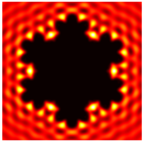
BOUNDARY ELEMENT METHODS, OBERWOLFACH, 6 FEBRUARY 2020

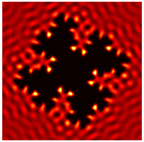
Convergence of Boundary Element Methods on Fractals



Simon Chandler-Wilde

http://www.personal.reading.ac.uk/~sms03snc/

University of Reading



Joint work with

D.P. Hewett (UCL), A. Moiola (Pavia), J. Besson (ENSTA)

Acoustic wave scattering by a planar screen

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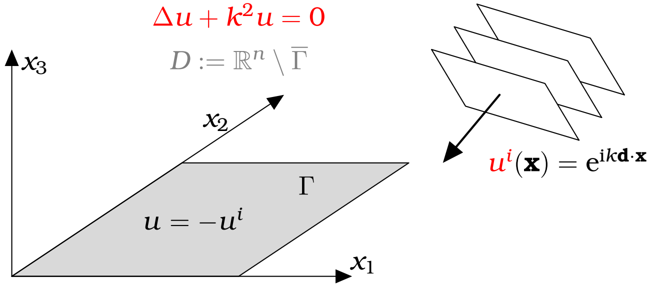
Scattering: incoming wave u^i hits flat screen Γ and generates field u.

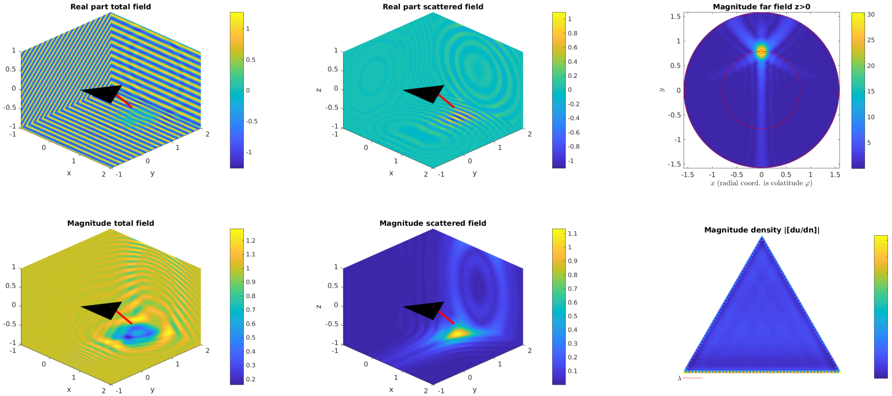
Γ bounded subset of $\Gamma_\infty := \{\mathbf{x} \in \mathbb{R}^n : x_n = 0\} \cong \mathbb{R}^{n-1}$, $n = 2, 3$



u satisfies Sommerfeld radiation condition (SRC) at infinity
(i.e. $\partial_r u - iku = o\left(r^{(1-n)/2}\right)$ uniformly as $r = |\mathbf{x}| \to \infty$).

Scattering by Lipschitz and rough screens

Incident field is plane wave $u^i(\mathbf{x}) = \mathrm{e}^{ik\mathbf{d}\cdot\mathbf{x}}$, $|\mathbf{d}| = 1$. $\qquad u^{tot} = u + u^i$



Classical problem when Γ is open and Lipschitz.

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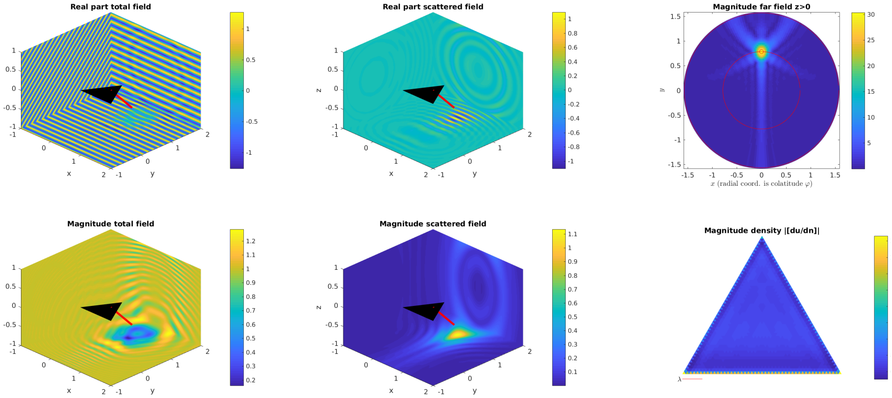
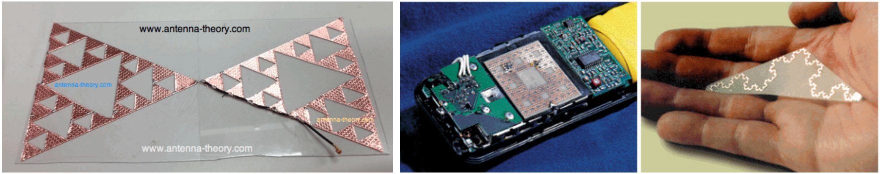


Classical problem when Γ is open and Lipschitz.

What happens for arbitrary (rougher than Lipschitz, e.g. fractal) Γ?

Waves and fractals: applications

Wideband fractal antennas



(Figures from `http://www.antenna-theory.com/antennas/fractal.php`)

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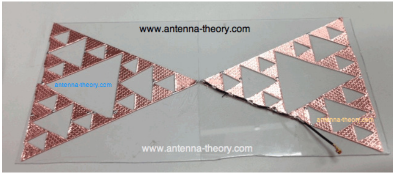

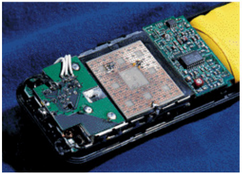
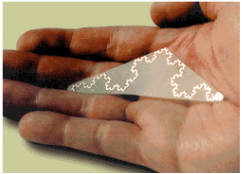

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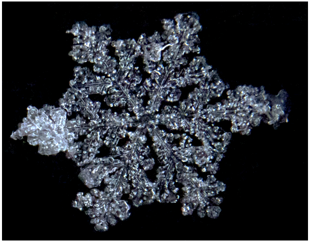
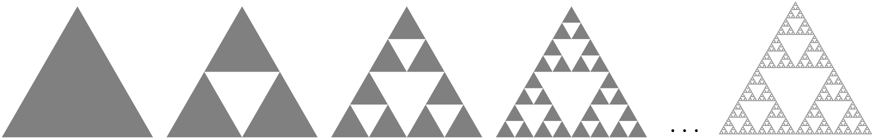
Scattering by ice crystals
in atmospheric physics
e.g. C. Westbrook



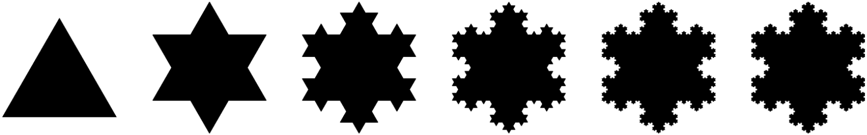
Fractal apertures in laser optics
e.g. J. Christian

Scattering by fractal screens



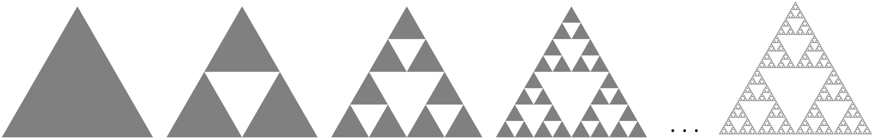
Lots of mathematical challenges:

- How to formulate well-posed BVPs?
  (What is the right function space setting? How to impose BCs?)
- Do solutions on prefractals converge to solutions on fractals?
- Do BEM solutions on prefractals converge?



Ideas and analysis relevant to BEM for any BIE/ΨDO on fractals or other rough sets – e.g. fractional Laplacian on rough sets?

Scattering by fractal screens



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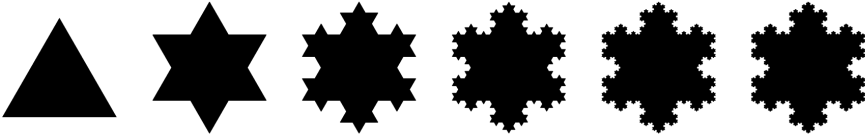
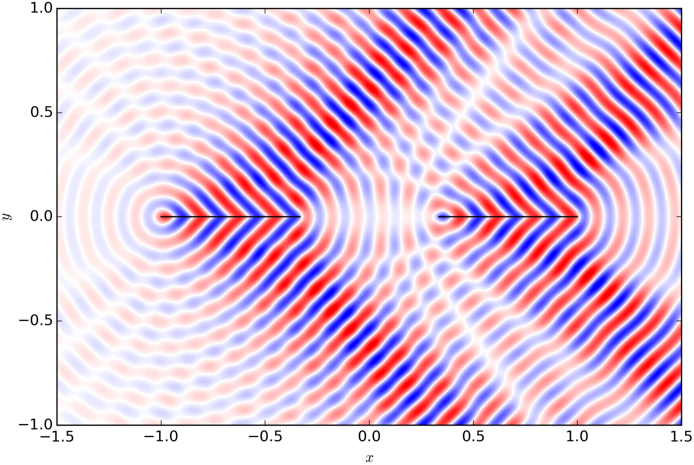


Ideas and analysis relevant to BEM for any BIE/ΨDO on fractals or other rough sets – e.g. fractional Laplacian on rough sets?
Previous BEM computations on sequences of prefractals, e.g. Jones, Ma, Rokhlin 1994, Panagiotopoulos, Panagouli 1996, but no proof that these converge to right limit.

Our method is: **solve by piecewise constant BEM on sequence of prefractals**: results for Cantor set



Γ_1 and $\mathrm{Re}\, u_1^h$

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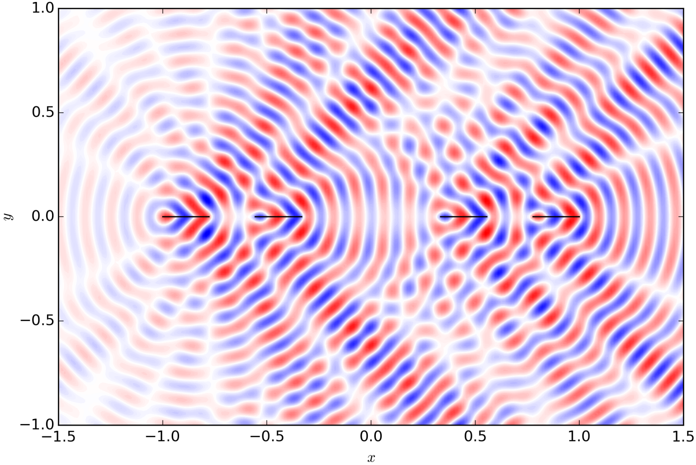

Γ_2 and $\mathrm{Re}\, u_2^h$

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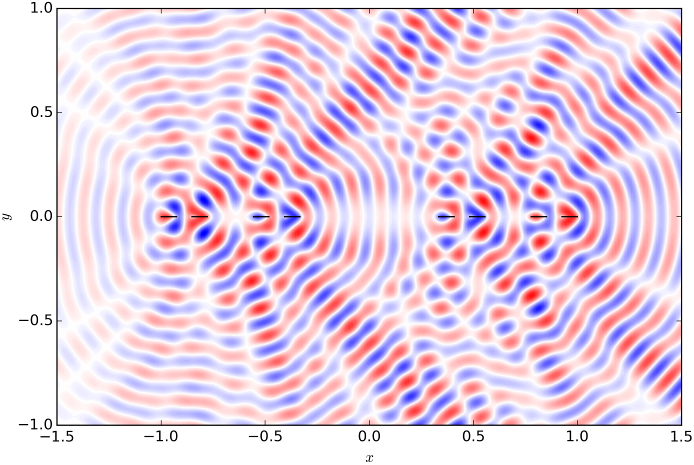

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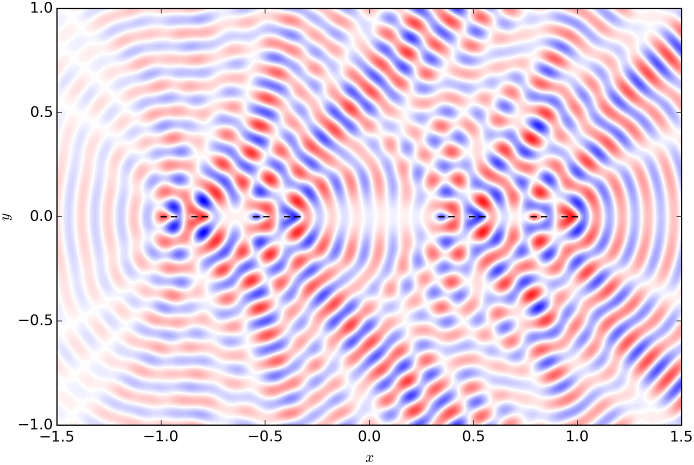

Γ_4 and $\mathrm{Re}\, u_4^h$

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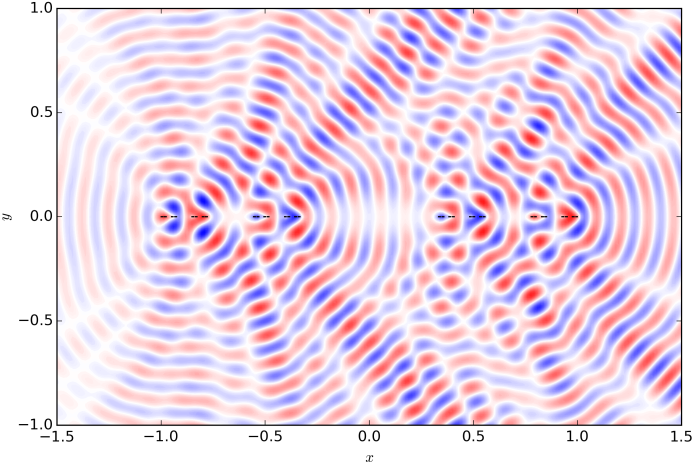
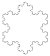


Γ_5 and $\mathrm{Re}\, u_5^h$

Outline

- Sobolev spaces on rough sets
- BVPs and BIEs
  - open screens 
  - compact screens 
- Abstract convergence framework, using Mosco convergence
- Prefractal to fractal convergence
- Convergence of BEM on sequences of prefractals
- Numerical examples
  - Cantor set
  - Cantor dust: dependence on Hausdorff dimension
  - Fractal apertures

Sobolev spaces on rough subsets of $\mathbb{R}^{n-1}$

We need Sobolev spaces on $\Gamma \subset \mathbb{R}^{n-1}$. For $s \in \mathbb{R}$ let

$$H^s(\mathbb{R}^{n-1}) = \left\{ u \in \mathcal{S}^*(\mathbb{R}^{n-1}) : \|u\|^2_{H^s(\mathbb{R}^{n-1})} := \int_{\mathbb{R}^{n-1}} (1+|\boldsymbol{\xi}|^2)^s |\hat{u}(\boldsymbol{\xi})|^2 \, \mathrm{d}\boldsymbol{\xi} < \infty \right\}$$

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For $\Gamma \subset \mathbb{R}^{n-1}$ open and $F \subset \mathbb{R}^{n-1}$ closed define (McLean)

$$H^s(\Gamma) := \{ u|_\Gamma : u \in H^s(\mathbb{R}^{n-1}) \}$$ restriction

$$\widetilde{H}^s(\Gamma) := \overline{C_0^\infty(\Gamma)}^{H^s(\mathbb{R}^{n-1})}$$ closure

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When Γ is Lipschitz it holds that

- $\widetilde{H}^s(\Gamma) \cong (H^{-s}(\Gamma))^*$ with equal norms
- $s \in \mathbb{N} \Rightarrow \|u\|^2_{H^s(\Gamma)} \sim \sum_{|\alpha| \le s} \int_\Gamma |\partial^\alpha u|^2$
- $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$ $\quad (\cong H^s_{00}(\Gamma),\ s \ge 0)$
- $H^{\pm 1/2}_{\partial\Gamma} = \{0\}$
- $\{H^s(\Gamma)\}_{s\in\mathbb{R}}$ and $\{\widetilde{H}^s(\Gamma)\}_{s\in\mathbb{R}}$ are interpolation scales.

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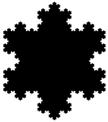
BVPs for open and compact screens

BVP D$^{op}(\Gamma)$ for open screens

Let $\Gamma \subset \Gamma_\infty$ be bounded & open.
Given $g \in H^{1/2}(\Gamma)$
(for instance, $g = -(\gamma^\pm u^i)|_\Gamma$),
find $u \in C^2(D) \cap W^{1,\mathrm{loc}}(D)$
satisfying

$$\Delta u + k^2 u = 0 \quad \text{in } D,$$
$$(\gamma^\pm u)|_\Gamma = g,$$
Sommerfeld RC.



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BVP D$^{co}(\Gamma)$ for compact scr.

Let $\Gamma \subset \Gamma_\infty$ be compact.
Given $g \in \widetilde{H}^{1/2}(\Gamma^c)^\perp$
(e.g., $g = -P_\Gamma \gamma^\pm u^i$),
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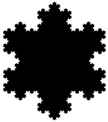

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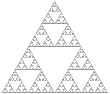

Orthogonal projection
$P_\Gamma : H^{1/2}(\Gamma_\infty) \to \widetilde{H}^{1/2}(\Gamma^c)^\perp$.

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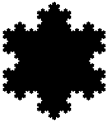

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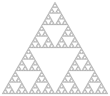

Orthogonal projection
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If Ω bdd open & $\widetilde{H}^{-1/2}(\Omega) = H^{-1/2}_{\overline{\Omega}}$, then D$^{op}(\Omega)$&D$^{co}(\overline{\Omega})$ are equivalent.

Well-posedness & boundary integral equations

Theorem (CW, H, M 2019)

If $\widetilde{H}^{-1/2}(\Gamma) = H^{-1/2}_{\overline{\Gamma}}$ then problem D$^{op}(\Gamma)$ has a unique solution u.

Theorem (CW, H 2019)

Problem D$^{co}(\Gamma)$ has a unique solution u.

Well-posedness & boundary integral equations

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Theorem (CW, H 2019)

Problem D$^{co}(\Gamma)$ has a unique solution u.

u satisfies the representation formula $\quad u(\mathbf{x}) = -\mathcal{S}_\Gamma\phi(\mathbf{x}), \mathbf{x} \in D$,
where $\phi = [\partial_\mathbf{n} u] := \partial_\mathbf{n}^+ u - \partial_\mathbf{n}^- u$ is the unique solution of BIE $S_\Gamma\phi = -g$.

$\mathcal{S}_\Gamma$ = single-layer potential,
S_Γ = single layer operator: cont. & coercive in $H^{-1/2}(\mathbb{R}^{n-1})$ norm.

$$\mathcal{S}_\Gamma\psi(\mathbf{x}) := \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\psi(\mathbf{x})\mathrm{d}s(\mathbf{y})$$

$$\mathcal{S}_\Gamma : \widetilde{H}^{-1/2}(\Gamma) \to C^2(D)\cap W^{1,loc}(\mathbb{R}^n) \qquad \mathcal{S}_\Gamma : H_{\overline{\Gamma}}^{-1/2} \to C^2(D) \cap W^{1,loc}(\mathbb{R}^n)$$

$$S_\Gamma\psi = (\gamma^\pm\mathcal{S}_\Gamma\psi)|_\Gamma \qquad S_\Gamma = P_\Gamma\gamma^\pm\mathcal{S}_\Gamma$$

$$S_\Gamma : \widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) \qquad S_\Gamma : H_{\overline{\Gamma}}^{-1/2} \to \widetilde{H}^{1/2}(\Gamma^c)^\perp$$

Φ is the Helmholtz fundamental solution ($\Phi(\mathbf{x},\mathbf{y}) = \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}$ for $n = 3$)

When is $\widetilde{H}^{-1/2}(\Gamma) = H_{\overline{\Gamma}}^{-1/2}$?

The previous theorems extend classical results for Lipschitz domains (Stephan & Wendland 1984, Stephan 1987).

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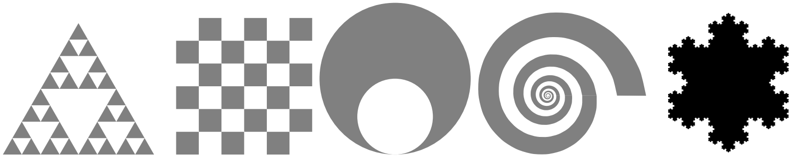
Sufficient conditions for $\widetilde{H}^{-1/2}(\Gamma) = H^{-1/2}_{\overline{\Gamma}}$ are that either

- Γ is C^0 (e.g. Lipschitz); or
- Γ is C^0 except at a set of countably many points $P \subset \partial\Gamma$ such that P has only finitely many limit points (C-W, H, M 2017); or
- $|\partial\Gamma| = 0$ and Γ is "thick", in the sense of Triebel (Caetano, H, M 2019).



$$(\widetilde{H}^{-1/2}(\Gamma) = H^{-1/2}_{\overline{\Gamma}} \iff C_0^\infty(\Gamma) \overset{\text{dense}}{\subset} \{v \in H^{-1/2}(\mathbb{R}^{n-1}) : \operatorname{supp} v \subset \overline{\Gamma}\})$$

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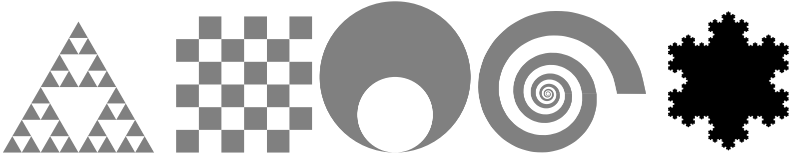

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Cases with $\widetilde{H}^{-1/2}(\Gamma) \neq H^{-1/2}_{\overline{\Gamma}}$ constructed using characterisation:

If $s \leq 0$, $\operatorname{int}(\overline{\Gamma})$ is C^0 then $\qquad \widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}} \iff H^{-s}_{\overline{\Gamma}\setminus\Gamma} = \{0\}$.

BIE Variational Formulation

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If Ω is the screen, then the (standard) variational formulation is:

Find $\phi \in H := \widetilde{H}^{-1/2}(\Omega)$ s.t.

$$a(\phi, \psi) = \langle g, \psi \rangle_{H^* \times H}, \quad \forall \psi \in H,$$

where

$$g := \gamma^{\pm} u^i|_{\Omega} \in H^{1/2}(\Omega) \cong H^*, \quad a(\phi, \psi) := \langle S_\Omega \phi, \psi \rangle_{H^* \times H}, \;\; \forall \phi, \psi \in H.$$

N.B. $a(\cdot, \cdot)$ is continuous and coercive (Ha Duong 1992, C-W, H 2015).

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If Ω is the screen, then the (standard) variational formulation is:

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$$a(\phi, \psi) = \langle g, \psi \rangle_{H^* \times H}, \quad \forall \psi \in H,$$

where

$$g := \gamma^{\pm} u^i|_\Omega \in H^{1/2}(\Omega) \cong H^*, \quad a(\phi, \psi) := \langle S_\Omega \phi, \psi \rangle_{H^* \times H}, \;\; \forall \phi, \psi \in H.$$

N.B. $a(\cdot, \cdot)$ is continuous and coercive (Ha Duong 1992, C-W, H 2015).

If $\Gamma \subset \Omega$ is the screen the variational formulation is:

Find $\phi \in V$ s.t.

$$a(\phi, \psi) = \langle g, \psi \rangle_{H^* \times H}, \quad \forall \psi \in V,$$

where

$$V := \begin{cases} \widetilde{H}^{-1/2}(\Gamma), & \Gamma \text{ open} \\ H_\Gamma^{-1/2}, & \Gamma \text{ compact} \end{cases}$$

N.B. Well-posed by Lax-Milgram.

Convergence of BEM: abstract framework

$a(\cdot, \cdot)$ is continuous and coercive on Hilbert space H, closed subspace $V \subset H$, $g \in H^*$.

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Approximating sequence. Given closed subspace $V_j \subset H$, find $\phi_j \in V_j$ s.t.

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Céa's Lemma. Suppose each $V_j \subset V$. Then

$$\phi_j \to \phi, \;\; \forall g \in H^* \Leftrightarrow V_j \xrightarrow{\mathcal{M}} V,$$

where $V_j \xrightarrow{\mathcal{M}} V$ means that

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Indeed,

$$\|\phi - \phi_j\| \leq c \inf_{\psi_j \in V_j} \|\phi - \psi_j\|.$$

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- $\forall (j_m)$ subseq. of $\mathbb{N}$, $v_{j_m} \in V_{j_m}$, $v_{j_m} \rightharpoonup v$, then $v \in V$ (weak closure)

This is Mosco convergence (Mosco 1969).

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Open Problem. Replacement for $\|\phi - \phi_j\| \leq c \inf_{\psi_j \in V_j} \|\phi - \psi_j\|$?
(This doesn't hold, for example, if $V_j \supset V$.)

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Elementary examples of Mosco convergence. $V_j \xrightarrow{\mathcal{M}} V$ if **either**

$$V_1 \subset V_2 \subset \ldots \text{ and } V = \bigcup_{j=1}^{\infty} V_j \quad \textbf{or} \quad V_1 \supset V_2 \supset \ldots \text{ and } V = \bigcap_{j=1}^{\infty} V_j$$

Prefractal to fractal convergence of BVPs

Let Γ_j be a sequence of "prefractals" approximating "fractal" Γ.
Denote ϕ_j and ϕ the corresponding variational BIE solutions.

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where $V_j = \begin{cases} \widetilde{H}^{-1/2}(\Gamma_j) & \Gamma_j \text{ open} \\ H^{-1/2}_{\Gamma_j} & \Gamma_j \text{ comp.} \end{cases}$ $\quad V = \begin{cases} \widetilde{H}^{-1/2}(\Gamma) & \Gamma \text{ open} \\ H^{-1/2}_{\Gamma} & \Gamma \text{ comp.} \end{cases}$

Definition of Mosco convergence: $H \supset W_j \xrightarrow{\mathcal{M}} W \subset H$ if

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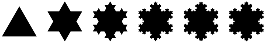
1 open $\Gamma_j \subset \Gamma_{j+1}$



2 compact $\Gamma_j \supset \Gamma_{j+1}$

3 non-nested $\Gamma_j \underset{\not\supset}{\not\subset} \Gamma_{j+1}$

The boundary element method (BEM)

Partition open prefractal Γ_j with pre-convex mesh

$$M_j = \{T_{j,1}, \ldots, T_{T_j,N_j}\},$$

where "pre-convex" means elements have disjoint convex hulls and $|\partial T_{j,l}| = 0$.

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Denote by $V_j^h \subset H^{-1/2}(\Gamma_\infty)$ the space of piecewise constants on M_j, and let ϕ_j^h denote the Galerkin BEM solution on Γ_j obtained by solving the variational problem on subspace V_j^h.

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Key approximation lemma (C-W, H, M, B 2019). For $-1 \leq s \leq 0$ and $0 \leq t \leq 1$, if $v \in H^t(\Gamma_j)$,

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We want to ensure BEM solution on Γ_j converges to BIE solution on Γ.

$$\begin{tikzcd} \phi_j^h \arrow[r, "h_j \to 0"] \arrow[rr, bend right, "?"'] & \phi_j \arrow[r, "j \to \infty"] & \phi \end{tikzcd}$$

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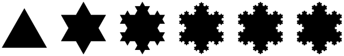
If $V_j^h \xrightarrow{\mathcal{M}} V$, (with either $V = \widetilde{H}^{-1/2}(\Gamma)$ or $V = H_\Gamma^{-1/2}$)
then BEM solution $\phi_j^h \to \phi$ in $H^{-1/2}(\Gamma_\infty)$ and $\mathcal{S}_{\Gamma_j}\phi_j^h \to u$ in $W^{1,\mathrm{loc}}(\mathbb{R}^n)$

BEM convergence: open screen

Assume all mesh elements have disjoint convex hulls and $|\partial T_{j,l}| = 0$.
(Allows multi-component elements!)

How to choose $(h_j)_{j=0}^{\infty}$ so that $V_j^h \xrightarrow{\mathcal{M}} V$?



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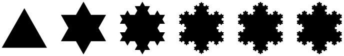

Theorem (CW, H, M 2019)

Let Γ, Γ_j be bounded open, $\Gamma_j \subset \Gamma_{j+1}$, $\Gamma = \bigcup_{j=0}^{\infty} \Gamma_j$.
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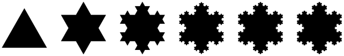

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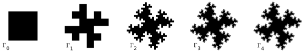
Proof: For $V_j^h \xrightarrow{\mathcal{M}} V = \widetilde{H}^{-1/2}(\Gamma) = \overline{C_0^\infty(\Gamma)}$ we have to show (i) strong approximability and (ii) weak closedness.
For (i), let $v \in C_0^\infty(\Gamma)$. Then $\exists j_*(v)$ s.t. $v \in C_0^\infty(\Gamma_j)$ for $j \geq j_*(v)$ and

$$\|\Pi_{L^2, V_j^h} v - v\|_{\widetilde{H}^{-1/2}(\Gamma)} \leq (h_j/\pi)^{1/2} \|v\|_{L^2(\Gamma_j)}.$$

For (ii), $V_j^h \subset \widetilde{H}^{-1/2}(\Gamma_j) \subset \widetilde{H}^{-1/2}(\Gamma)$. $\square$

Extends to some non-nested $\Gamma_j \overset{\not\subset}{\not\supset} \Gamma_{j+1}$, e.g.



BEM convergence: compact screen

When Γ is compact with empty interior and $\dim_{\mathrm{H}}\Gamma > 1$ this argument fails because $C_0^\infty(\Gamma^\circ) = \{0\}$ is not dense in $V = H_\Gamma^{-1/2} \neq \{0\}$.

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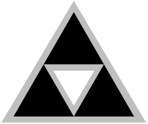
To obtain smooth approximation we mollify: this enlarges the support.
Currently only results for "thickened prefractals".



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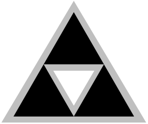

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If H_Γ^t is dense in $H_\Gamma^{-1/2}$ for $t \in (-1/2, 0)$ then $h_j = o(\epsilon_j^{-2t})$ suffices.

If Γ is d-set (e.g. IFS attractor), $h_j = o(\epsilon_j^\mu)$, $\mu > n - 1 - d$ is enough.

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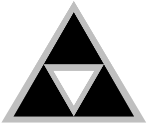

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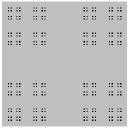
Proof of (i) (strong approx.): Let $v \in H_\Gamma^t$ and set $v_j := (\psi_{\varepsilon_j/2} * v)$, then

$$\|\Pi_{L^2, V_j^h} v_j - v_j\|_{\widetilde{H}^{-1/2}(\Gamma)} \leq (h_j/\pi)^{1/2} \|v_j\|_{L^2(\Gamma_j)} \leq (h_j/\pi)^{1/2} (\varepsilon_j/2)^t \|v\|_{H_\Gamma^t}.$$

Attractors of iterated function systems



Let $s_1, \dots, s_m : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be contracting similarities,
$s(U) := \bigcup_{m=1}^{\nu} s_m(U)$, for $U \subset \mathbb{R}^{n-1}$,
$\Gamma = s(\Gamma)$ the unique attractor (the fractal).

(Open set condition.)
Assume $O \neq \emptyset$ is open, convex, $s(O) \subset O$ and $s_m(O) \cap s_{m'}(O) = \emptyset$.
Define open prefractal sequence: $\Gamma_0 := O$, $\Gamma_{j+1} := s(\Gamma_j)$

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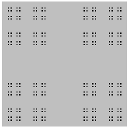

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Let $M_0 = \{T_{0,1}, \ldots, T_{0,N_0}\}$ be any convex mesh on Γ_0,
then define a convex mesh on Γ_j as

$$M_j := \left\{ s_{m_1} \circ \cdots \circ s_{m_j} (T_{0,l}) : 1 \leq m_{j'} \leq \nu \text{ for } j' = 1, \ldots, j \text{ and } 1 \leq l \leq N_0 \right\}.$$

Then Γ is a d-set, and BEM convergence holds if $\Gamma \subset O$.

The prefractals Γ_j are not the standard ones, but thickened.
Convergence extends to "pre-convex" meshes, each element with many components.

Cantor set

Cantor set is attractor of IFS with

$$s_1(t) = \alpha t, \quad s_2(t) = \alpha t + 1 - \alpha,$$

for some $\alpha \in (0, 1/2)$.

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BEM converges if we take

$$\Gamma_0 := (-\epsilon, 1+\epsilon), \quad \Gamma_{j+1} := s(\Gamma_j) := s_1(\Gamma_j) \cup s_2(\Gamma_j), \quad j = 0, 1, ...,$$

and mesh Γ_j so that the elements are the 2^j components of Γ_j, each of length $h_j = (1+2\epsilon)3^{-j}$.

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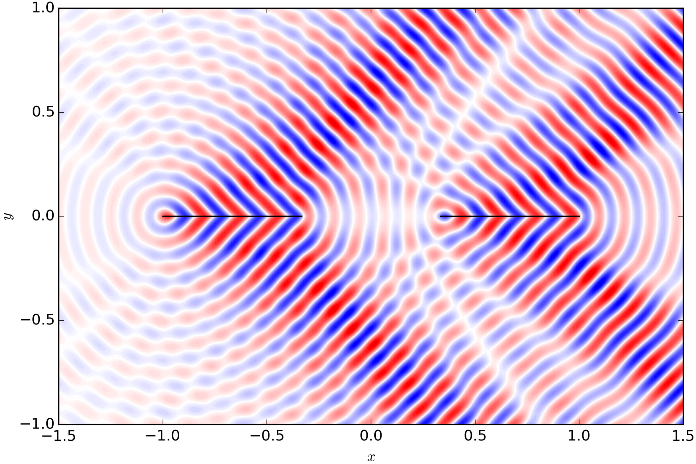
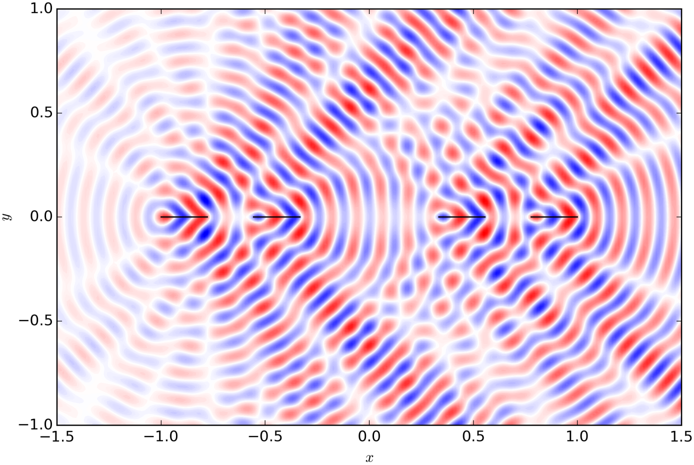
In fact BEM converges with only 1.3^j elements (and degrees of freedom) on Γ_j.

Cantor set: $\alpha = 1/3$



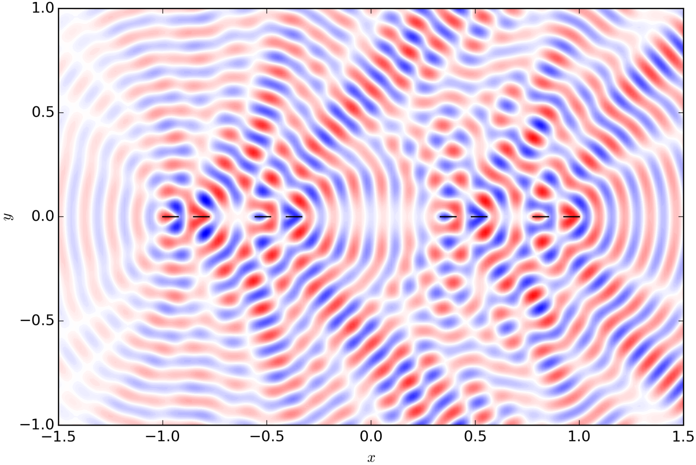
Γ_1 and $\mathrm{Re}\, u_1^h$

Cantor set: $\alpha = 1/3$



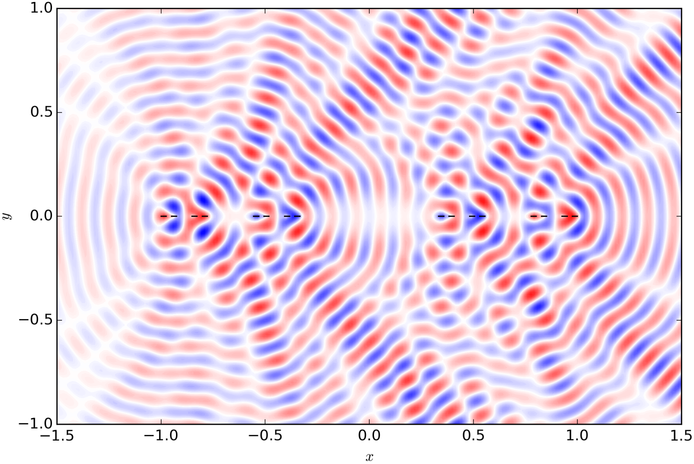
Γ_2 and $\mathrm{Re}\, u_2^h$

Cantor set: $\alpha = 1/3$



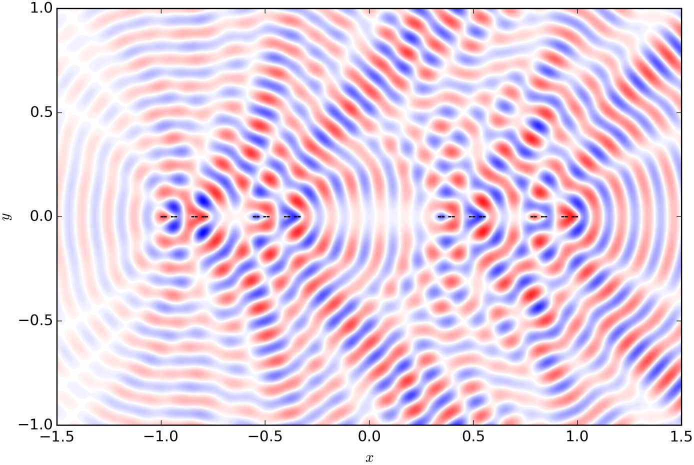
Γ_3 and $\mathrm{Re}\, u_3^h$

Cantor set: $\alpha = 1/3$



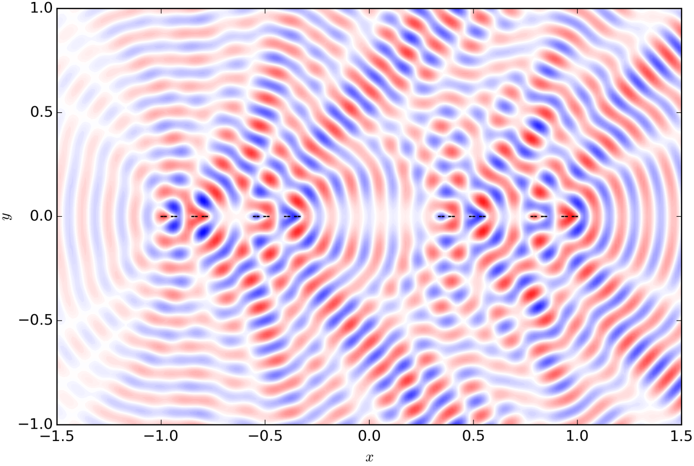
Γ_4 and $\mathrm{Re}\, u_4^h$

Cantor set: $\alpha = 1/3$



Γ_5 and $\mathrm{Re}\, u_5^h$

Cantor set: $\alpha = 1/3$



Γ_6 and $\mathrm{Re}\, u_6^h$

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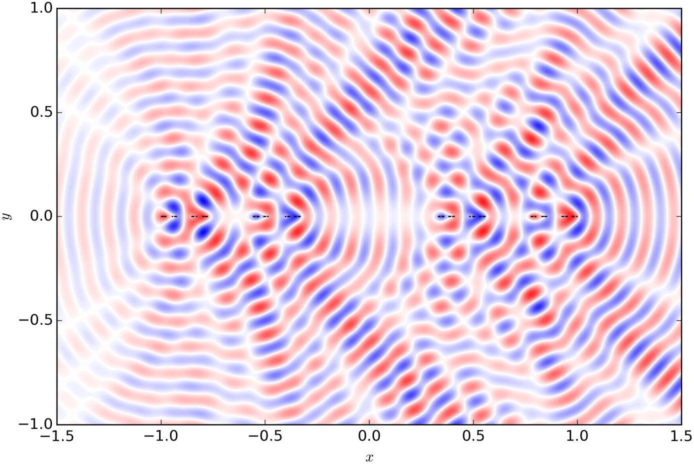
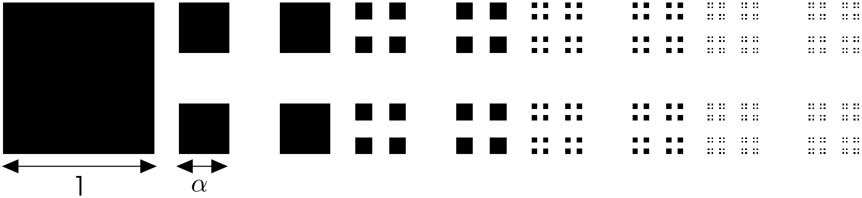


Γ_7 and $\operatorname{Re} u_7^h$

Cantor dust

Cantor dust is Cartesian product of 2 copies of Cantor set with parameter $0 < \alpha < 1/2$. Prefractals $\Gamma_0, \ldots, \Gamma_4$:



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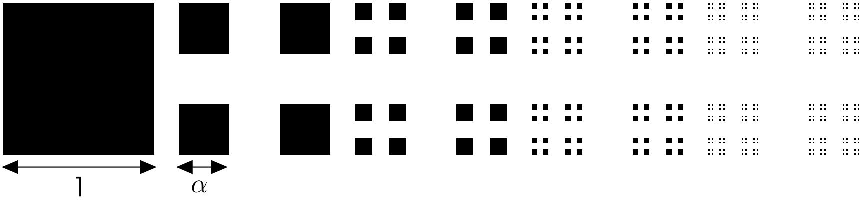
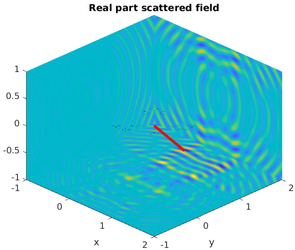


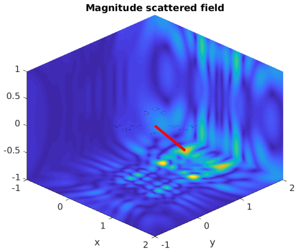
- Γ "audible" ($\phi \neq 0$) $\iff$ $\alpha > \frac{1}{4}$ $\iff$ $\dim_H(\Gamma) > 1$.
  ($\phi \neq 0 \iff \dim_H(\Gamma) > 1$ holds for all d-sets!)
- $H^{-1/2}_{\Gamma_j} \xrightarrow{\mathcal{M}} H^{-1/2}_{\Gamma}$, prefractal solutions ϕ_j converge to ϕ.
- BEM on thickened prefractals converge,
  1 DOF / prefractal component is enough.

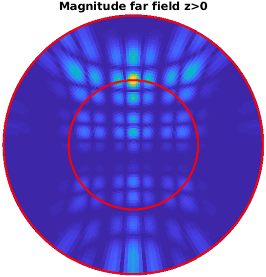
Actually BEM converges with even less than 1 DOF/component:
m_j components/element on Γ_j for $1 \leq m_j < 4^{(\frac{\log 4}{\log 1/\alpha} - 1)j}$.

Cantor dust: field plots

Prefractal level $j = 6$, $N_j = 4^6 = 4\,096$ DOFs, $k = 50$, $\alpha = 1/3$.







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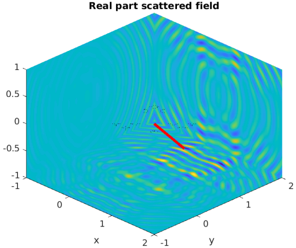
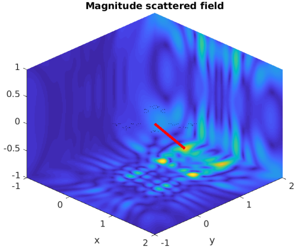
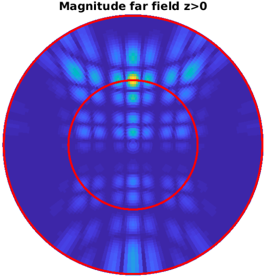
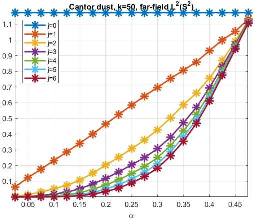




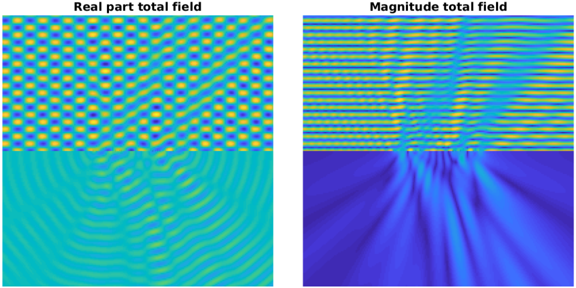




◀ L^2 norms of far-field, $\alpha \in (0.025, 0.475)$, prefractal levels $j = 0, \ldots, 6$.

Apertures

Field through bounded apertures in unbounded Neumann screens computed via Babinet's principle.



$n = 1$, Cantor set $\alpha = 1/3$, prefractal level 12:
field through 0-measure holes!

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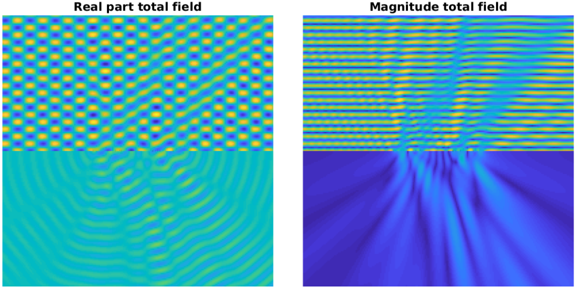

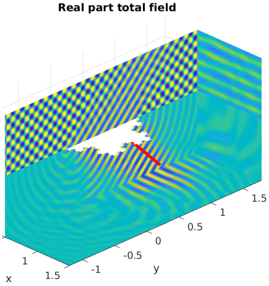

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Koch snowflake-shaped aperture △

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Open questions

- Impedance (Robin) bc's: Hewett and Gibbs in progress
- Regularity theory for the fractal solution
- Rates of convergence - something replacing best approximation?
- Convergence on standard prefractal sequences?
- Approximation **on** fractals - distributional elements?
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Thank you!