

# A high frequency boundary element method for scattering by convex polygons with impedance boundary conditions

S. N. Chandler-Wilde<sup>1</sup>, S. Langdon<sup>1,\*</sup>, and M. Mokgolele<sup>1</sup>

<sup>1</sup> *Department of Mathematics, University of Reading, Whiteknights PO Box 220, Reading RG6 6AX, U.K.*

---

**Abstract.** We consider scattering of a time harmonic incident plane wave by a convex polygon with piecewise constant impedance boundary conditions. Standard finite or boundary element methods require the number of degrees of freedom to grow at least linearly with respect to the frequency of the incident wave in order to maintain accuracy. Extending earlier work by Chandler-Wilde and Langdon for the sound soft problem, we propose a novel Galerkin boundary element method, with the approximation space consisting of the products of plane waves with piecewise polynomials supported on a graded mesh with smaller elements closer to the corners of the polygon. Theoretical analysis and numerical results suggest that the number of degrees of freedom required to achieve a prescribed level of accuracy grows only logarithmically with respect to the frequency of the incident wave.

**AMS subject classifications:** 35J05, 65N38, 65R20

**Key words:** boundary integral equation method, high frequency scattering, convex polygons, impedance boundary conditions

---

## 1 Introduction

In this paper we consider two-dimensional scattering of a time-harmonic incident plane wave  $u^i(\mathbf{x}) = e^{ik\mathbf{x}\cdot\mathbf{d}}$ ,  $\mathbf{x} \in \mathbb{R}^2$ , where the unit vector  $\mathbf{d}$  is the direction of propagation and  $k > 0$  is the wavenumber of the incident wave, by a bounded Lipschitz domain  $\Omega \subset \mathbb{R}^2$ , with impedance boundary conditions holding on  $\Gamma := \partial\Omega$ . Define  $D := \mathbb{R}^2 \setminus \bar{\Omega}$  to be the unbounded domain exterior to  $\Omega$ , let  $\gamma^+ : H^1(D) \rightarrow H^{1/2}(\Gamma)$  and  $\gamma^- : H^1(\Omega) \rightarrow H^{1/2}(\Gamma)$  denote the exterior and interior trace operators, respectively, and, where  $H^1(G, \Delta) := \{v \in H^1(G) : \Delta v \in L^2(G)\}$ , let  $\partial_{\mathbf{n}}^+ : H^1(D, \Delta) \rightarrow H^{-1/2}(\Gamma)$  and  $\partial_{\mathbf{n}}^- : H^1(\Omega, \Delta) \rightarrow H^{-1/2}(\Gamma)$  denote the exterior and interior normal derivative operators, respectively. (All of  $\gamma^\pm$  and  $\partial_{\mathbf{n}}^\pm$  are

---

\*Corresponding author. *Email addresses:* s.n.chandler-wilde@reading.ac.uk (S. N. Chandler-Wilde), s.langdon@reading.ac.uk (S. Langdon), m.mokgolele@yahoo.co.uk (M. Mokgolele)

well-defined as bounded linear operators, see [15], where also our various function space notations are defined.) Then the scattering problem we consider is: given  $\beta \in L^\infty(\Gamma)$ , find the total field  $u^t \in C^2(D) \cap H_{\text{loc}}^1(D)$  such that

$$\Delta u^t + k^2 u^t = 0 \quad \text{in } D, \quad (1.1)$$

$$\partial_{\mathbf{n}}^+ u^t + ik\beta\gamma^+ u^t = 0 \quad \text{on } \Gamma, \quad (1.2)$$

and such that the scattered field  $u^s := u^t - u^i$  satisfies the Sommerfeld radiation condition

$$\frac{\partial u^s}{\partial r}(\mathbf{x}) - ik u^s(\mathbf{x}) = o\left(r^{-1/2}\right), \quad (1.3)$$

as  $r := |\mathbf{x}| \rightarrow \infty$ , uniformly with respect to  $\mathbf{x}/|\mathbf{x}|$ .

It is a standard result that this boundary value problem is uniquely solvable if  $\text{Re}\beta \geq 0$ , which physically is a condition that the impedance boundary does not emit energy. (See [9] for a proof in the case that  $\Gamma$  is  $C^2$ , and [15, Lemma 9.9, Exercise 9.5] for the main ideas for the extension to the case of Lipschitz  $\Gamma$ .) Our concern in this paper is to develop a novel and very effective high frequency boundary element method for the particular case when  $\Omega$  is a convex polygon and  $\beta$  is constant on each side of  $\Gamma$ , corresponding to an obstacle made up from several, homogeneous, materials, each with a different relative surface admittance.

The problem (1.1)–(1.3) has received significant recent attention in the literature [1, 18, 21]. Standard boundary or finite element approximations suffer from the requirement that the number of degrees of freedom must increase at least linearly with respect to  $k$  in order to maintain accuracy. Although asymptotic schemes can provide reasonable approximations when  $k$  is very large [12], there thus exists a wide range of frequencies for which numerical schemes are prohibitively expensive whilst asymptotic approaches are insufficiently accurate. This difficulty for scattering problems has been well documented in the literature in recent years, and numerous novel approaches to reducing the computational cost for moderate to large  $k$  have been proposed. The scattering problem that has received the most attention in the literature is the sound soft problem, i.e. (1.1) and (1.3) with the boundary condition  $\gamma^+ u^t = 0$  on  $\Gamma$  replacing (1.2). Using a boundary element approach, with a hybrid approximation space consisting of the product of plane waves with piecewise polynomials, very efficient schemes have been developed for scattering by smooth obstacles [10] and by convex polygons [6], with in each case the number of degrees of freedom required to achieve a prescribed level of accuracy depending only very mildly on  $k$ . In contrast, the only numerical scheme for (1.1)–(1.3) that we are aware of that has been developed specifically for the purpose of efficiency at high frequencies is that in [18], where a circle of piecewise constant impedance is considered. There the approximation space is enriched with plane waves travelling in multiple directions; this reduces the number of degrees of freedom required per wavelength from ten to three (for engineering accuracy), but the cost still grows linearly with  $k$ . For a full review of recent developments in integral equation methods for high frequency scattering we refer to [3].

The approach we will take in this paper combines ideas from our earlier work for sound soft convex polygons [6] with ideas developed for solving a two-dimensional problem of high frequency scattering by an inhomogeneous half-plane of piecewise constant impedance [7, 14]. In [7] a method in the spirit of the geometrical theory of diffraction was applied to obtain a representation of the solution, with the known leading order behaviour being subtracted off, leaving only the remaining scattered field due to the discontinuities in the impedance boundary conditions to be approximated. This diffracted field was expressed as a product of oscillatory and non-oscillatory functions, with a rigorous error analysis, supported by numerical experiments, demonstrating that the number of degrees of freedom required to maintain accuracy as  $k \rightarrow \infty$  grows only logarithmically with respect to  $k$ . This approach was improved in [14], where derivation of sharper regularity estimates regarding the rate of decay of the scattered field away from impedance discontinuities led to error estimates independent of  $k$ .

The plan of this paper is as follows. In §2 we derive regularity results, demonstrating in particular that  $\gamma^+ u^t$  can be written as the known leading order physical optics solution plus the products of plane waves with unknown functions that are non-oscillatory, highly peaked near the corners of the polygon and rapidly decaying away from the corners. In §3, we discuss the boundary integral equation formulation of (1.1)–(1.3). We describe our approximation space and Galerkin boundary element method in §4, and present numerical results demonstrating the efficiency of our scheme at high frequencies in §5. Finally, in §6 we present some conclusions.

## 2 Regularity results

Our aim in this section is to investigate the regularity of  $u^t$ , deriving bounds on derivatives which are sufficiently explicit, in particular in their dependence on the wavenumber, so that we can prove the effectiveness of our novel boundary element approximation space. In this endeavour we will, as part of our arguments, relate all bounds on derivatives to

$$M := \sup_{x \in D} |u^t(x)|. \quad (2.1)$$

We note first of all that  $\gamma^+ u^t \in H^{1/2}(\Gamma) \subset L^2(\Gamma)$  so that the impedance boundary condition (1.2) implies that  $\partial_n^+ u^t \in L^2(\Gamma)$ . It follows from standard regularity results for elliptic problems in Lipschitz domains [15, Theorem 4.24] that  $\gamma^+ u^t \in H^1(\Gamma)$  and thus, from Theorem 6.12 and the accompanying discussion in [15], that  $u^t \in H_{\text{loc}}^{3/2}(D)$ , so that, by standard Sobolev imbedding theorems [15],  $u^t \in C(\bar{D})$  (as a consequence of which  $M < \infty$ ).

From this point on in the paper we restrict attention to the case shown in figure 1 where  $\Omega$  is a convex polygon. We write the boundary of the polygon as  $\Gamma = \cup_{j=1}^n \Gamma_j$ , where  $\Gamma_j$ ,  $j = 1, \dots, n$ , are the  $n$  sides of the polygon, with  $j$  increasing anticlockwise as shown in figure 1. We denote the corners of the polygon by  $P_j$ ,  $j = 1, \dots, n$ , and we set  $P_{n+1} = P_1$ , so that, for  $j = 1, \dots, n$ ,  $\Gamma_j$  is the line joining  $P_j$  with  $P_{j+1}$ . We denote the length

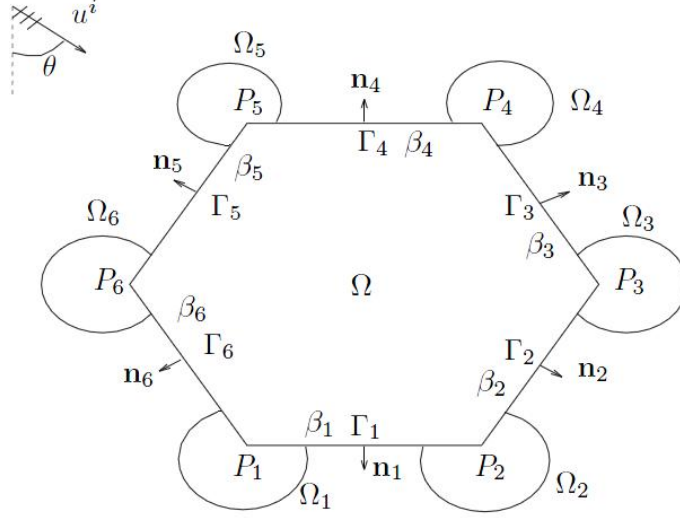


Figure 1: Notation for scattering by an impedance polygon

of  $\Gamma_j$  by  $L_j = |P_{j+1} - P_j|$ , the external angle at vertex  $P_j$  by  $\Omega_j \in (\pi, 2\pi)$ , and the outwards unit normal vector to  $\Gamma_j$  by  $\mathbf{n}_j$ . We let  $\theta \in [0, 2\pi)$  denote the angle of the incident plane wave direction  $\mathbf{d}$ , as measured anticlockwise from the downward vertical  $(0, -1)$ . We also assume from this point on that  $\beta$  takes a constant value on each side of the polygon  $\Gamma_j$ ; that is,  $\beta(\mathbf{x}) = \beta_j$ ,  $\mathbf{x} \in \Gamma_j$ ,  $j = 1, \dots, n$ , where  $\beta_j \in \mathbb{C}$ ,  $\text{Re}(\beta_j) > 0$ ,  $j = 1, \dots, n$ .

The following lemma is our first tool to derive explicit regularity estimates. In this lemma we use the notations  $B_\delta := \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| < \delta\}$ ,  $S_\delta := \{\mathbf{x} \in B_\delta : x_2 > 0\}$ , and  $\gamma_\delta := \{\mathbf{x} = (x_1, 0) : |x_1| < \delta\}$ , for  $\delta > 0$ .

**Lemma 2.1.** *Suppose that  $\delta > 0$  and  $N > 0$ , and that  $u \in H^1(S_\delta) \cap C^2(S_\delta) \cap C(\overline{S_\delta})$ ,  $m \in C(\overline{S_\delta})$ , and  $\alpha \in C^{0,\varepsilon}(\gamma_\delta)$ , for some  $\varepsilon \in (0, 1)$ , with  $\|m\|_\infty \leq N$  and  $\|\alpha\|_{C^{0,\varepsilon}(\gamma_\delta)} \leq N$ . Suppose also that  $\Delta u + mu = 0$  in  $S_\delta$  and  $\frac{\partial u}{\partial x_2} + i\alpha u = 0$  on  $\gamma_\delta$ . Then  $u \in C^1(S_\delta \cup \gamma_\delta)$  and, for some constant  $C > 0$  depending only on  $\delta$ ,  $N$ , and  $\varepsilon$ ,*

$$|\nabla u(\mathbf{x})| \leq C \sup_{\mathbf{y} \in S_\delta} |u(\mathbf{y})|, \quad \mathbf{x} \in S_{\delta/2}. \quad (2.2)$$

*Proof.* Let  $G(\mathbf{x}, \mathbf{y})$  be the Dirichlet Green's function for the Laplace operator for the ball  $B_\delta$ , given explicitly for example in [19, p.107], and let  $G^*(\mathbf{x}, \mathbf{y}) := G(\mathbf{x}, \mathbf{y}) + G(\mathbf{x}', \mathbf{y})$ , where, for  $\mathbf{x} = (x_1, x_2) \in B_\delta$ ,  $\mathbf{x}' := (x_1, -x_2)$ . Then, by applications of Green's second theorem to  $u$  and  $G^*(\mathbf{x}, \cdot)$ , and noting that  $G^*(\mathbf{x}, \cdot) = 0$  on  $\partial B_\delta$  and that  $\frac{\partial G^*(\mathbf{x}, \mathbf{y})}{\partial y_2} = 0$  and  $\frac{\partial u}{\partial x_2} + i\alpha u = 0$  on  $\gamma_\delta$ , one obtains that, for  $\mathbf{x} \in S_\delta$ , where  $\Gamma_\delta := \partial B_\delta \cap \partial S_\delta$ ,

$$u(\mathbf{x}) = i \int_{\gamma_\delta} G^*(\mathbf{x}, \mathbf{y}) \alpha(\mathbf{y}) u(\mathbf{y}) ds(\mathbf{y}) - \int_{\Gamma_\delta} \frac{\partial G^*(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} u(\mathbf{y}) ds(\mathbf{y}) + \int_{S_\delta} G^*(\mathbf{x}, \mathbf{y}) m(\mathbf{y}) u(\mathbf{y}) dy. \quad (2.3)$$

Taking the limit as  $\mathbf{x} \rightarrow \gamma_\delta$  we see that this equation holds also for  $\mathbf{x} \in \gamma_\delta$ . Now the second and third integrals in (2.3) are continuously differentiable in  $S_\delta \cup \gamma_\delta$ , with gradient whose magnitude is  $\leq CM^*$  in  $\overline{S_{3\delta/4}}$ , where  $M^* = \sup_{\mathbf{y} \in S_\delta} |u(\mathbf{y})|$  and  $C$  depends only on  $N$  and  $\delta$ . It follows from (2.3) (with  $\mathbf{x} \in \gamma_\delta$ ) and mapping properties of the single-layer potential (e.g. [9]) that  $u \in C^{0,\varepsilon}(\gamma_{p\delta})$ , for every  $p \in (0,1)$ , with  $\|u\|_{C^{0,\varepsilon}(\gamma_{3\delta/4})} \leq CM^*$ , where  $C$  here depends on all of  $\varepsilon$ ,  $N$ , and  $\delta$ . Again using (2.3) and standard mapping properties of the single-layer potential [9], we see that  $u \in C^1(S_\delta \cup \gamma_\delta)$  and that the bound (2.2) holds.  $\square$

Our first bounds on derivatives of  $u^t$  will be bounds on  $\nabla u^t$  in  $D$ . We note first of all that it follows from standard interior elliptic regularity estimates [11, Theorem 3.9, Lemma 4.1] that there exists an absolute constant  $C > 0$  such that, for every  $\varepsilon > 0$ ,

$$|\nabla u^t(\mathbf{x})| \leq C\varepsilon^{-1}(1+(k\varepsilon)^2)M \quad (2.4)$$

if  $\mathbf{x} \in D$  and the distance of  $\mathbf{x}$  from  $\Gamma$ ,  $\text{dist}(\mathbf{x}, \Gamma) > \varepsilon$ . Using Lemma 2.1 applied to  $u^t$  in domains  $\{\mathbf{x} \in D : |\mathbf{x} - \mathbf{x}^*| < 2c/k\}$  with  $\mathbf{x}^* \in \Gamma$ ,  $c > 0$ , and  $\text{dist}(\mathbf{x}^*, \{P_1, \dots, P_n\}) > 2c/k$ , we can extend this bound up to  $\Gamma$ , excluding the corner points  $P_j$ . Precisely, if  $|\beta_j| \leq B$ , for  $j = 1, \dots, n$ , then it follows from (2.4) and Lemma 2.1 that there exists  $C > 0$ , depending only on  $B$  and  $c$ , such that, if  $\mathbf{x} \in \bar{D}$  and  $\text{dist}(\mathbf{x}, \{P_1, \dots, P_n\}) > c/k$ , then

$$|\nabla u^t(\mathbf{x})| \leq CkM. \quad (2.5)$$

To obtain bounds which apply throughout  $D$  we make a further application of Lemma 2.1. Suppose  $\omega > 0$  is such that  $\pi + \omega < \Omega_j < 2\pi$ ,  $j = 1, \dots, n$ . Let us use that  $\mathbb{R}^2$  is isomorphic to the complex plane  $\mathbb{C}$ ; for  $\mathbf{x} = (x_1, x_2) \in D$  let  $\tilde{\mathbf{x}} := x_1 + ix_2 \in \mathbb{C}$  and, conversely, if  $z \in \mathbb{C}$ , let  $\hat{z} = (\text{Re}z, \text{Im}z) \in \mathbb{R}^2$ . In terms of this notation, set  $P_0 := P_n$  and let  $z_j := \tilde{P}_j$ ,  $j = 0, 1, \dots, n+1$ . Given a typical corner  $P_j$ , with  $1 \leq j \leq n$ , let  $c > 0$  be such that  $\text{dist}(P_j, \{P_{j+1}, P_{j-1}\}) > 2c/k$ . Let  $\theta_i := \arg(z_{i-1} - z_i)$ , for  $i = 1, \dots, n+1$ , so that  $\theta_j + \Omega_j = \arg(z_{j+1} - z_j)$ . Our application of Lemma 2.1 is to  $u \in H^1(S_{2c}) \cap C^2(S_{2c}) \cap C(\overline{S_{2c}})$  given by  $u(\mathbf{y}) = u^t(g(\tilde{\mathbf{y}}))$ ,  $\mathbf{y} \in S_{2c}$ , where the conformal mapping  $g$  is given by  $g(w) = z_j + (e^{i\theta_j} 2c/k)(w/(2c))^{\Omega_j/\pi}$ , with  $\arg(g(w) - z_j) \in [\theta_j, \theta_j + \Omega_j]$ . This function  $u$  satisfies the conditions of Lemma 2.1 with  $\delta = 2c$  and  $\varepsilon = \omega/\pi$ , and with  $m(\mathbf{y}) = k^2 |g'(\tilde{\mathbf{y}})|^2 = (\Omega_j/\pi)^2 |\mathbf{y}/(2c)|^{2(\Omega_j/\pi-1)}$ ,  $\mathbf{y} \in S_{2c}$ , and  $\alpha(\mathbf{y}) = k |g'(\tilde{\mathbf{y}})| \beta_j^*(y_1) = (\Omega_j/\pi) |y_1/(2c)|^{\Omega_j/\pi-1} \beta_j^*(y_1)$ ,  $\mathbf{y} \in \gamma_{2c}$ , where  $\beta_j^*(y_1) := \beta_{j-1}$ , for  $y_1 > 0$ ,  $:= \beta_j$ , for  $y_1 < 0$ . Thus, applying Lemma 2.1, it follows that there exists  $C > 0$  depending only on  $B$ ,  $c$ , and  $\omega$ , such that  $|\nabla u(\mathbf{y})| \leq CM$ , for  $\mathbf{y} \in S_c$ . Since, within distance  $2c/k$  of  $P_j$ ,  $|\nabla u^t(\mathbf{x})| = |f'(\tilde{\mathbf{x}})| |\nabla u(f(\tilde{\mathbf{x}}))| = k(k|\mathbf{x} - P_j|/(2c))^{\pi/\Omega_j-1} |\nabla u(f(\tilde{\mathbf{x}}))|$ , where  $f(z) := g^{-1}(z) = 2c[(z - z_j)ke^{-i\theta_j}/(2c)]^{\pi/\Omega_j}$ , we see that, if  $\mathbf{x} \in \bar{D}$  and  $|\mathbf{x} - P_j| < c/k$ , then

$$|\nabla u^t(\mathbf{x})| \leq Ck(k|\mathbf{x} - P_j|)^{\pi/\Omega_j-1} M, \quad (2.6)$$

where  $C$  depends only on  $B$ ,  $c$ , and  $\omega$ .

This analysis, leading to (2.6), applies in particular in the special case  $\beta = 0$  of Neumann boundary conditions on  $\Gamma$  when the behaviour (2.6) agrees with that predicted by

a separation of variables solution in polar coordinates local to the corner  $P_j$  (cf. [6, Theorem 2.3]). The bound (2.6) is also consistent with the well-known Malyuzhinets solution (see [17] and the references therein) for scattering by a wedge with impedance boundary conditions.

In order to deduce more detailed regularity estimates we combine ideas from [6] (for the related sound soft problem) and [7, 14] (for an impedance half-plane problem). These more detailed estimates are bounds on derivatives of all orders related to the trace of the total field  $\gamma^+ u^t$ , relevant to the analysis of boundary element methods based on a direct integral equation formulation obtained from Green's theorem (see §3 below).

For  $j = 1, \dots, n$ , let  $D_j$  denote the half-plane to one side of  $\Gamma_j$  given by  $D_j := \{\mathbf{x} \in \mathbb{R}^2 : (\mathbf{x} - P_j) \cdot \mathbf{n}_j > 0\} \subset D$ . The boundary  $\partial D_j$  of this half-plane consists of the line segment  $\Gamma_j$  and two semi-infinite line segments  $\Gamma_j^+$  and  $\Gamma_j^-$  to either side of  $\Gamma_j$ , as shown in figure 2. Let

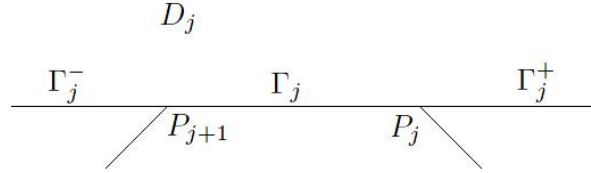


Figure 2: The half-plane  $D_j$  and its boundary.

$$\Phi(\mathbf{x}, \mathbf{y}) := \frac{i}{4} H_0^{(1)}(k|\mathbf{x} - \mathbf{y}|), \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^2, \quad \mathbf{x} \neq \mathbf{y}, \quad (2.7)$$

with  $H_0^{(1)}$  the Hankel function of the first kind of order zero, so that  $\Phi$  is the standard fundamental solution of the Helmholtz equation. Let  $G_j(\mathbf{x}, \mathbf{y})$  denote the impedance Green's function for the half plane  $D_j$  for impedance  $\beta_j$ , given explicitly by

$$G_j(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x}, \mathbf{y}) + \Phi(\mathbf{x}, \mathbf{y}') + P_{\beta_j}(k(\mathbf{x} - \mathbf{y}') \cdot \mathbf{s}_j, k(\mathbf{x} - \mathbf{y}') \cdot \mathbf{n}_j), \quad \text{for } \mathbf{x}, \mathbf{y} \in \overline{D_j}.$$

Here  $\mathbf{y}'$  denotes the image of  $\mathbf{y}$  on reflection in the line  $\partial D_j$ ,  $\mathbf{s}_j$  is a unit vector in the direction  $P_{j+1} - P_j$ , parallel to  $\Gamma_j$ , and, where  $U := \mathbb{R} \times [0, \infty)$  is the upper half-plane,  $P_{\beta^*} \in C(\overline{U}) \cap C^\infty(\overline{U} \setminus \{(0, 0)\})$  is defined (see e.g. [4, 5]), for  $\text{Re} \beta^* > 0$ , by

$$P_{\beta^*}(\xi, \eta) := -\frac{i\beta^*}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(i(\eta(1-s^2)^{1/2} - \xi s))}{(1-s^2)^{1/2}((1-s^2)^{1/2} + \beta^*)} ds, \quad \text{for } \xi \in \mathbb{R}, \eta \geq 0.$$

By applications of Green's theorems, utilising that both  $u^s$  and  $G_j(\mathbf{x}, \cdot)$  satisfy the Sommerfeld radiation conditions (see (1.3) and [4, (7)], respectively), and using the impedance boundary condition satisfied by  $G_j$ , that  $\frac{\partial G_j(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} + ik\beta_j G_j(\mathbf{x}, \mathbf{y}) = 0$  on  $\partial D_j$ , it can be deduced (see [16, Theorem 4.3] for details) that

$$u^s(\mathbf{x}) = - \int_{\partial D_j} G_j(\mathbf{x}, \mathbf{y}) \left( \frac{\partial u^s}{\partial \mathbf{n}}(\mathbf{y}) + ik\beta_j u^s(\mathbf{y}) \right) ds(\mathbf{y}), \quad \mathbf{x} \in D_j. \quad (2.8)$$

We note that the integral on the right-hand side is well-defined since  $u^t$  is continuous and bounded in  $\bar{D}$  and  $\nabla u^t$  satisfies the bounds (2.5) and (2.6), while  $G_j(\mathbf{x}, \cdot)$  decreases sufficiently rapidly at infinity [7, (2.10)] so that it is absolutely integrable on  $\partial D_j$ . We note also that, since the left and right hand sides of (2.8) are both continuous in  $\bar{D}_j$ , (2.8) holds in fact for all  $\mathbf{x} \in \bar{D}_j$ .

In the case that  $\Gamma_j$  is not illuminated by the incident wave (by this we mean the case that  $\mathbf{d} \cdot \mathbf{n}_j \geq 0$ ), it can be shown that (2.8) holds for  $\mathbf{x} \in \bar{D}_j$  also with  $u^s$  replaced by  $u^i$ . (The point is that (see [8, Remark 2.15, Theorem 2.19(ii)]), in the case  $\mathbf{d} \cdot \mathbf{n}_j \geq 0$ ,  $u^i$  can be approximated in  $\bar{D}_j$  by a bounded sequence of solutions of the Helmholtz equation which satisfy the Sommerfeld radiation condition and which converge uniformly on compact subsets of  $\bar{D}_j$  to  $u^i$ , so that (2.8) holds first for each member of this sequence and then, in the limit, also for  $u^i$ .) Adding the equations (2.8) satisfied by  $u^s$  and  $u^i$ , we see that

$$u^t(\mathbf{x}) = - \int_{\Gamma_j^- \cup \Gamma_j^+} G_j(\mathbf{x}, \mathbf{y}) \left( \frac{\partial u^t}{\partial \mathbf{n}}(\mathbf{y}) + ik\beta_j u^t(\mathbf{y}) \right) ds(\mathbf{y}), \quad \mathbf{x} \in \bar{D}_j, \quad (2.9)$$

if  $\Gamma_j$  is a shadow side, since  $u^t = u^i + u^s$  and  $\frac{\partial u^t}{\partial \mathbf{n}} + ik\beta_j u^t = 0$  on  $\Gamma_j$ . On illuminated sides (where  $\mathbf{d} \cdot \mathbf{n}_j < 0$ ), (2.8) still holds for  $u^s$  and we can follow the argument of [7, p.653] to deduce that

$$u^t(\mathbf{x}) = u_j^t(\mathbf{x}) - \int_{\Gamma_j^- \cup \Gamma_j^+} G_j(\mathbf{x}, \mathbf{y}) \left( \frac{\partial u^t}{\partial \mathbf{n}}(\mathbf{y}) + ik\beta_j u^t(\mathbf{y}) \right) ds(\mathbf{y}), \quad \mathbf{x} \in \bar{D}_j, \quad (2.10)$$

where  $u_j^t(\mathbf{x}) := u^i(\mathbf{x}) + R_{\beta_j}(\theta - \theta_{j+1})u^r(\mathbf{x})$ ,  $u^r$  is the plane wave  $u^r(\mathbf{x}) := \exp(ik(c_j + \mathbf{x} \cdot \mathbf{d}'_j))$ , with  $\mathbf{d}'_j := \mathbf{d} \cdot \mathbf{s}_j \mathbf{s}_j - \mathbf{d} \cdot \mathbf{n}_j \mathbf{n}_j$  and  $c_j := P_j \cdot (\mathbf{d} - \mathbf{d}'_j)$ , and  $R_{\beta_j}$  is the plane wave reflection coefficient for a plane of impedance  $\beta_j$ , defined by  $R_{\beta_j}(\theta^*) := (\cos \theta^* - \beta_j) / (\cos \theta^* + \beta_j)$ .

We now define  $\tilde{g}(\mathbf{x}) := \frac{\partial u^t}{\partial \mathbf{n}}(\mathbf{x}) + ik\beta_j u^t(\mathbf{x})$ ,  $\mathbf{x} \in \partial D_j$ , noting that  $\tilde{g}(\mathbf{x}) = 0$  for  $\mathbf{x} \in \Gamma_j$ . To write (2.10) and (2.9) more explicitly we introduce new dimensionless functions whose arguments can be interpreted as arc-length along  $\partial D_j$ . These functions are defined by  $\phi_j(s) := u^t(P_j + s\mathbf{s}_j)$  and  $g_j(s) := \tilde{g}(P_j + s\mathbf{s}_j)$ , for  $s \in \mathbb{R}$ , and we also define  $\psi_j(s) := u_j^t(P_j + s\mathbf{s}_j)$ , if  $\Gamma_j$  is a side illuminated by the incident field ( $\mathbf{d} \cdot \mathbf{n}_j < 0$ ), and set  $\psi_j(s) := 0$  otherwise. Also, let  $\kappa_j(t) := \frac{i}{2} H_0^{(1)}(|t|) + P_{\beta_j}(|t|, 0)$ , for  $t \in \mathbb{R} \setminus \{0\}$ , noting that  $G_j(\mathbf{x}, \mathbf{y}) = \kappa_j(k|\mathbf{x} - \mathbf{y}|)$  if  $\mathbf{x}, \mathbf{y} \in \partial D_j$  with  $\mathbf{x} \neq \mathbf{y}$ . Then equations (2.10) and (2.9), restricted to  $\Gamma_j$ , can be written as the statement that, for  $0 < s < L_j$ ,

$$\begin{aligned} \phi_j(s) &= \psi_j(s) - \int_{-\infty}^0 \kappa_j(k(s-t))g_j(t)dt - \int_{L_j}^{\infty} \kappa_j(k(s-t))g_j(t)dt \\ &= \psi_j(s) + e^{iks}v_j^+(s) + e^{-iks}v_j^-(L_j-s), \end{aligned} \quad (2.11)$$

where, defining  $\check{\kappa}_j(s) := e^{-i|s|}\kappa_j(s)$ ,  $s \in \mathbb{R} \setminus \{0\}$ , the functions  $v_j^+$  and  $v_j^-$  are defined by

$$v_j^+(s) := - \int_{-\infty}^0 \check{\kappa}_j(k(s-t))e^{-ikt}g_j(t)dt, \quad s \geq 0,$$

and

$$v_j^-(s) := - \int_{L_j}^{\infty} \check{\kappa}_j(k(t-L_j+s)) e^{ikt} g_j(t) dt = - e^{ikL_j} \int_{-\infty}^0 \check{\kappa}_j(k(s-t)) e^{-ikt} g_j(L_j-t) dt, \quad s \geq 0.$$

In the next theorem we establish that the functions  $v_j^\pm$  are at most slowly oscillatory, in that their derivatives are rapidly decaying at infinity.

**Theorem 2.1.** *Suppose that, for some  $B > 0$ ,  $c > 0$ , and  $\omega \in (0, \pi)$ , it holds that*

$$\operatorname{Re} \beta_j \geq B^{-1}, \quad |\beta_j| \leq B, \quad \pi + \omega \leq \Omega_j < 2\pi, \quad kL_j \geq 2c, \quad \text{for } j = 1, \dots, n, \quad (2.12)$$

and set  $\alpha_j^+ := \pi/\Omega_j$ , for  $j = 1, \dots, n$ , and  $\alpha_j^- := \alpha_{j+1}^+$ ,  $j = 1, \dots, n-1$ ,  $\alpha_n^- := \alpha_1^+$ . Then there exist constants  $C_m$ ,  $m = 0, 1, \dots$ , dependent only on  $m$ ,  $B$ ,  $\omega$ , and  $c$ , such that, for  $j = 1, \dots, n$  and  $m \in \mathbb{N}$ ,

$$|v_j^\pm(s)| \leq \begin{cases} C_0 M, & 0 < ks \leq 1, \\ C_0 M (ks)^{-1/2}, & ks \geq 1, \end{cases} \quad |v_j^{\pm(m)}(s)| \leq \begin{cases} C_m k^m M (ks)^{\alpha_j^\pm - m}, & 0 < ks \leq 1, \\ C_m k^m M (ks)^{-1/2 - m}, & ks \geq 1. \end{cases} \quad (2.13)$$

*Proof.* We shall prove only the bounds on  $v_j^{+(m)}$ ; the bounds on  $v_j^{- (m)}$  follow analogously. Throughout the proof let  $C_m$  denote a positive constant, depending only on  $m$ ,  $B$ ,  $\omega$ , and  $c$ , not necessarily the same at each occurrence.

For  $s \geq 0$  and  $m = 0, 1, \dots$ ,

$$v_j^{+(m)}(s) = -k^m \int_{-\infty}^0 \check{\kappa}_j^{(m)}(k(s-t)) e^{-ikt} g_j(t) dt.$$

From (2.5) and (2.6), respectively, it follows that  $|g_j(t)| \leq CkM$  if  $t \leq -c/k$ , while  $|g_j(t)| \leq Ck(k|t|)^{\alpha_j^+ - 1}M$  if  $-c/k < t < 0$ , where the constant  $C > 0$  depends only on  $B$ ,  $c$ , and  $\omega$ . Thus it is easily seen that

$$|v_j^{+(m)}(s)| \leq Ck^m M [A_1(ks) + A_2(ks)], \quad s \geq 0,$$

where, for  $\sigma \geq 0$ ,

$$A_1(\sigma) := \int_{-\infty}^{-c} |\check{\kappa}_j^{(m)}(\sigma - \tau)| d\tau, \quad A_2(\sigma) := \int_{-c}^0 |\check{\kappa}_j^{(m)}(\sigma - \tau)| |\tau|^{\alpha_j^+ - 1} d\tau.$$

It is shown in [14, lemma 2.2] that  $|\check{\kappa}_j(t)| \leq C_0(1 + |\log t|)$ , for  $0 < t \leq 1$ , while  $|\check{\kappa}_j(t)| \leq C_0 t^{-3/2}$  for  $t > 1$ . Using these bounds it is easily seen that, in the case  $m = 0$ ,  $A_1(\sigma) + A_2(\sigma) \leq C_0$ , for  $0 \leq \sigma \leq 1$ ,  $\leq C_0 \sigma^{-1/2}$  for  $\sigma > 1$ . From this bound the bound (2.13) on  $|v_j^+|$  follows.

To bound  $A_1$  and  $A_2$  when  $m \in \mathbb{N}$  we use the bounds from [14, lemma 2.2] that  $|\check{\kappa}_j^{(m)}(t)| \leq C_m t^{-m}$ , for  $0 < t \leq 1$ , while  $|\check{\kappa}_j^{(m)}(t)| \leq C_m t^{-3/2 - m}$  for  $t > 1$ . Using these bounds

it is easily seen that  $A_1(\sigma) \leq C_m(c+\sigma)^{-1/2-m}$ , for  $\sigma \geq 0$ , and that  $A_2(\sigma) \leq C_m\sigma^{-1/2-m}$  for  $\sigma \geq c$ , while, for  $0 < \sigma < c$ ,

$$\begin{aligned} A_2(\sigma) &\leq C_m \int_{-c}^0 (\sigma-\tau)^{-m} |\tau|^{\alpha_j^+-1} d\tau \\ &= C_m \left( \int_{\sigma}^c (\sigma+\tau)^{-m} \tau^{\alpha_j^+-1} d\tau + \int_0^{\sigma} (\sigma+\tau)^{-m} \tau^{\alpha_j^+-1} d\tau \right) \\ &\leq C_m \left( \int_{\sigma}^c \tau^{\alpha_j^+-1-m} d\tau + \sigma^{-m} \int_0^{\sigma} \tau^{\alpha_j^+-1} d\tau \right) \leq C_m \sigma^{\alpha_j^+-m}. \end{aligned}$$

From these bounds the bound (2.13) on  $|v_j^{+(m)}|$  for  $m \in \mathbb{N}$  follows.  $\square$

### 3 Boundary integral equation formulation

Where  $\Phi$  is defined by (2.7), applying Green's representation theorem [15] to  $u^s$  gives

$$u^s(\mathbf{x}) = \int_{\Gamma} \left( \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} \gamma^+ u^s(\mathbf{y}) - \Phi(\mathbf{x}, \mathbf{y}) \partial_{\mathbf{n}}^+ u^s(\mathbf{y}) \right) ds(\mathbf{y}), \quad \mathbf{x} \in D. \quad (3.1)$$

Applying Green's second theorem [15] to  $\Phi(\mathbf{x}, \cdot)$  and  $u^i$  in  $\Omega$  we see that

$$0 = \int_{\Gamma} \left( \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} u^i(\mathbf{y}) - \Phi(\mathbf{x}, \mathbf{y}) \frac{\partial u^i}{\partial \mathbf{n}}(\mathbf{y}) \right) ds(\mathbf{y}), \quad \mathbf{x} \in D. \quad (3.2)$$

Then adding (3.1) and (3.2) and using the boundary condition (1.2), we find that

$$u^t(\mathbf{x}) = u^i(\mathbf{x}) + \int_{\Gamma} \left( \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} + ik\beta(\mathbf{y})\Phi(\mathbf{x}, \mathbf{y}) \right) \gamma^+ u^t(\mathbf{y}) ds(\mathbf{y}), \quad \mathbf{x} \in D. \quad (3.3)$$

Applying the trace operator  $\gamma^+$  and using the jump relations [15, Theorem 6.11], we obtain a standard boundary integral equation (cf. [9, Section 3.9]) for  $\gamma^+ u^t$ , that

$$\frac{1}{2} \gamma^+ u^t(\mathbf{x}) = u^i(\mathbf{x}) + \int_{\Gamma} \left( \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} + ik\beta(\mathbf{y})\Phi(\mathbf{x}, \mathbf{y}) \right) \gamma^+ u^t(\mathbf{y}) ds(\mathbf{y}), \quad \mathbf{x} \in \Gamma \setminus \{P_1, \dots, P_n\}. \quad (3.4)$$

It is well known [9, 20] that, while (3.4) is uniquely solvable for all but a countable set of positive wavenumbers  $k$ , with the associated linear operator bounded and invertible on  $H^s(\Gamma)$ , for  $0 \leq s \leq 1$ , in particular on  $L^2(\Gamma)$ , (3.4) is not uniquely solvable for all wavenumbers. Precisely, if  $k$  is such that the Dirichlet problem for the Helmholtz equation in the interior region  $\Omega$  has a non-trivial solution  $u_D$  ( $k$  is a so-called *irregular frequency*), then (3.4) has infinitely many solutions. To avoid this problem, the standard solution is to use a combined-layer formulation [2, 9], taking a linear combination of (3.4) with the equation that we get by applying the normal derivative operator  $\partial_{\mathbf{n}}^+$  to (3.3),

which gives a more elaborate integral equation formulation, involving hypersingular integral operators. This is the approach used in the boundary element method of [21], for example. Another option is to use the so-called CHIEF scheme, the basic idea of which is to overdetermine the solution of (3.4) with equations arising from enforcing a version of (3.3) at interior points  $\mathbf{x} \in \Omega$  called the CHIEF points. This scheme, proposed in [20], is used, for example, in [18].

In the next section we will describe a Galerkin boundary element method for solving (3.4) numerically, and will see in §5 that this method seems to work well even when  $k$  is an irregular frequency. That we do not see problems at irregular frequencies seems to be associated with our novel approximation space, specifically designed to accurately approximate  $\gamma^+ u^t$ . Rather than requiring a fixed number of degrees of freedom per wavelength, which is the case for conventional solvers, our scheme allows us to resolve many wavelengths with each degree of freedom. As a result, our approximation space will not approximate arbitrary oscillatory functions accurately. In particular, we expect that the solutions to the homogeneous version of (3.4), which are oscillatory but with a different phase to  $\gamma^+ u^t$ , will be poorly approximated with our scheme, and so anticipate that, provided  $k$  is large enough compared to the total number of degrees of freedom, solving (3.4) with the boundary element scheme we propose should approximate  $\gamma^+ u^t$  well, avoiding problems of nonuniqueness. For numerical results supporting this claim see §5, and for an elaboration of this argument for the related sound-soft problem, supported by extensive numerical results, see [13].

It is important to note, in any case, that, while we show numerical results for (3.4) in §5, the novel boundary element approximation space we design is just as relevant to any other direct integral equation formulation, in particular to the combined-layer integral equation formulation, in which  $\gamma^+ u^t$  is also the unknown.

## 4 Galerkin boundary element method

Our aim now is to use the regularity results in Theorem 2.1 to design an optimal approximation space for  $\gamma^+ u^t$ , which can be used for the numerical solution of (3.4). We represent  $\mathbf{x} \in \Gamma$  parametrically by

$$\mathbf{x}(s) = P_j + \left( s - \tilde{L}_{j-1} \right) \left( \frac{P_{j+1} - P_j}{L_j} \right), \quad \text{for } s \in (\tilde{L}_{j-1}, \tilde{L}_j), \quad j = 1, \dots, n,$$

where  $\tilde{L}_j := \sum_{m=1}^j L_m$ , and then write (3.4) in parametric form as

$$\phi(s) - 2 \int_0^L K(s, t) \phi(t) dt = 2f(s), \quad (4.1)$$

where  $\phi(s) = u^t(\mathbf{x}(s))$ ,  $L = \sum_{j=1}^n L_j$ ,

$$K(s, t) := \left( \frac{\partial \Phi(\mathbf{x}(s), \mathbf{x}(t))}{\partial \mathbf{n}(\mathbf{x}(t))} + ik\beta(\mathbf{x}(t))\Phi(\mathbf{x}(s), \mathbf{x}(t)) \right),$$

and  $f(s) = u^i(\mathbf{x}(s))$ . The first step in our numerical method is to separate off the explicitly known high frequency leading order behaviour which we denote by  $\Psi(s)$ . From (2.11) and Theorem 2.1 it is clear that this leading order behaviour is

$$\Psi(s) := \begin{cases} u_j^t(\mathbf{x}(s)) & \text{on illuminated sides} \\ 0 & \text{on shadow sides.} \end{cases}$$

Introducing the new unknown  $\varphi = \phi - \Psi$ , and substituting into (4.1), we have

$$\varphi(s) - 2 \int_0^L K(s,t) \varphi(t) dt = 2f(s) - \Psi(s) + 2 \int_0^L K(s,t) \Psi(t) dt, \quad (4.2)$$

which we can write in operator form as

$$(I - \mathcal{K})\varphi = F, \quad (4.3)$$

where  $\mathcal{K}v(s) := 2 \int_0^L K(s,t)v(t) dt$ ,  $F(s) := 2f(s) - \Psi(s) + 2 \int_0^L K(s,t)\Psi(t) dt$ , and  $I$  is the identity operator. Thinking of (4.3) as an operator equation on  $L^2(0,L)$ , this is the equation that we are going to solve for the unknown  $\varphi$  by a Galerkin boundary element method.

We now design our Galerkin approximation space  $V_{N,\nu} \subset L^2(0,L)$  in such a way as to efficiently represent  $\varphi$ , based on the representation (2.11) and the bounds in Theorem 2.1 (note that the notations in (2.11) and (4.2) are related by

$$\varphi(\tilde{L}_{j-1} + s) = \phi_j(s) - \psi_j(s), \quad \text{for } 0 \leq s \leq L_j, \quad j = 1, \dots, n). \quad (4.4)$$

Our estimates in Theorem 2.1 are similar to those for the same scattering problem but with sound-soft boundary conditions [6, Theorem 3.3, Corollary 3.4], but with different exponents for  $0 < ks \leq 1$ . Hence our approximation space is similar to (although not the same as) that defined in [6]. To describe this approximation space we begin by defining a composite graded mesh on a finite interval  $[0,A]$ , which comprises a polynomial grading near 0 and a geometric grading on the rest of the interval  $[0,A]$ . This mesh will be a component in the boundary element mesh that we will use on each side of the polygon.

**Definition 4.1.** For  $A > \lambda > 0$ ,  $q \geq 1$ ,  $N = 2, 3, \dots$ , we define  $N_1 := \lceil Nq \rceil$  and  $N_2 := \left\lceil \frac{-\log(A/\lambda)}{q \log(1 - 1/N_1)} \right\rceil$ , where, for  $s \in \mathbb{R}$ ,  $\lceil s \rceil$  denotes the smallest integer greater than or equal to  $s$ . The mesh  $\Lambda_{N,A,\lambda,q} := (y_0, \dots, y_{N_1+N_2})$  then consists of the points

$$y_i := \lambda \left( \frac{i}{N_1} \right)^q, \quad i = 0, \dots, N_1, \quad \text{and} \quad y_{N_1+j} := \lambda \left( \frac{A}{\lambda} \right)^{j/N_2}, \quad j = 1, \dots, N_2. \quad (4.5)$$

For large  $N$ ,  $N_1$  and  $N_2$  are both proportional to  $N$ , and the definitions of  $N_1$  and  $N_2$  ensure a smooth transition between the two parts of the mesh; in particular  $y_{N_1} - y_{N_1-1}$  and  $y_{N_1+1} - y_{N_1}$  are both asymptotically equal to  $\lambda/N$  as  $N \rightarrow \infty$ .

We now show that piecewise polynomials supported on the graded mesh defined in (4.5) are well-suited to the approximation of  $v_j^\pm$ . For  $a < b$  let  $\|\cdot\|_{2,(a,b)}$  denote the norm

on  $L^2(a,b)$ ,  $\|g\|_{2,(a,b)} := \left\{ \int_a^b |g(s)|^2 ds \right\}^{1/2}$ . For  $A > \lambda > 0$ ,  $\nu \in \mathbb{N} \cup \{0\}$ ,  $q \geq 1$ , where  $y_i$ ,  $i = 0, 1, \dots, N_1 + N_2$ , are the points of the mesh in Definition 4.1, let  $\Pi_{N,\nu} \subset L^2(0,A)$  denote the set of piecewise polynomials

$$\Pi_{N,\nu} := \{ \sigma \in L^2(0,A) : \sigma|_{(y_{j-1}, y_j)} \text{ is a polynomial of degree } \leq \nu \text{ for } j = 1, \dots, N_1 + N_2 \},$$

and let  $P_N$  be the orthogonal projection operator from  $L^2(0,A)$  to  $\Pi_{N,\nu}$ , so that setting  $p = P_N g$  minimises  $\|g - p\|_{2,(0,A)}$  over all  $p \in \Pi_{N,\nu}$ . Our error estimates for our boundary element method approximation space are based on the following theorem (cf. [6, Theorem 4.2]). We omit the proof which is a minor variant of the proof of [6, Theorem 4.2], referring the reader to [16] for details. Note that the relevance of this result is that, by Theorem 2.1,  $v_j^\pm$  satisfies the conditions of this theorem with  $\alpha = \alpha_j^\pm$ .

**Theorem 4.1.** *Suppose that  $g \in C^\infty(0,\infty)$ ,  $k > 0$ ,  $A > \lambda := 2\pi/k$  and  $\alpha \in [\frac{1}{2}, 1]$ , and that there exist constants  $c_m > 0$ ,  $m = 0, 1, 2, \dots$ , such that, for  $m \in \mathbb{N}$ ,*

$$|g(s)| \leq \begin{cases} c_0, & 0 < ks \leq 1, \\ c_0 (ks)^{-1/2}, & ks \geq 1, \end{cases} \quad |g^{(m)}(s)| \leq \begin{cases} c_m k^m (ks)^{\alpha-m}, & 0 < ks \leq 1, \\ c_m k^m (ks)^{-1/2-m}, & ks \geq 1. \end{cases}$$

Then, where  $q := (2\nu + 3)/(1 + 2\alpha)$ , there exists a constant  $C_\nu^*$ , dependent only on  $\nu$ , such that for  $N = 2, 3, \dots$ ,

$$\|g - P_N g\|_{2,(0,A)} \leq \frac{C_\nu^* \bar{c}_\nu (1 + \log(A/\lambda))^{1/2}}{k^{1/2} N^{\nu+1}},$$

where  $\bar{c}_\nu := \max(c_1, c_{\nu+1})$ .

From this point on we assume that  $L_j \geq \lambda$ , for  $j = 1, \dots, n$ , where  $\lambda := 2\pi/k$  is the wavelength. Setting  $\alpha_j = \pi/\Omega_j$ ,  $j = 1, \dots, n$ , and  $\alpha_{n+1} := \alpha_1$ , we define  $q_j := (2\nu + 3)/(1 + 2\alpha_j) \in (2\nu/3 + 1, \nu + 3/2)$ , for  $j = 1, \dots, n+1$ , and the two meshes  $\Gamma_j^+ := \tilde{L}_{j-1} + \Lambda_{N, L_j, \lambda, q_j}$ ,  $\Gamma_j^- := \tilde{L}_j - \Lambda_{N, L_j, \lambda, q_{j+1}}$ , for  $j = 1, \dots, n$ . Letting  $e_\pm(s) := e^{\pm iks}$ ,  $s \in [0, L]$ , we then define the approximation space associated with each mesh as

$$V_{\Gamma_j^+, \nu} := \{ \sigma e_+ : \sigma \in \Pi_{\Gamma_j^+, \nu} \}, \quad V_{\Gamma_j^-, \nu} := \{ \sigma e_- : \sigma \in \Pi_{\Gamma_j^-, \nu} \},$$

for  $j = 1, \dots, n$ , where

$$\begin{aligned} \Pi_{\Gamma_j^+, \nu} &:= \{ \sigma \in L^2(0, L) : \sigma|_{(\tilde{L}_{j-1} + y_{m-1}, \tilde{L}_{j-1} + y_m)} \text{ is a polynomial of degree } \leq \nu, \\ &\quad \text{for } m = 1, \dots, N_1^+ + N_2^+, \text{ and } \sigma|_{(0, \tilde{L}_{j-1}) \cup (\tilde{L}_j, L)} = 0 \}, \\ \Pi_{\Gamma_j^-, \nu} &:= \{ \sigma \in L^2(0, L) : \sigma|_{(\tilde{L}_j - \tilde{y}_m, \tilde{L}_j - \tilde{y}_{m-1})} \text{ is a polynomial of degree } \leq \nu, \\ &\quad \text{for } m = 1, \dots, N_1^- + N_2^-, \text{ and } \sigma|_{(0, \tilde{L}_{j-1}) \cup (\tilde{L}_j, L)} = 0 \}, \end{aligned}$$

with  $0 = y_0 < y_1 < \dots < y_{N_1^+ + N_2^+} = L_j$  the points of the mesh  $\Lambda_{N, L_j, \lambda, q_j}$  and  $0 = \tilde{y}_0 < \tilde{y}_1 < \dots < \tilde{y}_{N_1^- + N_2^-} = L_j$  the points of the mesh  $\Lambda_{N, L_j, \lambda, q_{j+1}}$ , and  $(N_1^+, N_2^+)$  and  $(N_1^-, N_2^-)$  the values of

$(N_1, N_2)$  for the meshes  $\Lambda_{N, L_j, \lambda, q_j}$  and  $\Lambda_{N, L_j, \lambda, q_{j+1}}$ , respectively. Our approximation space  $V_{N, \nu}$  is then the linear span of  $\bigcup_{j=1, \dots, n} \{V_{\Gamma_j^+, \nu} \cup V_{\Gamma_j^-, \nu}\}$ . The total number of the degrees of freedom is  $M_N = (\nu + 1) \sum_{j=1}^n N_j^*$ , where  $N_j^*$  is the sum of the values of  $N_1 + N_2$  (the number of subintervals) for the meshes  $\Lambda_{N, L_j, \lambda, q_j}$  and  $\Lambda_{N, L_j, \lambda, q_{j+1}}$ . Since  $-1/\log(1 - 1/N_1) < N_1$ , for  $N_1 \in \mathbb{N}$ , and  $1 < q_j < \nu + 3/2$ , we see that  $N_j^* < (Nq_j + 1 + (Nq_j + 1)\log(L_j/\lambda)/q_j + 1) + (Nq_{j+1} + 1 + (Nq_{j+1} + 1)\log(L_j/\lambda)/q_{j+1} + 1) < (2\nu + 3)N + 4 + 2(N + 1)\log(L_j/\lambda)$ , so that

$$M_N < (\nu + 1)n \left( (2\nu + 3)N + 4 + 2(N + 1)\log\left(\frac{\bar{L}}{\lambda}\right) \right) < (\nu + 1)nN \left( 2\nu + 5 + 3\log\left(\frac{k\bar{L}}{2\pi}\right) \right), \quad (4.6)$$

where  $\bar{L} := (L_1 \dots L_n)^{1/n}$ .

It follows from equations (2.11) and (4.4), Theorem 2.1 (applied with  $c = \pi$ ), and Theorem 4.1 that  $\varphi$  can be approximated very well by an element of the approximation space  $V_{N, \nu}$ . Precisely, these equations and theorems imply that, if the conditions of Theorem 2.1 are satisfied, then on each interval  $(\tilde{L}_{j-1}, \tilde{L}_j)$  (corresponding to side  $\Gamma_j$ ), there exist elements  $\sigma_j^+$  and  $\sigma_j^-$  of  $\Pi_{\Gamma_j^+, \nu}$  and  $\Pi_{\Gamma_j^-, \nu}$ , respectively, such that, for some constant  $C_\nu > 0$ , depending only on  $\nu$ ,  $B$ , and  $\omega$ ,

$$\|(\sigma_j^+ \mathbf{e}_+ + \sigma_j^- \mathbf{e}_-) - \varphi\|_{2, (\tilde{L}_{j-1}, \tilde{L}_j)} \leq \frac{C_\nu M (1 + \log(L_j/\lambda))^{1/2}}{k^{1/2} N^{\nu+1}}.$$

It follows that, where  $\phi_N := \sum_{j=1}^n (\sigma_j^+ \mathbf{e}_+ + \sigma_j^- \mathbf{e}_-) \in V_{N, \nu}$ ,

$$\|\phi_N - \varphi\|_{2, (0, L)}^2 = \sum_{j=1}^n \|(\sigma_j^+ \mathbf{e}_+ + \sigma_j^- \mathbf{e}_-) - \varphi\|_{2, (\tilde{L}_{j-1}, \tilde{L}_j)}^2 \leq \frac{C_\nu^2 M^2 n (1 + \log(\bar{L}/\lambda))}{k N^{2(\nu+1)}}.$$

Combining this bound with (4.6) we see that we have shown the following theorem.

**Theorem 4.2.** *Suppose that, for  $j = 1, \dots, n$ ,  $kL_j \geq 2\pi$  and that  $\beta_j$  and  $\Omega_j$  satisfy (2.12). Then, where  $\phi_N$  denotes the best approximation to  $\varphi$  from the approximation space  $V_{N, \nu} \subset L^2(0, L)$ , it holds that*

$$\|\phi_N - \varphi\|_{2, (0, L)} \leq C_\nu M \frac{n^{1/2} \left(1 + \log\left(\frac{k\bar{L}}{2\pi}\right)\right)^{1/2}}{k^{1/2} N^{\nu+1}} \leq C'_\nu M \frac{n^{\nu+3/2} \left(1 + \log\left(\frac{k\bar{L}}{2\pi}\right)\right)^{\nu+3/2}}{k^{1/2} M_N^{\nu+1}},$$

where the constants  $C_\nu$  and  $C'_\nu$  depend only on  $\nu$ ,  $B$ , and  $\omega$ .

We note that this bound makes clear that, assuming  $M$  defined by (2.1) is bounded as  $k \rightarrow \infty$ , to achieve any desired  $L^2$  error in the best approximation from  $V_{N, \nu}$ , it is enough to keep the degrees of freedom  $M_N$  fixed as  $k \rightarrow \infty$ . Further, keeping  $N$  fixed (which means increasing  $M_N$  logarithmically as  $k$  increases) will come close to keeping  $k^{1/2} \|\phi_N - \varphi\|_{2, (0, L)}$  bounded as  $k$  increases.

Of course we have no way of computing the best approximation  $\varphi_N$ ; to select what we hope is something close to the best approximation from  $V_{N,\nu}$  we use the Galerkin method. Let  $(\cdot, \cdot)$  denote the usual inner product on  $L^2(0, L)$ , defined by  $(\chi_1, \chi_2) := \int_0^L \chi_1(s) \bar{\chi}_2(s) ds$ . Then our Galerkin method approximation  $\varphi_N \in V_{\Gamma,\nu}$  is defined by

$$(\varphi_N, \rho) - (\mathcal{K}\varphi_N, \rho) = (F, \rho), \quad \text{for all } \rho \in V_{\Gamma,\nu}. \quad (4.7)$$

For brevity we give further detail only for the case  $\nu=0$ . Writing  $\varphi_N$  as a linear combination of the basis functions of  $V_{\Gamma,0}$ , we have

$$\varphi(s) \approx \varphi_N(s) = \sum_{j=1}^{M_N} v_j \rho_j(s), \quad 0 \leq s \leq L, \quad (4.8)$$

where  $\rho_j$  is the  $j$ th basis function,  $M_N$  is the dimension of  $V_{\Gamma,0}$ , and the constants  $v_j$ ,  $j=1, \dots, M_N$ , satisfy a linear system equivalent to (4.7). To make explicit this linear system, for  $p=1, \dots, n$ , let  $n_p^\pm$  be the number of points in the mesh  $\Gamma_p^\pm$ . Denote the points of the mesh  $\Gamma_p^\pm$  by  $s_{p,l}^\pm$ , for  $l=1, \dots, n_p^\pm$ ,  $p=1, \dots, n$ , with  $s_{p,l}^\pm < \dots < s_{p,n_p^\pm}^\pm$ . Setting  $n_1 := 0$ ,  $n_p := \sum_{j=1}^{p-1} (n_j^+ + n_j^-)$  for  $p=2, \dots, n-1$ , define, for  $p=1, \dots, n$ ,  $\rho_{n_p+j}(s) := e^{iks} \chi_{(s_{p,j-1}^+, s_{p,j}^+)}(s) / \sqrt{s_{p,j}^+ - s_{p,j-1}^+}$ ,  $j=1, \dots, n_p^+$ ,  $\rho_{n_p+n_p^+ + j}(s) := e^{-iks} \chi_{(s_{p,j-1}^-, s_{p,j}^-)}(s) / \sqrt{s_{p,j}^- - s_{p,j-1}^-}$ ,  $j=1, \dots, n_p^-$ , where  $\chi_{(a,b)}$  denotes the characteristic function of the interval  $(a,b)$ . Substituting (4.8) into (4.7) gives the linear system

$$\sum_{j=1}^{M_N} [(\rho_j, \rho_m) - (\mathcal{K}\rho_j, \rho_m)] v_j = (F, \rho_m), \quad m=1, \dots, M_N. \quad (4.9)$$

For implementation details, namely discussion of how to compute the integrals in (4.9), we refer to [16].

## 5 Numerical results

As a numerical example we take the scattering object,  $\Omega$ , to be a square, with vertices  $(0,0)$ ,  $(2\pi,0)$ ,  $(2\pi,2\pi)$  and  $(0,2\pi)$ , so that  $L=8\pi$ . We take  $\beta_j=1$  on each side  $\Gamma_j$ ,  $j=1, \dots, n$ , and incident angle  $\theta=\pi/4$ . We solve (4.9) to find the coefficients  $v_j$ ,  $j=1, \dots, M_N$ , and then use (4.8) to compute  $\varphi_N$ . In Table 1 we show both absolute and relative estimated  $L^2$  errors for  $k=5$  and 320 (all  $L^2$  norms approximated by discrete  $L^2$  norms, sampling at 1000000 evenly spaced points on the boundary), tabulated as a function of  $N$ . (Recall that  $N$  is the local number of degrees of freedom per wavelength in each of the graded meshes we use at distance  $\lambda$  from the corner where the mesh is refined.) We also show the number of degrees of freedom,  $M_N$ , and  $M_N/(L/\lambda)$ , the average number of degrees of freedom per wavelength (which needs to be in the range 5-10 for standard low order boundary element methods to achieve engineering accuracy). Note that, for the same values of

$k$	$N$	$M_N$	$M_N\lambda/L$	$\ \varphi_{128} - \varphi_N\ _2$	$\ \varphi_{128} - \varphi_N\ _2 / \ \varphi_{128}\ _2$	EOC	$\text{cond}_2 A$
5	4	56	2.8	$1.3053 \times 10^{-1}$	$3.0383 \times 10^{-1}$	0.9	$1.82 \times 10^1$
	8	104	5.2	$7.0707 \times 10^{-2}$	$1.6458 \times 10^{-1}$	0.8	$8.50 \times 10^1$
	16	200	10.0	$3.9573 \times 10^{-2}$	$9.2110 \times 10^{-2}$	0.9	$4.66 \times 10^2$
	32	376	18.8	$2.1229 \times 10^{-2}$	$4.9412 \times 10^{-2}$	1.0	$2.13 \times 10^3$
	64	752	37.6	$1.0731 \times 10^{-2}$	$2.4978 \times 10^{-2}$	-	$1.45 \times 10^4$
320	4	120	0.094	$1.6285 \times 10^{-2}$	$3.0284 \times 10^{-1}$	0.9	$2.97 \times 10^1$
	8	248	0.194	$8.7732 \times 10^{-3}$	$1.6315 \times 10^{-1}$	0.9	$4.04 \times 10^1$
	16	472	0.369	$4.8664 \times 10^{-3}$	$9.0494 \times 10^{-2}$	1.0	$4.62 \times 10^1$
	32	904	0.706	$2.4775 \times 10^{-3}$	$4.6072 \times 10^{-2}$	1.1	$5.19 \times 10^1$
	64	1832	1.431	$1.1782 \times 10^{-3}$	$2.1910 \times 10^{-2}$	-	$1.34 \times 10^2$

Table 1: Errors in the Galerkin method approximation, scattering by square,  $k=5$  and  $k=320$ 

$N$ , greater accuracy is achieved for  $k=320$  than for  $k=5$ , even though the average numbers of degrees of freedom per wavelength are very small for  $k=320$ . We also tabulate the estimated order of convergence (EOC), given by  $\text{EOC} := \log_2(\|\varphi - \varphi_N\|_2 / \|\varphi - \varphi_{2N}\|_2)$ . We expect, given that  $\nu=0$ ,  $\text{EOC} \approx 1$  from Theorem 4.2. This is clearly seen in the table. Note also that the 2-norm condition number,  $\text{cond}_2 A$ , of the matrix  $A = [(\rho_j, \rho_m) - (\mathcal{K}\rho_j, \rho_m)]$  only increases very moderately as  $N$  increases for the larger value of  $k$ .

Table 2 shows the errors for fixed  $N=32$  and increasing  $k$ . The 2-norm condition num-

$k$	$M_N$	$M_N\lambda/L$	$k^{1/2}\ \varphi_{128} - \varphi_{32}\ _2$	$\ \varphi_{128} - \varphi_{32}\ _2 / \ \varphi_{128}\ _2$	$\text{cond}_2 A$
5	376	18.80	$4.7469 \times 10^{-2}$	$4.9412 \times 10^{-2}$	$2.13 \times 10^3$
10	464	11.60	$4.7033 \times 10^{-2}$	$4.9022 \times 10^{-2}$	$6.16 \times 10^2$
20	552	6.90	$4.7047 \times 10^{-2}$	$4.9006 \times 10^{-2}$	$5.06 \times 10^2$
40	640	4.00	$4.6715 \times 10^{-2}$	$4.8627 \times 10^{-2}$	$2.19 \times 10^2$
80	728	2.28	$4.7897 \times 10^{-2}$	$4.9871 \times 10^{-2}$	$7.35 \times 10^1$
160	816	1.28	$4.6208 \times 10^{-2}$	$4.8177 \times 10^{-2}$	$3.81 \times 10^1$
320	904	0.71	$4.4319 \times 10^{-2}$	$4.6072 \times 10^{-2}$	$5.19 \times 10^1$

Table 2: Errors in the Galerkin method approximation, scattering by square,  $N=32$ 

ber is small and decreasing as  $k$  increases. As predicted by Theorem 4.2,  $k^{1/2}\|\varphi_{128} - \varphi_N\|_2$  remains approximately fixed as  $k$  increases. The relative  $L^2$  error  $\|\varphi_{128} - \varphi_N\|_2 / \|\varphi_{128}\|_2$  also remains roughly constant as  $k$  increases. At the same time, the number of degrees of freedom  $M_N$ , bounded above by (4.6), grows only in proportion to  $\log k$  as  $k$  increases, and the average degrees of freedom per wavelength reduces rapidly as  $k$  increases. These observations indicate the effectiveness and robustness of our new boundary element scheme, particularly at high frequencies.

We remark, finally, that all the values of  $k$  for which we show results in the tables

above are irregular frequencies in the sense of §3, i.e. they are values of  $k$  for which the Dirichlet problem for the Helmholtz equation in the interior  $\Omega$  has a non-trivial solution  $u_D$  and so the integral equation (3.4) has infinitely many solutions. (For this numerical example these irregular frequencies are  $k = \sqrt{m^2 + n^2}$ , for  $m, n \in \mathbb{N}$ , with the corresponding non-trivial solutions  $u_D$  given by  $u_D(\mathbf{x}) = \sin mx_1 \sin nx_2$ .) This lack of uniqueness at a continuous level does not appear to translate to the discrete level; the Galerkin method with our approximation space, carefully tailored to  $\varphi$ , seems to select the right solution – see the discussion in §3 and [13].

## 6 Conclusions

In this paper we have derived regularity estimates for scattering by a convex polygon with piecewise constant impedance boundary conditions, and we have used these to derive a novel approximation space for  $\gamma^+ u^t$  consisting of the products of plane waves with piecewise polynomials supported on a graded mesh, with larger elements further from the corners of the polygon. We have implemented a Galerkin boundary element method; numerical results suggest that the number of degrees of freedom required to achieve a prescribed level of accuracy need grow only logarithmically with respect to the frequency of the incident wave. These numerical experiments are backed up by rigorous error bounds on the error in the best approximation from the approximation space.

## Acknowledgments

Mokgolele’s PhD studies at the University of Reading were supported by the Botswana College of Agriculture. Chandler-Wilde and Langdon were partially supported by EP-SRC grant EP/F067798/1.

## References

- [1] J. M. L. Bernard, A spectral approach for scattering by impedance polygons, *Quart. J. Mech. Appl. Math.*, 59 (2006), 517–550.
- [2] A. J. Burton and G. F. Miller, The applications of integral equations to the numerical solution of some exterior boundary-value problems, *Proc. R. Soc. Lond. Ser. A*, 323 (1971), 201–210.
- [3] S. N. Chandler-Wilde and I. G. Graham, Boundary integral methods in high frequency scattering, in “Highly Oscillatory Problems”, B. Engquist, T. Fokas, E. Hairer, A. Iserles, editors, CUP (2009), 154–193.
- [4] S. N. Chandler-Wilde and D. C. Hothersall, On the Green Function for Two-Dimensional Acoustic Propagation above a Homogeneous Impedance Plane, Research Report, Dept. of Civil Engineering, University of Bradford, UK, 1991.
- [5] S. N. Chandler-Wilde and D. C. Hothersall, A uniformly valid far field asymptotic expansion of the Green function for two-dimensional propagation above a homogeneous impedance plane, *J. Sound Vibration*, 182 (1995), 665–675.

- [6] S. N. Chandler-Wilde and S. Langdon, A Galerkin boundary element method for high frequency scattering by convex polygons, *SIAM J. Numer. Anal.*, 45 (2007), 610–640.
- [7] S. N. Chandler-Wilde, S. Langdon and L. Ritter, A high-wavenumber boundary-element method for an acoustic scattering problem, *Phil. Trans. R. Soc. Lond. A*, 362 (2004), 647–671.
- [8] S. N. Chandler-Wilde and B. Zhang, Electromagnetic scattering by an inhomogeneous conducting or dielectric layer on a perfectly conducting plate, *Proc. R. Soc. Lond. A*, 454 (1998), 519–542.
- [9] D. Colton and R. Kress, *Integral Equation Methods in Scattering Theory*, John Wiley, 1983.
- [10] V. Dominguez, I. G. Graham, and V. P. Smyshlyaev, A hybrid numerical-asymptotic boundary integral method for high-frequency acoustic scattering, *Numer. Math.*, 106 (2007), 471–510.
- [11] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, 1977.
- [12] M. I. Herman and J. L. Volakis, High frequency scattering from polygonal impedance cylinders and strips, *IEEE Trans. Antennas and Propagation*, 36 (1988), 679–689.
- [13] C. Howarth, *Integral Equation Formulations for Scattering Problems*, MSc thesis, University of Reading, 2009.
- [14] S. Langdon and S. N. Chandler-Wilde, A wavenumber independent boundary element method for an acoustic scattering problem, *SIAM J. Numer. Anal.*, 43 (2006), 2450–2477.
- [15] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, CUP, 2000.
- [16] M. Mokgolele, *Numerical Solution of High Frequency Acoustic Scattering Problems*, PhD Thesis, University of Reading, 2009.
- [17] A. V. Osipov and A. N. Norris, The Malyuzhinets theory for scattering from wedge boundaries: a review, *Wave Motion*, 29 (1999), 313–340.
- [18] E. Perrey-Debain, J. Trevelyan and P. Bettess, On wave boundary elements for radiation and scattering problems with piecewise constant impedance, *IEEE Trans. Antennas and Propagation*, 53 (2005), 876–879.
- [19] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Springer, 1999.
- [20] H. A. Schenck, Improved integral formulation for acoustic radiation problems, *J. Acoust. Soc. Am.*, 44 (1968), 41–58.
- [21] P. Ylä-Oijala and S. Järvenpää, Iterative solution of high-order boundary element method for acoustic impedance boundary value problems, *J. Sound Vibration*, 291 (2006), 824–843.