Hybrid numerical asymptotic approximation for multiple scattering problems

A. Gibbs\(^1\), S.N. Chandler-Wilde\(^2\), S. Langdon\(^3\), A. Moiola\(^4\)

\(^1\)Department of Mathematics and Statistics, University of Reading, UK
\(^2\)Department of Mathematics and Statistics, University of Reading, UK
\(^3\)Department of Mathematics and Statistics, University of Reading, UK
\(^4\)Department of Mathematics and Statistics, University of Reading, UK

Email: a.j.gibbs@pgr.reading.ac.uk

Abstract

Standard numerical schemes for scattering problems have a computational cost that grows at least in direct proportion to the frequency of the incident wave. For many problems of scattering by single obstacles, it has been shown that a careful choice of approximation space, utilising knowledge of high frequency asymptotics, can lead to numerical schemes whose computational cost is independent of frequency. Here, we extend these ideas to multiple scattering configurations, focusing in particular on the case of two scatterers, with one much larger than the other.

Keywords: Multiple scattering, BEM, high frequency, Helmholtz equation, Hybrid Numerical Asymptotic method

1 Introduction

We consider the problem of scattering of a time-harmonic incident wave \(u^i(x) := \exp(ikx \cdot d)\) propagating in direction \(d\) with wavenumber \(k > 0\), by multiple sound-soft scatterers in two-dimensions. For simplicity, here we consider the case of two scatterers as shown in Figure 1. We assume the larger scatterer is a convex polygon with boundary \(\Gamma\) and denote the boundary of the smaller scatterer by \(\gamma\). What follows may also be applied to more general cases where \(\gamma\) is the union of many scatterers, which are not required to be convex or polygonal. Our boundary value problem is: Find \(u \in H^1_{\text{loc}}(D) \cap C^2(D)\), such that

\[
(\Delta + k^2)u = 0 \text{ in } D, \quad u = 0 \text{ on } \Gamma \cup \gamma,
\]

where \(D \subset \mathbb{R}^2\) is the complement of the scatterers and the scattered field \(u^s := u - u^i\) satisfies the Sommerfeld radiation condition.

Figure 1: \(\text{Re}(u)\) in \(D\), scattering by two triangles. \(|\Gamma| = 6\pi, |\gamma| = 3\pi/5, k = 10, d = 2^{-1/2}(1,1)^T\).

Using the standard Green’s representation formula (see e.g. [3]), the problem reduces to finding \(\partial u / \partial n\) on \(\Gamma \cup \gamma\). We use a direct formulation

\[
\mathcal{A}\left[\frac{\partial u}{\partial n}\right] = f \text{ on } \Gamma \cup \gamma,
\]

where \(\mathcal{A}\) is the standard combined layer integral operator as in (for example) [1] and \(f\) consists of known boundary data. For problems with a low wavenumber \(k\), a standard BEM can approximate \(\partial u / \partial n\) with piecewise polynomials \(\varphi_i\):

\[
\frac{\partial u}{\partial n}(x) \approx \sum_{i=1}^{M_\gamma} \beta_i \varphi_i(x) \text{ on } \gamma, \quad \alpha_i \in \mathbb{C}.
\]

With this approach, the number of degrees of freedom \(M_\gamma\) must increase at least linearly with \(k\) to maintain accuracy.

2 HNA ansatz for a convex polygon \(\Gamma\)

For a single convex scatterer, the Hybrid Numerical Asymptotic (HNA) method [3] is used
to enrich the approximation space with carefully chosen oscillatory basis functions, designed to capture the oscillations of the diffracted waves. The HNA approximation is

$$\frac{\partial u}{\partial n}(x) \approx \Psi(x) + \sum_{\ell=1}^{M_\Gamma} \alpha_\ell \phi_\ell(x) e^{i k \psi_\ell(x)}, \quad x \in \Gamma,$$

where $\Psi$ is the geometrical optics approximation which represents the reflected wave. The phases $\psi_\ell$ are chosen a priori, only the (non-oscillatory) amplitudes of these oscillations are approximated by the piecewise polynomials $\phi_\ell$ with $M_\Gamma$ degrees of freedom.

3 Extension of the HNA ansatz to multiple scatterers

The HNA ansatz can be extended to account for multiple scatterers [2], with an additional term representing the contribution to the solution on $\Gamma$ from the solution on $\gamma$. The approximation on $\Gamma$ becomes

$$\frac{\partial u}{\partial n}|_\Gamma(x) \approx \Psi(x) + \sum_{\ell=1}^{M_\Gamma} \alpha_\ell \phi_\ell(x) e^{i k \psi_\ell(x)} + \sum_{i=1}^{M_\gamma} \beta_i \mathcal{G}\varphi_i(x),$$

(4)

where $\mathcal{G} : H^{-1/2}(\gamma) \rightarrow H^{-1/2}(\Gamma)$ is defined by

$$\mathcal{G}\varphi(x) := -2 \int_{\gamma \cap U_j} \frac{\partial \Phi(x,y)}{\partial n(x)} \varphi(y) \, ds(y), \quad x \in \Gamma_j,$$

for each side $\Gamma_j$ of $\Gamma$. Here $\Phi$ is the fundamental solution to the Helmholtz equation and $U_j$ is the half-plane with $\Gamma_j$ situated along its boundary, such that $\Gamma \not\subset U_j$ (as in Figure 2). The term $\mathcal{G}[\partial u/\partial n|_\gamma]$ represents the contribution from the solution on $\gamma$ to the solution on $\Gamma$.

4 Numerical results

We solve for the unknown on the large obstacle $\Gamma$ and small obstacle $\gamma$ simultaneously, approximating $\partial u/\partial n$ using (4) on $\Gamma$ and (3) on $\gamma$, with piecewise polynomials $\varphi_i$ and $\phi_\ell$ on a graded mesh. A Galerkin method is used with the BIE (2) as in [1]. For each $k$ we observe exponential convergence as the number of degrees of freedom increases, whilst the error does not appear to grow with $k$ for fixed degrees of freedom.

References

