Laplace's Equation on a Perturbed Half-plane Domain

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Solving Water Wave Problems via Integral Equations

Our problem

- Half-plane (2D)
- Non-periodic
- Non-overturning

The current state of play

- Half-domain (2D, 3D)
- Periodic
- Overturning
- Proven numerics (Dold, Hou, Beale, etc)

Lyapunov Surface

Free surface is assumed to be the graph of a Lyapunov function

- Non-overturning
- $f \in BC^{1,\alpha}(\mathbb{R})$, $\alpha \in (0,1)$
- $f_- \leqslant f(x_1) \leqslant f_+$
- $\Gamma = \{(x_1, f(x_1))\}$
- $\Omega = \{x : x_2 \leqslant f(x_1)\}$

where $x = (x_1, x_2) \in \mathbb{R}^2$.

Boundary Value Problem

Given $\phi_0 \in BC(\Gamma)$, find $\phi \in BC(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$\phi = \phi_0 \text{ on } \Gamma$$

and

$$\Delta\phi = 0 \text{ in } \Omega.$$

Solution

Fundamental Solution of Laplace's Equation in 2D

$$\Phi(x, y) \quad := \quad \frac{1}{2\pi} \ln \left| \frac{1}{x - y} \right|, \qquad x \neq y$$

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Double Layer Potential Solution

$$\phi(x) := \int_\Gamma \frac{\partial \Phi_H(x,y)}{\partial \mathbf{n}(y)}\mu(y)\mathrm{d}s, \qquad x \in \Omega$$

Older

- Closed Domains
- $C^{1,\alpha}$ Boundaries
- Jump Relations
- No Mapping

(Günter, Mikhlin, 1930s-)

Modern

- Closed Domains
- C^2 Boundaries
- Jump Relations
- Mapping $C^{0,\alpha}$ to $C^{1,\alpha}$

(Colton, Kress, 1980s-)

Helmholtz Equation

- Half-plane Domains
- $C^{1,\alpha}$ Boundaries
- Jump Relations
- No Mapping

(Chandler-Wilde, Ross, 1990s-)

Properties

We have shown that:

Decay

$$\left|\frac{\partial \Phi_H(x,y)}{\partial \mathbf{n}(y)}\right| \leqslant \frac{C}{|x-y|^2}, \qquad x \in \Omega_H, y \in \Gamma$$

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Boundedness

$$|\phi(x)| = \left|\int_\Gamma \frac{\partial \Phi_H(x,y)}{\partial \mathbf{n}(y)} \mu(y) \mathrm{d}s(y)\right| \leqslant C_\Gamma, \qquad x \in \Omega_H$$

Properties

Jump Relations

$$\phi_{\pm}(x) \quad := \quad \int_{\Gamma} \frac{\partial \Phi_H(x,y)}{\partial \mathbf{n}(y)} \mu(y) \mathrm{d}s(y) \pm \frac{1}{2}\mu(x), \qquad x \in \Gamma$$

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Continuous Normal Derivative

$$\mathbf{n}(x).\left(\nabla\phi(x+h\mathbf{n}(x)) - \nabla\phi(x-h\mathbf{n}(x))\right) \quad \to \quad 0, \qquad \text{as } h \to 0$$

Boundary Integral Equation

Define integral operator K by

$$(K\psi)(x) := 2\int_\Gamma \frac{\partial \Phi_H(x,y)}{\partial \mathbf{n}(y)}\psi(y)\mathrm{d}s(y), \qquad x \in \Gamma$$

for $\psi \in BC(\Gamma)$.

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To give the Boundary Integral Equation on Γ

$$(I - K)\mu = -2\phi_0.$$

Mapping Properties

We have also shown that:

$$
\begin{aligned}
K : BC(\Gamma) &\rightarrow BC^{0,\alpha}(\Gamma) \\
K : BC^{0,\beta}(\Gamma) &\rightarrow BC^{0,\gamma}(\Gamma), \qquad \gamma = \min\{\alpha + \beta, 1\} \\
K : BC^{0,\beta}(\Gamma) &\rightarrow BC^{1,\delta}(\Gamma), \qquad \delta = \alpha + \beta - 1 > 0
\end{aligned}
$$

where $\beta \in (0, 1)$.

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where $\beta \in (0,1)$.
If $\mu \in BC^{1,\delta}(\Gamma)$ then $\phi \in BC^{1,\delta}(\Omega)$.

Unbounded Maximum-Minimum Principle

If ϕ is harmonic in Ω then

$$\sup_{x \in \Omega} \phi(x) \quad \leqslant \quad \sup_{x \in \Gamma} \phi(x).$$

Hence the BVP has at most one solution.

Injectivity of the BIE

$(I - K)\mu = 0$

- Equivalence, BVP uniqueness $\Rightarrow \phi_- = 0$
- Continuous Normal Derivative $\Rightarrow \frac{\partial \phi}{\partial \mathbf{n}} = 0$ on Γ
- Mapping, $K^n \mu = \mu \Rightarrow \mu \in C^{1,\delta}$
- Solve mixed BVP in infinite strip domain $\Rightarrow \phi_+ = 0$
- Jump relations $\Rightarrow \mu = 0$

Hence $(I - K)$ is injective.

The Well-posedness

Arens, et al [2003], showed that an injective integral equation (over the real line) is also surjective.

Hence Dirichlet boundary value problem has exactly one solution.

Numerical Results

Outline

- Logarithmic singularity outside of Ω
- Determine ϕ_0
- Determine exact test point value
- Nyström method
- Solve IE for μ, using LU decomposition
- Use μ to calculate value at test point

Numerical Results

Parameters

- Truncated to n periods of 2π, $n = 1, 2, 3, 4, 5$
- p points per period, $p = 16, 32, 64, 128, 256$
- Flat boundary

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Expected Results

- Meier, et al [2000-01]
- Smooth surfaces - superalgebraic convergence

Boundary: Flat

Source: (0,0.5)

Test: (0,-1)

Error

	16	32	64	128	256
1	5.14126e-005	5.9323e-005	5.87816e-005	5.86455e-005	5.86114e-005
2	-9.89577e-006	1.31289e-007	1.31915e-007	1.31679e-007	1.3162e-007
3	-1.00287e-005	-1.1045e-009	4.58856e-010	4.58121e-010	4.57937e-010
4	-1.00284e-005	-1.56421e-009	1.94447e-012	1.94164e-012	1.94077e-012
5	-1.00281e-005	-1.5661e-009	9.08995e-015	9.14546e-015	9.38138e-015

Relative Error

	16	32	64	128	256
1	0.000632378	0.000729676	0.000723017	0.000721343	0.000720924
2	-0.000121719	1.61486e-006	1.62256e-006	1.61966e-006	1.61894e-006
3	-0.000123354	-1.35855e-008	5.64395e-009	5.63491e-009	5.63265e-009
4	-0.00012335	-1.92398e-008	2.39171e-011	2.38823e-011	2.38715e-011
5	-0.000123347	-1.92632e-008	1.11807e-013	1.1249e-013	1.15392e-013

Boundary: Flat

Source: (0.2,0.5)

Test: (-0.15,-0.75)

Error

	16	32	64	128	256
1	5.50577e-005	4.72648e-005	4.68266e-005	4.67165e-005	4.66889e-005
2	6.13642e-006	1.03105e-007	1.03587e-007	1.03402e-007	1.03356e-007
3	6.02715e-006	-8.5953e-010	3.59811e-010	3.59235e-010	3.59091e-010
4	6.02618e-006	-1.22001e-009	1.52474e-012	1.5226e-012	1.52203e-012
5	6.02596e-006	-1.2215e-009	7.10543e-015	7.09155e-015	7.25808e-015

Relative Error

	16	32	64	128	256
1	0.000615545	0.00052842	0.000523522	0.00052229	0.000521982
2	6.86052e-005	1.15271e-006	1.1581e-006	1.15603e-006	1.15551e-006
3	6.73835e-005	-9.60955e-009	4.02269e-009	4.01625e-009	4.01464e-009
4	6.73727e-005	-1.36397e-008	1.70466e-011	1.70227e-011	1.70163e-011
5	6.73703e-005	-1.36564e-008	7.94387e-014	7.92835e-014	8.11454e-014

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Future work

- Introduce time-stepping...