real axis. In the numerical computations below we use a simple numerical integration scheme, making the substitution \( r = s^2/(1 - s^2) \), which
2 18 1.6466 × 10^{-1} 1.17 1.6761 × 10^{-1} 1.25
4 46 7.3159 × 10^{-2} 1.04 7.0485 × 10^{-2} 0.86
8 106 3.5568 × 10^{-2} 1.02 3.8766 × 10^{-2} 1.15
16 240 1.7482 × 10^{-2} 1.04 1.7484 × 10^{-2} 1.11
32 530 8.5103 × 10^{-3} 8.0838 × 10^{-3}

Table 1: \( \| \Phi - \Phi_N \|_2 / \| \Phi \|_2 \) for \( m=160 \) and \( 5120 \), and increasing \( N \).

changes the range of integration to \((0, 1)\) and removes the branch point singularity at \( r = 0 \). We then apply the composite midpoint rule, with \( M \) evenly distributed points on \((0, 1)\), choosing \( M = 500 \).

4 Numerical results

As a numerical example, we take \( \theta = \pi / 4, \ n = 1, \) and

\[
\beta(s) = \begin{cases} 
0.505 - 0.3i, & s \in [-m\lambda, m\lambda], \\
0.1 & s \notin [-m\lambda, m\lambda],
\end{cases}
\]

for \( m=160, 320, 640, 1280, 2560 \) and \( 5120 \), where \( k = 1 \) and \( \lambda = 2\pi \) is the wavelength. This experiment is equivalent to fixing the interval \([a, b] = [t_0, t_1] \) and decreasing the wavelength. Assumption (8) is not satisfied, but we demonstrate here that even in this case the numerical scheme still appears to be stable and convergent. For each value of \( m \), we compute \( \Phi_N \) with \( \nu = 0, \alpha = 50 \) (chosen experimentally) and \( N=2, 4, 8, 16, 32 \). For the purpose of computing errors, we compute the “exact” solution \( \Phi \) with \( N = 128 \) and \( \alpha = 1000 \). For \( m=160 \) and \( 5120 \) the relative \( L_2 \) errors \( \| \Phi - \Phi_N \|_2 / \| \Phi \|_2 \) are shown in table 1. (We approximate \( \| \cdot \|_2 \) by the discrete \( L_2 \) norm, sampling at 100000 evenly spaced points in the relevant interval for the function whose norm is to be evaluated.) The estimated order of convergence is given by \( \text{EOC} := \log_2(\| \Phi - \Phi_N \|_2 / \| \Phi - \Phi_{2N} \|_2) \). The fact that \( \text{EOC} \approx 1 \) is consistent with what we would expect in the case that (8) did hold, in which case the bounds of
<table>
<thead>
<tr>
<th>((b - a)/\lambda)</th>
<th>(M_N)</th>
<th>(|\Phi - \Phi_{16}|/|\Phi|)</th>
<th>(|\Phi - \Phi_{16}|)</th>
</tr>
</thead>
<tbody>
<tr>
<td>320</td>
<td>240</td>
<td>1.7482 \times 10^{-2}</td>
<td>1.3798 \times 10^{-2}</td>
</tr>
<tr>
<td>640</td>
<td>240</td>
<td>1.7198 \times 10^{-2}</td>
<td>1.3546 \times 10^{-2}</td>
</tr>
<tr>
<td>1280</td>
<td>240</td>
<td>1.6448 \times 10^{-2}</td>
<td>1.2902 \times 10^{-2}</td>
</tr>
<tr>
<td>2560</td>
<td>240</td>
<td>1.6217 \times 10^{-2}</td>
<td>1.2619 \times 10^{-2}</td>
</tr>
<tr>
<td>5120</td>
<td>240</td>
<td>1.5481 \times 10^{-2}</td>
<td>1.1856 \times 10^{-2}</td>
</tr>
<tr>
<td>10240</td>
<td>240</td>
<td>1.7484 \times 10^{-2}</td>
<td>1.2979 \times 10^{-2}</td>
</tr>
</tbody>
</table>

Table 2: \(\|\Phi - \Phi_{16}\|/\|\Phi\|\) for increasing interval length.

Theorem 2.1 would apply. The number of degrees of freedom is the same for the two cases \(b - a = 320\lambda\) and \(b - a = 10240\lambda\), and yet the relative \(L_2\) errors are almost equal.

In Table 2 we fix \(N = 16\) and show \(\|\Phi - \Phi_{16}\|/\|\Phi\|\) and also \(\|\Phi - \Phi_{16}\|\) for increasing values of \(m = (b - a)/2\lambda\). The number of degrees of freedom remains constant, but the relative and actual \(L_2\) error also both remain roughly constant as \(m\) grows. For \(m = 5120\) the interval is of length greater than ten thousand wavelengths, and yet we achieve almost one per cent relative error with only 240 degrees of freedom.

References

