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The Impedance Boundary Value Problem for the Helmholtz Equation in a Half-Plane

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Dedicated to Professor George C. Hsiao on the occasion of his 60th birthday.

We prove unique existence of solution for the impedance (or third) boundary value problem for the Helmholtz equation in a half-plane with arbitrary L_∞ boundary data. This problem is of interest as a model of outdoor sound propagation over inhomogeneous flat terrain and as a model of rough surface scattering. To formulate the problem and prove uniqueness of solution we introduce a novel radiation condition, a generalization of that used in plane wave scattering by one-dimensional diffraction gratings. To prove existence of solution and a limiting absorption principle we first reformulate the problem as an equivalent second kind boundary integral equation to which we apply a form of Fredholm alternative, utilizing recent results on the solvability of integral equations on the real line in [5]. © 1997 by B. G. Teubner Stuttgart–John Wiley & Sons Ltd.

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1. Introduction

In this paper we prove unique existence of solution for the Helmholtz equation,

$$\Delta u + k^2 u = 0, \tag{1}$$

in the half-plane $U = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_2 > 0\}$, with impedance boundary condition

$$\frac{\partial u}{\partial n} - \mathrm{i}k\beta u = f \tag{2}$$

on $\Gamma = \{(x_1, 0) \colon x_1 \in \mathbb{R}\}$, and with arbitrary L_∞ boundary data β and f. In equation (2) n is the normal to Γ directed out of U.

This boundary value problem has been utilized (e.g. [14, 8]) as a model of mono-frequency ($\mathrm{e}^{-\mathrm{i}\omega t}$ time dependence) outdoor sound propagation over flat inhomogeneous terrain. In this context u is the scattered or reflected part of the acoustic field (the total field u_t, the sum of the incident and scattered fields, satisfies the

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homogeneous impedance boundary condition, $\partial u_t/\partial n - ik\beta u_t = 0$ on Γ). The function β is the relative surface admittance of the ground surface, and is a function of the angular frequency ω and of local properties of the ground surface.

Usually, it is the case when β is piecewise constant which is of interest, β taking a different value for each ground surface type (grassland, forest floor, road pavement, etc.). If, as is common, the ground surface is modelled as a rigid frame porous half-space, β takes values strictly in the sector of the complex plane $\mathrm{Re}\beta \geqslant 0$, $\mathrm{Im}\,\beta \leqslant 0$, $|\beta| \leqslant 1$ [1, 7]. In any case, if the ground surface is to absorb rather than emit energy, the condition $\mathrm{Re}\,\beta \geqslant 0$ must be satisfied.

The impedance boundary value problem in a half-plane is also of interest as a model of the scattering of an incident plane acoustic or electromagnetic wave by an infinite rough surface. (For the rough surface problem the Helmholtz equation holds in a region $D = \{(x_1, x_2) \in \mathbb{R}^2\colon x_2 > g(x_1)\}$, for some bounded and at least Lipschitz continuous function g.) Firstly, it is a model in that it has been suggested that, for certain parameter regimes [15, 21], scattering by a rough surface is adequately approximated by the impedance condition (2) applied on a flat surface.

Perhaps more significantly, this problem is also a model of rough surface scattering in the sense that, although geometrically simpler, it shares with it many theoretical features and difficulties. Specifically, for scattering of a plane wave by both an infinite rough surface and a flat impedance boundary, it is appropriate to look for the scattered field on the boundary in a function space of bounded continuous functions (rather than in $L_2(\Gamma)$, say), in that no decay of the scattered field with distance along the boundary can be expected. Further, for both problems, even for the specific case of plane wave incidence, the correct mathematical formulation is unclear, in particular the formulation of an appropriate radiation condition.

In section 3 of the paper we make a preliminary study of the much simpler Dirichlet boundary value problem in the half-plane with L_∞ boundary data on Γ. For the Dirichlet problem the solution can be given explicitly as a double-layer potential on Γ, with density the given L_∞ boundary data. For the case when the wave number $k > 0$ we construct an example to show that, although this solution is the physically correct one, in that it is the unique solution satisfying a limiting absorption principle, it may grow algebraically at a rate not exceeding $h^{1/2}$, where h is the distance from the boundary. To cope with the L_∞ boundary data we suppose that the solution on $\Gamma_h = \{(x_1, h)\colon x_1 \in \mathbb{R}\}$ approaches the data on Γ in a weak$*$ sense as $h \to 0^+$.

This study of the Dirichlet problem is of assistance in formulating the impedance boundary value problem for $k > 0$ in section 4. Specifically, as proposed in [6], as a radiation condition for the impedance problem we suppose that, in some half-plane $U_h = \{(x_1, x_2) \in \mathbb{R}^2\colon x_2 > h\}$, with $h \geqslant 0$, the solution u can be written as a double-layer potential on the boundary Γ_h, with some L_∞ density, so that u satisfies a Dirichlet problem in the half-plane U_h. This radiation condition appears to be novel and is shown in [6] to be a generalization of the usual radiation condition for plane wave scattering by a one-dimensional diffraction grating [18, 16].

Using this radiation condition we reformulate the boundary value problem as an equivalent second kind boundary integral equation and prove, for the case $\mathrm{Re}\,\beta \geqslant \eta > 0$, in what is the longest section of the paper, uniqueness of solution. Existence of solution follows from a form of Fredholm alternative applied to the boundary integral equation, using recent results on the solvability of integral equations on the real line in [5]. The argument depends on the local compactness of the boundary integral operator and is a generalization of the classical method, based on compactness of the integral operator, for proving existence of solution for scattering by smooth bounded obstacles via a second kind integral equation formulation and the Fredholm alternative [12]. We also obtain continuous dependence of the solution u in a weighted norm on the boundary data f, specifically that

$$\sup_{x \in \bar{U}} |\exp(x_2 \operatorname{Im} k)(1 + x_2)^{-1/2} u(x)| \leqslant C_P \|f\|_\infty. \tag{3}$$

The constant C_P, independent of f, is also independent of β, provided β takes values in the set P, where P is an arbitrary compact subset of the right-hand complex plane. Having proved unique existence of solution for $\operatorname{Re} k > 0$, $\operatorname{Im} k \geqslant 0$, it is straightforward to show that the solution selected by the radiation condition for $k > 0$ satisfies a limiting absorption principle.

It is anticipated that the radiation condition and uniqueness and existence proofs introduced will be useful in formulating and proving unique existence of solution for a wider range of problems of acoustic and electromagnetic wave scattering by unbounded obstacles, especially plane wave scattering by infinite one-dimensional rough surfaces [13, 11]. A related study, but restricted to the case $\operatorname{Im} k > 0$, of acoustic scattering by a sound-hard infinite cylinder, has been made in [17].

2. Notation and Preliminaries

Throughout, $x = (x_1, x_2)$, $y = (y_1, y_2)$ will denote points in $\mathbb{R}^2$. For $h \geqslant 0$, U_h will denote the half-plane, $U_h = \{x\colon x_2 > h\}$, and Γ_h its boundary, $\Gamma_h = \{x\colon x_2 = h\}$. We will abbreviate U_0 and Γ_0 by U and Γ, respectively. For all $x \in \bar{U}_h$, x'_h will denote the image of x in Γ_h, i.e. $x'_h := (x_1, 2h - x_2)$. We abbreviate x'_0 by x'.

For the most part our function space notation is standard. For $S \subset \mathbb{R}^2$, $C(S)$ will denote the set of functions continuous on S, and $BC(S)$ the set of functions bounded and continuous on S. The set $BC(S)$ with the normal vector space operations and the supremum norm, $\|\psi\|_\infty := \sup_{x \in S} |\psi(x)|$, forms a Banach space. We will also use $\|\cdot\|_\infty$ to denote the essential supremum norm on $L_\infty(\mathbb{R})$.

For $\varepsilon \geqslant 0$, let

$$L_{2,\varepsilon}(\mathbb{R}) := \left\{ \psi \in L_2^{\mathrm{loc}}(\mathbb{R})\colon \|\psi\|_{2,\varepsilon} := \left\{ \int_{-\infty}^{+\infty} (1 + t^2)^{-\varepsilon} |\psi(t)|^2 \, \mathrm{d}t \right\}^{1/2} < \infty \right\},$$

$$L'_{2,\varepsilon}(\mathbb{R}) := \left\{ \psi \in L_2^{\mathrm{loc}}(\mathbb{R})\colon \|\psi\|'_{2,\varepsilon} := \left\{ \sup_{A > 0} (1 + A^2)^{-\varepsilon} \int_{-A}^{A} |\psi(t)|^2 \, \mathrm{d}t \right\}^{1/2} < \infty \right\}.$$

Clearly, $L_2(\mathbb{R}) = L_{2,0}(\mathbb{R}) = L'_{2,0}(\mathbb{R})$. We also note the following imbeddings.

Lemma 2.1. *For* $0 \leqslant \varepsilon_1 < \varepsilon_2$,

$$L_{2,\varepsilon_1}(\mathbb{R}) \subset L'_{2,\varepsilon_1}(\mathbb{R}) \subset L_{2,\varepsilon_2}(\mathbb{R}),$$

and the imbeddings are continuous.

Proof. For $\psi \in L_2^{\mathrm{loc}}(\mathbb{R})$ and $0 \leqslant \varepsilon_1 < \varepsilon_2$,

$$(1 + A^2)^{-\varepsilon_1} \int_{-A}^{A} |\psi(t)|^2 \, \mathrm{d}t \leqslant \int_{-A}^{A} (1 + t^2)^{-\varepsilon_1} |\psi(t)|^2 \, \mathrm{d}t \tag{4}$$

and

$$\int_{-A}^{A}(1+s^2)^{-\varepsilon_2}|\psi(s)|^2\,\mathrm{d}s=(1+A^2)^{-\varepsilon_2}\int_{-A}^{A}|\psi(t)|^2\,\mathrm{d}t$$

$$+\,2\varepsilon_2\int_{-A}^{A}\left\{\int_0^s|\psi(t)|^2\,\mathrm{d}t\right\}s(1+s^2)^{-1-\varepsilon_2}\,\mathrm{d}s, \tag{5}$$

equation (5) obtained by integration by parts. Equations (4) and (5) give the continuity of the imbeddings $L_{2,\varepsilon_1}(\mathbb{R})\subset L'_{2,\varepsilon_1}(\mathbb{R})$ and $L'_{2,\varepsilon_1}(\mathbb{R})\subset L_{2,\varepsilon_2}(\mathbb{R})$, respectively. $\square$

For $u\in C(U)$ and $h>0$, define $u_h\in C(\mathbb{R})$ by $u_h(s):=u((s,h))$, $s\in\mathbb{R}$, so that u_h is the restriction of u to Γ_h. If $u\in C(\bar{U})$ then we can define u_0 by the same formula with $h=0$. If $u\in C^1(U)$, define also $u_h^*\in C(\mathbb{R})$ by $u_h^*(s)=\partial u((s,h))/\partial h$, $s\in\mathbb{R}$, so that u_h^* is the restriction of $\partial u/\partial x_2$ to Γ_h.

We will make use of the notion of weak$*$ convergence in $L_\infty(\mathbb{R})$ (see e.g. [20]). For a sequence $\{\psi_n\}\subset L_\infty(\mathbb{R})$ and $\psi\in L_\infty(\mathbb{R})$ we will say that $\{\psi_n\}$ converges weak$*$ to ψ, and write $\psi_n\overset{w^*}{\to}\psi$, if

$$\int_{-\infty}^{+\infty}\psi_n\varphi\to\int_{-\infty}^{+\infty}\psi\varphi,\quad\forall\varphi\in L_1(\mathbb{R}). \tag{6}$$

Since $L_1(\mathbb{R})$ is a Banach space, it follows from the uniform boundedness theorem that if $\psi_n\overset{w^*}{\to}\psi$ then $\{\psi_n\}$ is bounded in $L_\infty(\mathbb{R})$ and $\|\psi\|_\infty\leqslant\sup_n\|\psi_n\|_\infty$. There follows a useful characterization of weak$*$ convergence, that

$$\psi_n\overset{w^*}{\to}\psi\Leftrightarrow\sup_n\|\psi_n\|_\infty<\infty,\ \int_{-\infty}^{+\infty}\psi_n\varphi\to\int_{-\infty}^{+\infty}\psi\varphi,\quad\forall\varphi\in C_0^\infty(\mathbb{R}), \tag{7}$$

where $C_0^\infty(\mathbb{R})$ denotes the set of C^∞ compactly supported functions on $\mathbb{R}$. Thus, $\psi_n\overset{w^*}{\to}\psi$ if and only if $\{\psi_n\}$ is bounded in $L_\infty(\mathbb{R})$ and $\{\psi_n\}$ converges to ψ in a distributional sense.

Many of the equations presented can be written compactly using a convolution notation. For $\varphi\in L_1(\mathbb{R})$ and $\psi\in L_p(\mathbb{R})$ define $\varphi*\psi$ by

$$\varphi*\psi(s):=\int_{-\infty}^{+\infty}\varphi(s-t)\psi(t)\,\mathrm{d}t. \tag{8}$$

From Young's Theorem, $\varphi*\psi(s)$, defined by (8), exists for almost all $s\in\mathbb{R}$, and $\varphi*\psi\in L_p(\mathbb{R})$ with

$$\|\varphi*\psi\|_p\leqslant\|\varphi\|_1\,\|\psi\|_p. \tag{9}$$

For $p=\infty$ we have that $\varphi*\psi(s)$ is well-defined for every $s\in\mathbb{R}$ and that $\varphi*\psi\in BC(\mathbb{R})$.

It follows from the above remarks that, if $\kappa\in L_1(\mathbb{R})$, then the operator $\mathscr{K}$, defined by $\mathscr{K}\psi=\kappa*\psi$, is a bounded operator on $L_p(\mathbb{R})$, for $1\leqslant p\leqslant\infty$, and is a bounded operator from $L_\infty(\mathbb{R})$ to BC$(\mathbb{R})$. $\mathscr{K}$ also has the following mapping properties.

Lemma 2.2. [19]. *If* $\int_{-\infty}^{+\infty}(1+|t|)^q|\kappa(t)|\,\mathrm{d}t<\infty$ *for some* $q\geqslant 0$ *then* $\mathscr{K}$ *is a bounded operator on* $L_{2,\varepsilon}(\mathbb{R})$ *for* $0\leqslant\varepsilon\leqslant q$.

For $\{\psi_n\} \subset BC(\mathbb{R})$, $\psi \in BC(\mathbb{R})$, say that ψ_n *converges strictly to* ψ and write $\psi_n \xrightarrow{s} \psi$ if $\sup_n \|\psi_n\|_\infty < \infty$ and $\psi_n(s) \to \psi(s)$ uniformly on finite intervals of $\mathbb{R}$. This is convergence of $\{\psi_n\}$ to ψ in the strict topology of [2]. Clearly, $\psi_n \xrightarrow{s} \psi \Rightarrow \psi_n \xrightarrow{w^*} \psi$. The following lemma follows immediately from [5, Theorem 4.2 (iv)] and (7).

Lemma 2.3. *If $\kappa \in L_1(\mathbb{R})$ and $\psi_n \xrightarrow{w^*} \psi$ then $\kappa * \psi \xrightarrow{s} \kappa * \psi$.*

It is straightforward to see that Lemma 2.3 can be extended further to show that, if $\{\kappa_n\} \subset L_1(\mathbb{R})$, $\kappa \in L_1(\mathbb{R})$, then

$$\|\kappa_n - \kappa\|_1 \to 0, \ \psi_n \xrightarrow{w^*} \psi \Rightarrow \kappa_n * \psi_n \xrightarrow{s} \kappa * \psi. \tag{10}$$

Let $\mathscr{F}$ denote the operation of Fourier transformation on $\mathbb{R}$, defined, for $\psi \in L_1(\mathbb{R})$, by

$$\mathscr{F}\psi(\xi) = \int_{-\infty}^{+\infty} \psi(s) \mathrm{e}^{-\mathrm{i}s\xi} \, \mathrm{d}s, \quad \xi \in \mathbb{R},$$

and abbreviate $\mathscr{F}\psi$ by $\hat{\psi}$. If $\varphi, \psi \in L_1(\mathbb{R})$ then $\hat{\varphi}\hat{\psi} \in BC(\mathbb{R})$ and $\mathscr{F}(\varphi * \psi) = \hat{\varphi}\hat{\psi}$.

We introduce a few further notations. For $x \in \mathbb{R}^2$ and $A > 0$, let $B_A(x)$ denote the open ball, $B_A(x) := \{y \in \mathbb{R}^2 \colon |y - x| < A\}$. Let

$$\Phi(x, y) := \frac{\mathrm{i}}{4} H_0^{(1)}(k|x - y|). \quad x, y \in \mathbb{R}^2, x \neq y,$$

so that Φ is the standard fundamental solution of the Helmholtz equation in $\mathbb{R}^2$. For $A > 0$, let $\chi_{[-A, A]} \in L_\infty(\mathbb{R})$ denote the characteristic function of the interval $[-A, A]$.

3. The Dirichlet problem

We first consider the Dirichlet boundary value problem for the Helmholtz equation in a half-plane, for which a solution can be given explicitly. We study a general form of the Dirichlet problem, with arbitrary L_∞ boundary data, so that the solution u is not necessarily continuous in $\bar{U}$, and u_h converges to f, the given boundary values of u, only in a weak sense as $h \to 0$. The solution u is not necessarily in the Sobolev space $H^1_{\mathrm{loc}}(U)$ either, for if f has a simple discontinuity, for example if $f = \chi_{[-1,1]}$, then $f \notin H^{1/2}_{\mathrm{loc}}(\mathbb{R})$ and, by the trace theorem, ∇u is not locally square integrable in U.

Consider then the following boundary value problem:

BVP1. *Given $f \in L_\infty(\mathbb{R})$ and $k \in \mathbb{C}$ with* $\operatorname{Im} k \geqslant 0$, $\operatorname{Re} k > 0$, *find $u \in C^2(U)$ satisfying*

(*i*) the *Helmholtz equation*, $\Delta u + k^2 u = 0$ in U;

(*ii*) *for some $a \in \mathbb{R}$ and every $h > 0$,*

$$\sup_{x \in U_h} |(1 + x_2)^a u(x)| < \infty; \tag{11}$$

(*iii*) $u_h \xrightarrow{w^*} f$ *as $h \to 0$.*

Remark 3.1. If $u \in C^2(U)$ satisfies (i) and (ii) then, by standard local regularity arguments, $u \in C^\infty(U)$ and the same bound as (11) holds for all the derivatives of u. In particular, for all $h > 0$,

$$\sup_{x \in U_h} |(1 + x_2)^a \nabla u(x)| < \infty. \tag{12}$$

Further, if $u \in C^2(U)$ satisfies (11), then $u_h \in BC(\mathbb{R})$ for every $h > 0$, and, if also $u_h \overset{w*}{\to} f$, then (see (7)), $\|u_h\|_\infty = O(1)$ as $h \to 0$, so that (11) holds with $h = 0$.

The above boundary value problem contains no radiation condition and is not uniquely solvable when $k > 0$; for example, $u(x) = \sin(kx_2)$ satisfies BVP1 with $f = 0$ when $k > 0$; though not when $\operatorname{Im} k > 0$ for then (11) is violated.

To write down a particular solution of BVP1 we introduce the Dirichlet Green's function, $G_{D,h}$, for the half-plane U_h. For $h > 0$ define

$$G_{D,h}(x, y) := \Phi(x, y) - \Phi(x'_h, y), \quad x, y \in \overline{U_h},\ x \neq y.$$

For $\operatorname{Im} k > 0$ (for which $G_{D,h}(x, y)$ decays exponentially as $|x - y| \to \infty$) we can obtain a form of Green's representation theorem for u, the solution of BVP1 (cf. [12]). Applying Green's second theorem to u and $G_{D,h}(x,.)$ in the region $U_h \cap B_R(0) \setminus \overline{B_\varepsilon(x)}$, and letting $\varepsilon \to 0$ and $R \to \infty$ (and noting, from (11) and (12), that u and ∇u have at most algebraic growth at infinity), we obtain that

$$u(x) = \int_{\Gamma_h} \frac{\partial G_{D,h}(x, y)}{\partial y_2} u(y)\, ds(y) = 2 \int_{\Gamma_h} \frac{\partial \Phi(x, y)}{\partial y_2} u(y)\, ds(y), \quad x \in U_h. \tag{13}$$

Defining, for $h > 0$,

$$\kappa_h(s) := 2 \left. \frac{\partial \Phi((s, h), y)}{\partial y_2} \right|_{y=0} = \frac{ihk H_1^{(1)}(k\sqrt{(s^2 + h^2)})}{2\sqrt{(s^2 + h^2)}}, \quad s \in \mathbb{R}, \tag{14}$$

(13) can be written more compactly as

$$u_H = \kappa_{H-h} * u_h, \quad H > h. \tag{15}$$

From standard asymptotic properties of the Hankel function it is easy to establish that, for $0 < h \leqslant 1$ and some constant $C > 0$,

$$|\kappa_h(s)| \leqslant \begin{cases} C \frac{h}{s^2 + h^2}, & |s| \leqslant 1, \\ C\,|s|^{-3/2}, & |s| \geqslant 1, \end{cases} \tag{16}$$

while, for $h \geqslant 1$,

$$|\kappa_h(s)| \leqslant \frac{Ch \exp(-\operatorname{Im} kh)}{(s^2 + h^2)^{3/4}}, \quad s \in \mathbb{R}. \tag{17}$$

Since the Hankel function, $H_1^{(1)}(z)$, is continuous in the quadrant $\operatorname{Im} z \geqslant 0$, $\operatorname{Re} z > 0$, it follows from the dominated convergence theorem that, for $h > 0$, $\kappa_h \in L_1(\mathbb{R})$ and depends continuously in norm on h, and

$$\|\kappa_h\|_1 = O(1),\ h \to 0, \ \|\kappa_h\|_1 = O(h^{1/2} \exp(-\operatorname{Im} kh)), h \to \infty. \tag{18}$$

Since $u_h \overset{w^*}{\to} f$ as $h \to 0$, it follows from (15) and (10) that

$$u_h = \kappa_h * f, \quad h > 0, \tag{19}$$

i.e. that

$$u(x) = 2\int_\Gamma \frac{\partial \Phi(x, y)}{\partial y_2} f(y)\, \mathrm{d}s(y), \quad x \in U. \tag{20}$$

We have shown the following result.

Theorem 3.1. *If* u *satisfies* $BVP1$ *and* $\operatorname{Im} k > 0$ *then* u *is given by* (20).

The following converse of Theorem 3.1 holds for all k with $\operatorname{Im} k \geqslant 0$, $\operatorname{Re} k > 0$.

Theorem 3.2. *If* $f \in L_\infty(\mathbb{R})$ *then* u, *defined by* (20), *satisfies* $BVP1$. *Further, for some constant* $C > 0$ *independent of* f,

$$\sup_{x \in U} |\exp(x_2 \operatorname{Im} k)(1 + x_2)^{-1/2} u(x)| \leqslant C \|f\|_\infty. \tag{21}$$

Also, if $f \in BC(\mathbb{R})$, *then* $u \in C(\bar{U})$ *and* $u_0 = f$.

Proof. To see that $u \in C^2(U)$ and satisfies the Helmholtz equation, let $f_n = \chi_{[-n,n]} f$, $n \in \mathbf{N}$. Define $u^{(n)}$ by (20) with f replaced by f_n. Then, clearly, $u^{(n)}$, a standard double-layer potential, satisfies $u^{(n)} \in C^2(U)$ and $\Delta u^{(n)} + k^2 u^{(n)} = 0$ in U. Further, using (16) and (17), we can see that $u^{(n)}$ converges to u uniformly on compact subsets of u, so that also $u \in C^2(U)$ and $\Delta u + k^2 u = 0$ in U. The bound (21), with $C := \sup_{h>0} \exp(h \operatorname{Im} k)(1 + h)^{-1/2} \|\kappa_h\|_1$, follows from (9), (19) and (18). Thus, u satisfies conditions (i) and (ii) of BVP1.

From (16) and standard jump relations for double-layer potentials [12],

$$\varphi \in BC(\mathbb{R}) \Rightarrow \kappa_h * \varphi \overset{s}{\to} \varphi \text{ as } h \to 0. \tag{22}$$

Thus, if $f \in BC(\mathbb{R})$, then $u \in C(\bar{U})$ and $u_0 = f$. In the general case $f \in L_\infty(\mathbb{R})$ we have from (21) that $\|u_h\|_\infty = O(1)$ as $h \to 0$. Thus, by the characterization (7), to show that $u_h \overset{w^*}{\to} f$ we need only show that $I_h := \int_{-\infty}^{+\infty} (\kappa_h * f - f)\varphi \to 0$ as $h \to 0$, for every $\varphi \in C_0^\infty(\mathbb{R})$.

Now, if $\varphi \in C_0^\infty(\mathbb{R})$ and $f \in L_\infty(\mathbb{R})$,

$$\begin{aligned} I_h &= \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} \kappa_h(s - t) f(t)\, \mathrm{d}t - f(s) \right\} \varphi(s)\, \mathrm{d}s \\ &= \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} \kappa_h(s - t) \varphi(s)\, \mathrm{d}s - \varphi(t) \right\} f(t)\, \mathrm{d}t \\ &= \int_{-\infty}^{+\infty} (\kappa_h * \varphi - \varphi) f, \end{aligned}$$

using Fubini's theorem to justify reversing the order of integration. Let A be chosen sufficiently large so that the support of φ is contained in $[-A, A]$. Then, for $B \geqslant A$,

$$|I_h| \leqslant \int_{\mathbb{R}\setminus[-B,B]} |(\kappa_h * \varphi) f| + \|f\|_\infty \int_{-B}^{B} |\kappa_h * \varphi - \varphi|. \tag{23}$$

Now, for every $B \geqslant A$ the second integral $\to 0$ as $h \to 0$, by (22), and it follows from (16) that, for $0 < h \leqslant 1$ and $|s| \geqslant A + 1$,

$$|\kappa_h * \varphi(s)| \leqslant 2A \|\varphi\|_\infty C(|s| - A)^{-3/2},$$

so that the first integral in (23) tends to zero as $B \to \infty$, uniformly in h. Thus, $I_h \to 0$ as $h \to 0$. □

We have shown in the case $\operatorname{Im} k > 0$ that BVP1 has precisely one solution, given by (20). In the case $k > 0$, in which (20) is not the unique solution of BVP1, it is sensible to select it as the 'physically correct' solution since it satisfies the *limiting absorption principle* given in the next theorem. Temporarily, within this theorem, let $u^{(\lambda)}$ denote the solution of BVP1 given by (20) when $k = \lambda$.

Theorem 3.3. *For $k > 0$ and all $x \in U$, $u^{(k+i\varepsilon)}(x) \to u^{(k)}(x)$ as $\varepsilon \to 0^+$.*

Proof. Temporarily, denote κ_h by $\kappa_h^{(k)}$ to indicate its dependence on k. Then, by the dominated convergence theorem, $\kappa_h^{(k)} \in L_1(\mathbb{R})$ depends continuously in norm on k in $\operatorname{Im} k \geqslant 0$, $\operatorname{Re} k > 0$ (note that (16) and (17) hold with the constant C independent of k provided that k is restricted to a compact subset of the first quadrant). But, from (9) and (19), $\|u_h^{(k+i\varepsilon)} - u_h^{(k)}\|_\infty \leqslant \|\kappa_h^{(k+i\varepsilon)} - \kappa_h^{(k)}\|_1 \|f\|_\infty$, and the result follows. □

Although it satisfies the above limiting absorption condition, the solution (20) for $k > 0$ does not have all the characteristics we associate with a radiating wave. Specifically, the bound (21) suggests that, if $k > 0$, $u(x)$ may increase in magnitude as $x_2 \to \infty$ (though $u(x)$ must decrease exponentially as $x_2 \to \infty$ if $\operatorname{Im} k > 0$).

To see that an increase can be achieved in the case $k > 0$ consider the following construction. For $h > 0$ define $f_h \in L_\infty(\mathbb{R})$ by $f_h(t) = \exp(i(\pi/4 - k\sqrt{t^2 + h^2}))$, $t \in \mathbb{R}$. Choose a sequence $\{a_n\} \subset \mathbb{R}$ such that $0 = a_1 < a_2 < \cdots$ and $a_{n+1}/a_n \to \infty$ as $n \to \infty$, and define $f \in L_\infty(\mathbb{R})$ with $\|f\|_\infty = 1$ by

$$f(t) := f_{a_n}(t), \quad a_{n-1} \leqslant |t| < a_{n+1},\ n = 2, 4, \ldots \tag{24}$$

Let u denote the solution to the Dirichlet problem defined by equation (20). Then at $x = (0, h)$, for $h > 0$,

$$\begin{aligned} u(x) &= \frac{ihk}{2} \int_{-\infty}^{+\infty} \frac{H_1^{(1)}(k\sqrt{(t^2 + h^2)})}{\sqrt{(t^2 + h^2)}} f(t)\,dt \\ &= \frac{hk^{1/2} e^{-i\pi/4}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\exp(ik\sqrt{(t^2 + h^2)})}{(t^2 + h^2)^{3/4}} f(t)\,dt + O(h^{-1/2}) \end{aligned}$$

as $h \to \infty$, since $H_1^{(1)}(s) = \sqrt{(2/\pi s)}\exp(\mathrm{i}(s - 3\pi/4)) + O(s^{-3/2})$ as $s \to \infty$. Now, at $x = x_n := (0, a_n)$, for $n = 2, 4, \ldots, u(x_n) = b_n + c_n + O(a_n^{-1/2})$ as $n \to \infty$, where

$$b_n := \frac{a_n k^{1/2} \mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\exp(\mathrm{i}k\sqrt{(t^2 + a_n^2)})}{(t^2 + a_n^2)^{3/4}} f_{a_n}(t)\,\mathrm{d}t,$$

$$c_n := \frac{a_n k^{1/2} \mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\exp(\mathrm{i}k\sqrt{(t^2 + a_n^2)})}{(t^2 + a_n^2)^{3/4}} (f(t) - f_{a_n}(t))\,\mathrm{d}t.$$

Now $b_n = I\sqrt{(2a_n k/\pi)}$, where $I = \int_0^\infty (1 + s^2)^{-3/4}\,\mathrm{d}s = \sqrt{\pi}\,\Gamma(\frac{1}{4})/(2\Gamma(\frac{3}{4}))$, and

$$|c_n| \leqslant 2\sqrt{\frac{(2a_n k)}{\pi}}\left(\int_0^{a_{n-1}/a_n} \frac{\mathrm{d}s}{(s^2 + 1)^{3/4}} + \int_{a_{n+1}/a_n}^{+\infty} \frac{\mathrm{d}s}{(s^2 + 1)^{3/4}}\right),$$

since $\|f - f_{a_n}\|_\infty \leqslant 2$ and $f(t) - f_{a_n}(t)$ vanishes on $a_{n-1} < |t| < a_{n+1}$. Since $a_{n+1}/a_n \to \infty$ as $n \to \infty$, $c_n = o(a_n^{1/2})$ and so

$$u(x_n) \sim \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\sqrt{\frac{a_n k}{2}} \tag{25}$$

as $n \to \infty$. Thus, the exponent $-\frac{1}{2}$ in (21) cannot be improved in the case $k > 0$.

The above construction can be modified to produce the same growth with Dirichlet data in $C^\infty(\mathbb{R})$. With the same definition of u and f, for some H in the range $0 < H < a_2$, let $\tilde{f} := u_H = \kappa_H * f$. Then, by Remark 3.1, $\tilde{f}$ and all its derivatives are in $BC(\mathbb{R})$. From (14) and standard tables of Fourier transforms,

$$\hat{\kappa}_h(\xi) = \exp(\mathrm{i}h\sqrt{(k^2 - \xi^2)}), \quad \xi \in \mathbb{R}, \tag{26}$$

where $\mathrm{Re}\sqrt{k^2 - \xi^2}$, $\mathrm{Im}\sqrt{k^2 - \xi^2} \geqslant 0$. Clearly, for $H > h > 0$, $\hat{\kappa}_{H-h}\hat{\kappa}_h = \hat{\kappa}_H$, so that

$$\kappa_H = \kappa_{H-h} * \kappa_h, \quad H > h > 0. \tag{27}$$

Defining $\tilde{u}$ by the right-hand side of (20) with f replaced by $\tilde{f}$, and defining $\tilde{x}_n := (0, a_n - H)$, it follows from (27) that $\tilde{u}(\tilde{x}_n) = u(x_n)$ so that $\tilde{u}(\tilde{x}_n) \sim (\Gamma(\frac{1}{4})/\Gamma(\frac{3}{4}))\sqrt{a_n k/2}$ as $n \to \infty$.

If u satisfies (19) and $f \in L_2(\mathbb{R}) \cap L_\infty(\mathbb{R})$ then (recall (9)), $u_h \in L_2(\mathbb{R}) \cap L_\infty(\mathbb{R})$, $h > 0$. Further, $\hat{f} \in L_2(\mathbb{R})$ and, taking Fourier transforms in (19),

$$\hat{u}_h(\xi) = \hat{\kappa}_h(\xi)\hat{f}(\xi) = \exp(\mathrm{i}h\sqrt{(k^2 - \xi^2)})\hat{f}(\xi), \tag{28}$$

for almost all $\xi \in \mathbb{R}$. Thus, taking inverse Fourier transforms (note that $\exp(\mathrm{i}h\sqrt{k^2 - \xi^2})\hat{f}(\xi) \in L_1(\mathbb{R})$ for $h > 0$),

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(\mathrm{i}(x_2\sqrt{(k^2 - \xi^2)} + x_1\xi))\hat{f}(\xi)\,\mathrm{d}\xi, \quad x \in U. \tag{29}$$

For $k > 0$ this is a representation of the solution (20) as a linear combination of upward propagating plane waves ($\exp(\mathrm{i}(x_2\sqrt{k^2 - \xi^2} + x_1\xi))$ for $|\xi| \leqslant k$) and evanescent surface waves ($\exp(\mathrm{i}(x_2\sqrt{k^2 - \xi^2} + x_1\xi))$ for $|\xi| > k$) and shows that if $f \in L_2(\mathbb{R}) \cap L_\infty(\mathbb{R})$ then u, given by (20), satisfies $\|u_h\|_\infty = O(1)$ as $h \to \infty$, so that u is bounded in U. (In fact, a careful analysis of (29) [4] gives further that $\|u_h\|_\infty = o(1)$ as $h \to \infty$.)

The following lemma (cf. [3]), which follows from the above results, will be very useful in the next section.

Lemma 3.4. *If u satisfies* (19) *with* $f \in L_2(\mathbb{R}) \cap L_\infty(\mathbb{R})$, *then* u_h, $u_h^* \in L_2(\mathbb{R}) \cap BC(\mathbb{R})$ *for every* $h > 0$ *and, for* $k > 0$,

$$\operatorname{Im} \int_{-\infty}^{+\infty} \bar{u}_h u_h^* = \int_{-k}^{k} \sqrt{(k^2 - \xi^2)}\, |\hat{u}_h(\xi)|^2 \, \mathrm{d}\xi. \tag{30}$$

Proof. We have seen above that if u satisfies (19) then $u_h \in L_2(\mathbb{R}) \cap BC(\mathbb{R})$, for every $h > 0$, and u satisfies (29). For $h > 0$ let $G_h(\xi) := \mathrm{i}\sqrt{(k^2 - \xi^2)} \exp(\mathrm{i}h\sqrt{(k^2 - \xi^2)}) \hat{f}(\xi)$, $\xi \in \mathbb{R}$, so that $G_h \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, and $|G_H(\xi)| \leqslant |G_h(\xi)|$, $\xi \in \mathbb{R}$, for $H > h > 0$. Using the dominated convergence theorem to justify interchange of differentiation and integration, if follows from (29) that

$$\frac{\partial u(x)}{\partial x_2} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G_{x_2}(\xi)\, \mathrm{e}^{\mathrm{i}x_1\xi} \, \mathrm{d}\xi, \quad x \in U.$$

Hence $\hat{u}_h^* = G_h$, $h > 0$, so that $u_h^* \in L_2(\mathbb{R}) \cap BC(\mathbb{R})$ for every $h > 0$ and

$$\hat{u}_h^*(\xi) = \mathrm{i}\sqrt{(k^2 - \xi^2)}\, \hat{u}_h(\xi), \quad \xi \in \mathbb{R}. \tag{31}$$

Hence, and by Parseval's theorem,

$$\int_{-\infty}^{+\infty} \bar{u}_h u_h^* = \int_{-\infty}^{+\infty} \hat{\bar{u}}_h \hat{u}_h^* = \int_{-\infty}^{+\infty} \mathrm{i}\sqrt{(k^2 - \xi^2)}\, |\hat{u}_h(\xi)|^2 \, \mathrm{d}\xi,$$

and the result follows. □

Finally, we point out that if u is given by (19) then, for $h > 0$, every derivative of u has a similar representation. Specifically, define

$$w(x) := 2 \left. \frac{\partial \Phi(x, y)}{\partial y_2} \right|_{y=0} = \kappa_{x_2}(x_1), \quad x \in U. \tag{32}$$

Then by direct calculation we can establish that $\Delta w + k^2 w = 0$ in U and, arguing as in the proof of (17), we have that, for every $H > h > 0$,

$$\sup_{x \in U_h \setminus U_H} |x|^{-3/2} |w(x)| < \infty. \tag{33}$$

By standard local regularity results it follows that $w \in C^\infty(U)$ and that all the derivatives of w also satisfy (33). Thus, if $f \in L_\infty(\mathbb{R})$, $n, m \in \mathbf{N} \cup \{0\}$, and $h > 0$,

$$\frac{\partial^{n+m}}{\partial x_1^n \partial x_2^m} (\kappa_{x_2} * f(x_1)) = \frac{\partial^{n+m}}{\partial x_1^n \partial x_2^m} \int_{-\infty}^{+\infty} \kappa_{x_2}(x_1 - t) f(t) \, \mathrm{d}t \tag{34}$$

$$= \int_{-\infty}^{+\infty} \tilde{\kappa}_{x_2}(x_1 - t) f(t) \, \mathrm{d}t$$

$$= \tilde{\kappa}_{x_2} * f(x_1), \tag{35}$$

for $x \in U$, where

$$\tilde{\kappa}_{x_2}(x_1) := \frac{\partial^{n+m}}{\partial x_1^n \partial x_2^m} \kappa_{x_2}(x_1) = \frac{\partial^{n+m}}{\partial x_1^n \partial x_2^m} w(x). \tag{36}$$

The bound (33) on the derivatives of w ensures that $\tilde{\kappa}_{x_2} \in L_1(\mathbb{R})$ and justifies the interchange of differentiation and integration in (34) above.

4. The impedance boundary value problem

We consider next the impedance boundary value problem in the half-plane U with impedance boundary condition (2) on Γ. As in the previous section the boundary condition is understood in a weak$*$ sense (see BVP2 below), with L_∞ boundary data.

While this problem has much in common with the Dirichlet problem just studied (in particular similar behaviour of the solution $u(x)$ as $x_2 \to \infty$ can be expected), it is less straightforward in that it is no longer possible to write down an explicit expression for the solution u, as done in (20) for the Dirichlet case, except for a few very specific choices for the boundary admittance β.

We are particularly concerned in this section with the (more difficult) case $k > 0$, for which a radiation condition is required. To obtain a radiation condition we point out that, in each half-plane U_h, u satisfies a Dirichlet problem with boundary data u_h. It makes sense then to suppose that, for some $h \geqslant 0$ and $\varphi \in L_\infty(\mathbb{R})$,

$$u(x) = 2 \int_{\Gamma_h} \frac{\partial \Phi(x, y)}{\partial y_2} \varphi(y) \, \mathrm{d}s(y), \quad x \in U_h, \tag{37}$$

since, as shown in section 3, with the choice $\varphi = u_h$, (37) is the unique solution of the Dirichet problem in U_h satisfying the limiting absorption principle given in Theorem 3.3. It is shown in [6] that this radiation condition is a generalization of the usual radiation condition utilized in the study of plane wave diffraction by one-dimensional periodic gratings [18, 16], and reduces to this radiation condition when φ is quasi-periodic in the sense of [18, 16]. In section 4.4 we show that the solution selected by the radiation condition (37) for $k > 0$ is the unique solution satisfying a limiting absorption principle.

Consider then the following impedance boundary value problem for the Helmholtz equation:

BVP2. *Given f, $\beta \in L_\infty(\mathbb{R})$ and $k \in \mathbb{C}$ with* $\mathrm{Im}\, k \geqslant 0$, $\mathrm{Re}\, k > 0$, *find $u \in C^2(U) \cap C(\bar{U})$ satisfying*

(*i*) $\Delta u + k^2 u = 0$ *in U;*

(*ii*) *for some $a \in \mathbb{R}$,*

$$\sup_{x \in U} |(1 + x_2)^a u(x)| < \infty; \tag{38}$$

(*iii*) $u_h^* \xrightarrow{w^*} f - \mathrm{i}k\beta u_0$ *as* $h \to 0$;

(*iv*) *the radiation condition* (37), *for some $h \geqslant 0$ and $\varphi \in L_\infty(\mathbb{R})$.*

Remark 4.1. Note that (cf. Remark 3.1), (i) and (ii) imply that also, for all $h > 0$,

$$\sup_{x \in U_h} |(1 + x_2)^a \nabla u(x)| < \infty, \tag{39}$$

so that $u_h^* \in BC(\mathbb{R})$ for all $h > 0$. Further, (iii) implies that $\|u_h^*\|_\infty = O(1)$ as $h \to 0$ so that $\sup_{0<h\leqslant 1} \|u_h^*\|_\infty < \infty$.

Remark 4.2. There is some redundancy in the above formulation, in that, if (iv) is satisfied and $u \in BC(\bar{U} \setminus U_H) \cap C^2(U \setminus U_H)$ for some $H > h$, then, by Theorem 3.2, automatically $u \in C^2(U) \cap C(\bar{U})$ and (ii) is satisfied for all $a \leqslant -\frac{1}{2}$.

Remark 4.3. If $\operatorname{Im} k > 0$ then the radiation condition (iv) is superfluous as it follows from the assumption that $u \in C(\bar{U}) \cap C^2(U)$ and from (i) and (ii), using Theorem 3.1.

4.1. *An integral equation formulation*

As an aid in proving uniqueness and existence of solution of BVP2 and as a tool for numerical computation we reformulate BVP2 as a boundary integral equation. The fundamental solution of the Helmholtz equation which satisfies BVP2 with $f \equiv 0$ and $\beta \equiv 1$ (and a Dirac delta function inhomogeneity in the Helmholtz equation) is given by [10]

$$G(x, y) = \Phi(x, y) + \Phi(x, y') + \hat{P}(x - y'), \tag{40}$$

where, for $x \in \bar{U}$,

$$\hat{P}(x) := -\frac{\mathrm{i}k}{2\pi} \int_{-\infty}^{+\infty} \frac{\exp(\mathrm{i}[x_1 s + x_2 \sqrt{(k^2 - s^2)}])}{\sqrt{(k^2 - s^2)}\,(\sqrt{(k^2 - s^2)} + k)}\, \mathrm{d}s \tag{41}$$

$$= \frac{\mathrm{e}^{\mathrm{i}k|x|}}{\pi} \int_0^\infty \frac{t^{-1/2} \mathrm{e}^{-k|x|t}(1 + \gamma(1 + \mathrm{i}t))}{\sqrt{(t - 2\mathrm{i})}\,(t - \mathrm{i}(1 + \gamma))^2}\, \mathrm{d}t, \tag{42}$$

with $\gamma = x_2 / |x|$ and all square roots here and throughout taken with $\operatorname{Re} \sqrt{\ } \geqslant 0$, $\operatorname{Im} \sqrt{\ } \geqslant 0$. The equivalence of (41) and (42) for $k > 0$ is shown in [10], and follows for $\operatorname{Im} k > 0$, $\operatorname{Re} k > 0$ by the uniqueness of analytic continuation.

Applying the dominated convergence theorem to (41) it is easy to see that $\hat{P} \in C(\bar{U})$. From (42) it follows that $\hat{P} \in C^\infty(\bar{U} \setminus \{0\})$ and satisfies the Sommerfeld radiation and boundedness conditions in $\bar{U}$ [9]. The impedance condition satisfied by G is a consequence of the equation [10, equation (39)],

$$\frac{\partial \hat{P}(y)}{\partial y_2} + \mathrm{i}kG(0, y) = 0, \quad y \in \bar{U},\ y \neq 0. \tag{43}$$

A further property of G that we shall need is that [11], given $C > 0$,

$$\nabla_y G(x, y),\ G(x, y) = O(|x - y|^{-3/2}) \text{ as } |x - y| \to \infty, \tag{44}$$

uniformly in $x, y \in \bar{U}$, with $0 \leqslant x_2, y_2 \leqslant C$. This rapid rate of decrease (very important in the arguments which follow) holds only provided the vertical co-ordinates, x_2 and y_2, are restricted as indicated. (If x and y are unrestricted then $G(x, y) = O(|x - y|^{-1/2})$ as $|x - y| \to \infty$, the same behaviour as that of Φ.) In physical terms the more rapid rate of decay (44) is due to the energy-absorbing nature of the boundary condition (2) when $\mathrm{Re}\beta > 0$.

To derive the boundary integral equation, suppose that u satisfies BVP2 (in particular (37) for some $h > 0$) and take $x \in U$. Choose h_1, h_2 such that $0 < h_1 < x_2 < h_2$ and $h_2 > h$, and apply Green's second theorem in the bounded region $S_{A,\varepsilon} := \{y \in U_{h_1} \setminus \bar{U}_{h_2}: |y_1| < A\} \setminus \overline{B_\varepsilon(x)}$ to $G(x,.)$ and u to obtain

$$0 = \int_{\partial S_{A,\varepsilon}} \left(\frac{\partial u}{\partial n}(y) G(x, y) - u(y) \frac{\partial G(x, y)}{\partial n(y)} \right) \mathrm{d}s(y),$$

where n is the outward-directed normal on $\partial S_{A,\varepsilon}$. Letting $\varepsilon \to 0$ and $A \to \infty$ (note that u and (see Remark 4.1) ∇u are bounded in $U_{h_1} \setminus U_{h_2}$ so that the integrals over the vertical sides of $\partial S_{A,\varepsilon}$ vanish as $A \to \infty$), we obtain that

$$u(x) = \int_{\Gamma_{h_1} \cup \Gamma_{h_2}} \left(\frac{\partial u}{\partial n}(y) G(x, y) - u(y) \frac{\partial G(x, y)}{\partial n(y)} \right) \mathrm{d}s(y). \tag{45}$$

As in the proof of Theorem 3.2, let $\varphi_n = \chi_{[-n,n]}\varphi$, $n \in \mathbb{N}$, and define $u^{(n)}$ *by* (37) with φ replaced by φ_n. Then, for each n the double-layer $u^{(n)} \in C^2(U_h)$ and satisfies the Helmholtz equation and Sommerfeld radiation and boundedness conditions, so that, applying Green's second theorem to $G(x,.)$ and $u^{(n)}$ in $U_{h_2} \cap B_R(0) \subset U_h$, and letting $R \to \infty$ we obtain

$$\int_{\Gamma_{h_2}} \left(\frac{\partial u^{(n)}}{\partial n}(y) G(x, y) - u^{(n)}(y) \frac{\partial G(x, y)}{\partial n(y)} \right) \mathrm{d}s(y) = 0. \tag{46}$$

Now, by (21), the functions $u^{(n)}$ are uniformly bounded on $U_h \setminus U_H$ for every $H > h_2$, and therefore so are the functions $\nabla u^{(n)}$, $n \in \mathbb{N}$, on Γ_{h_2}. Further, $u^{(n)}$ converges to u uniformly on compact subsets of U_h (and therefore so also does $\nabla u^{(n)}$ converge to ∇u). Thus, and bearing in mind (44), it follows that the integral in (46) converges to the same integral with $u^{(n)}$ replaced by u as $n \to \infty$, and so the integral over Γ_{h_2} in (45) vanishes. Thus, for every $h > 0$,

$$u(x) = \int_{\Gamma_h} \left(u(y) \frac{\partial G(x, y)}{\partial y_2} - \frac{\partial u(y)}{\partial y_2} G(x, y) \right) \mathrm{d}s(y), \quad x \in U_h. \tag{47}$$

Since $u \in BC(\bar{U} \setminus U_H)$ for every $H > 0$, and $G(x,.) \in C^\infty(\bar{U} \setminus \{x\})$ and satisfies (44), it follows from the dominated convergence theorem that

$$\int_{\Gamma_h} u(y) \frac{\partial G(x, y)}{\partial y_2} \mathrm{d}s(y)$$

depends continuously on h, for $0 \leqslant h < x_2$. Also, in view of the impedance boundary condition satisfied by G (equation (43)),

$$\int_{\Gamma_h} u(y) \frac{\partial G(x, y)}{\partial y_2} \mathrm{d}s(y) = -\mathrm{i}k \int_{\Gamma} u(y) G(x, y) \mathrm{d}s(y), \tag{48}$$

for $h = 0$.

Define $\lambda_h \in L_1(\mathbb{R})$, for $h \geqslant 0$, by $\lambda_h(s) = G((s,h),0)$, $s \in \mathbb{R}$. Since G, given by (40), satisfies (44), and $\hat{P} \in C(\bar{U})$, it follows by the dominated convergence theorem that λ_h depends continuously in norm on h in $L_1(\mathbb{R})$ for $h \geqslant 0$. Thus, and since $u_h^* \xrightarrow{w^*} f - \mathrm{i}k\beta u_0$ as $h \to 0$, it follows from (10) that

$$\int_{\Gamma_h} \frac{\partial u(y)}{\partial y_2} G(x,y)\,\mathrm{d}s(y) = \lambda_{x_2-h} * u_h^*(x_1)$$

$$\to \lambda_{x_2} * (f - \mathrm{i}k\beta u_0)(x_1), \tag{49}$$

as $h \to 0$. Thus, letting $h \to 0$ in (47), and noting (48) and (49), it follows that, in convolution form,

$$u_h = \lambda_h * (\mathrm{i}k(\beta - 1)u_0 - f), \; h > 0. \tag{50}$$

But, by (9), for $h \geqslant 0$ the right-hand side of (50) depends continuously on h in $BC(\mathbb{R})$. Since also $u \in C(\bar{U})$, it follows that (50) holds also for $h = 0$. We have shown the following result.

Theorem 4.1. *If u satisfies BVP2 then*

$$u(x) = \int_{\Gamma} G(x,y)\,(\mathrm{i}k(\beta(y) - 1)u(y) - f(y))\,\mathrm{d}s(y), \; x \in \bar{U}. \tag{51}$$

The following converse result is easily established using the arguments in [6] and the results of the previous section. Together, Theorems 4.1 and 4.2 establish the equivalence of BVP2 and the boundary integral equation (51).

Theorem 4.2. *If u satisfies* (51) *and $u_0 \in BC(\mathbb{R})$ then u satisfies BVP2.*

Proof. If u satisfies (51) then, in convolution form,

$$u_h = \lambda_h * (\mathrm{i}k(\beta - 1)u_0 - f), \; h \geqslant 0. \tag{52}$$

Now it follows from (41) and a standard Fourier transform of the Hankel function [10, equation (12)], that

$$\hat{\lambda}_h(\xi) = \frac{\mathrm{i}\exp(\mathrm{i}h\sqrt{(k^2 - \xi^2)})}{\sqrt{(k^2 - \xi^2)} + k}, \quad \xi \in \mathbb{R}, \tag{53}$$

so that $\hat{\lambda}_h = \hat{\kappa}_h \hat{\lambda}_0$ and

$$\lambda_h = \kappa_h * \lambda_0, \quad h > 0. \tag{54}$$

Thus, from (52),

$$u_h = \kappa_h * \lambda_0 * (\mathrm{i}k(\beta - 1)u_0 - f) = \kappa_h * u_0, \; h > 0. \tag{55}$$

It follows from Theorem 3.2 that u, defined by (51), satisfies the Dirichlet BVP1 with boundary data $u_0 \in BC(\mathbb{R})$, so that we have shown that u satisfies all the conditions of BVP2, except for the impedance boundary condition (iii).

Differentiating (51) and noting that, for $x \in U$, $y \in \Gamma$, using (43),

$$\frac{\partial G(x,y)}{\partial x_2} = 2\frac{\partial \Phi(x,y)}{\partial x_2} + \frac{\partial \hat{P}(x-y')}{\partial x_2}$$

$$= -2\frac{\partial \Phi(x,y)}{\partial y_2} - \mathrm{i}kG(x,y),$$

we obtain, in convolution form, that

$$u_h^* = -(\kappa_h + \mathrm{i}k\lambda_h) * (\mathrm{i}k(\beta-1)u_0 - f), \ h > 0,$$

$$= \kappa_h * (f - \mathrm{i}k\beta u_0), \ h > 0, \tag{56}$$

by (54) and (52). Thus, by Theorem 3.2, $\partial u/\partial x_2$ satisfies the Dirichlet BVP1 with boundary data $f - \mathrm{i}k\beta u_0$, and so we have shown the impedance condition (iii). □

From (55) and Theorem 3.2 we have also the following corollary.

Corollary 4.3. *If u satisfies BVP2 then, for some constant $\tilde{C} > 0$ independent of β and f,*

$$\sup_{x \in \bar{U}} |\exp(x_2 \operatorname{Im} k)(1 + x_2)^{-1/2} u(x)| \leqslant \tilde{C} \|u_0\|_\infty.$$

4.2. *Uniqueness of solution*

It is pointed out in [6] (and see Section 4.3) that, if $\|1-\beta\|_\infty$ is sufficiently small, then uniqueness and existence of solution for the integral equation (51) (and thus, by Theorems 4.1 and 4.2, for BVP2) follows easily from a Neumann series argument. In this section we prove uniqueness (and in the next section existence) of solution for more general variations of the function β. The cases considered are: (a) $\operatorname{Im} k > 0$, $\operatorname{Re}\beta \geqslant 0$; (b) $k > 0$, $\operatorname{Re}\beta \geqslant \eta$, for some $\eta > 0$. Recall that, in physical terms, the conditions $\operatorname{Im} k \geqslant 0$ and $\operatorname{Re}\beta \geqslant 0$ ensure that energy is not generated in the medium and on the boundary, respectively, while if $\operatorname{Im} k > 0$ ($\operatorname{Re}\beta > 0$) then the medium (boundary) absorbs energy.

We require first the following estimate of the first derivatives of u.

Lemma 4.4. *If u satisfies BVP2 then, for some $C > 0$,*

$$|\nabla u(x)| \leqslant C \ln(1/x_2), \ x_1 \in \mathbb{R}, \ 0 \leqslant x_2 \leqslant \tfrac{1}{2}. \tag{57}$$

Proof. We already know (see Remark 4.1) that this bound certainly holds for $\partial u(x)/\partial x_2$.

To see that it holds for $\partial u(x)/\partial x_1$, note that, by Theorem 4.1, u satisfies the integral equation (51), so that, utilizing (44) to justify interchange of differentiation and integration,

$$\frac{\partial u(x)}{\partial x_1} = \int_\Gamma \frac{\partial G(x,y)}{\partial x_1}\varphi(y)\,\mathrm{d}s(y), \ x \in U,$$

with $\varphi = \mathrm{i}k(\beta - 1)u_0 - f \in L_\infty(\mathbb{R})$. Thus,

$$\left|\frac{\partial u(x)}{\partial x_1}\right| \leqslant \int_\Gamma \left|\frac{\partial G(x,y)}{\partial x_1}\right| \mathrm{d}s(y)\, \|\varphi\|_\infty. \tag{58}$$

Now, for some $C > 0$,

$$\int_{\Gamma \backslash B_1(x)} \left|\frac{\partial G(x,y)}{\partial x_1}\right| \mathrm{d}s(y) \leqslant C,$$

for $0 \leqslant x_2 \leqslant \frac{1}{2}$, $x_1 \in \mathbb{R}$, by (44) and since $\hat{P} \in C^\infty(\bar{U} \backslash \{0\})$. Further, for $y \in \Gamma$, $x \in U$, $\partial(\Phi(x,y) - \Phi_0(x,y))/\partial x_1$ is bounded, where $\Phi_0(x,y) = -(1/2\pi)\ln|x-y|$, and it is shown in [9] that $\partial \hat{P}(x)/\partial x_1$ is bounded in $\bar{U}$. Thus, for some $C > 0$,

$$\int_{\Gamma \cap B_1(x)} \left|\frac{\partial G(x,y)}{\partial x_1} - 2\frac{\partial \Phi_0(x,y)}{\partial x_1}\right| \mathrm{d}s(y) \leqslant C,$$

for $0 \leqslant x_2 \leqslant \frac{1}{2}$, $x_1 \in \mathbb{R}$. Finally,

$$\begin{aligned} 2\int_{\Gamma \cap B_1(x)} \left|\frac{\partial \Phi_0(x,y)}{\partial x_1}\right| \mathrm{d}s(y) &\leqslant \frac{1}{\pi}\int_{x_1 - 1}^{x_1 + 1} \frac{|x_1 - y_1|}{|x - y|^2}\,\mathrm{d}y_1 \\ &= \frac{2}{\pi}\int_0^1 \frac{y_1}{x_2^2 + y_1^2}\,\mathrm{d}y_1 \\ &\leqslant \frac{1}{\pi}(\ln(\tfrac{5}{4}) + 2\ln(1/x_2)). \end{aligned}$$

Combining these inequalities we obtain the desired result. □

The following theorem deals with the simpler case $\operatorname{Im} k > 0$. The proof does not require the radiation condition (37) which we have already seen (Remark 4.3) is superfluous when $\operatorname{Im} k > 0$.

Theorem 4.5. *If u satisfies BVP2 with $f \equiv 0$, $\operatorname{Im} k > 0$, and $\operatorname{Re}\beta \geqslant 0$, then $u \equiv 0$.*

Proof. Choose ε in the range $0 < \varepsilon < \operatorname{Im} k$ and let $F(x) = \exp(-2\varepsilon\sqrt{1 + |x|^2})$, $x \in \bar{U}$. Choose $h > 0$, $R > h$ and apply Green's first theorem [12] to u and $\bar{u}F$ in $T := U_h \cap B_R(0)$ to obtain that

$$\int_T (\nabla(\bar{u}F)\cdot\nabla u + F\bar{u}\Delta u)\,\mathrm{d}x = \int_{\partial T} F\bar{u}\frac{\partial u}{\partial n}\,\mathrm{d}s,$$

where n is the outward normal to ∂T. Recall that $\Delta u = -k^2 u$, and take the limit $R \to \infty$, noting that, by (38) and (39), any increase in u and ∇u at infinity is dominated by the decay in F, to obtain that

$$\begin{aligned} \int_{U_h} (\nabla(\bar{u}F)\cdot\nabla u - k^2 F|u|^2)\,\mathrm{d}x &= \int_{\Gamma_h} F\bar{u}\frac{\partial u}{\partial n}\,\mathrm{d}s \\ &\to \mathrm{i}k\int_\Gamma F\beta|u|^2\,\mathrm{d}s \end{aligned} \tag{59}$$

as $h \to 0$, using condition (iii) of BVP2. Let

$$p = |k| \left\{ \int_U F|u|^2 \, dx \right\}^{1/2}, \qquad q = \left\{ \int_U F|\nabla u|^2 \, dx \right\}^{1/2},$$

the integrals well-defined in view of the bounds (38) and (39) and Lemma 4.4. Taking the limit $h \to 0$ in equation (59), multiplying by $\bar{k}$, and taking the imaginary part, we obtain that

$$\operatorname{Im} k(p^2 + q^2) + |k|^2 \int_\Gamma \operatorname{Re}(\beta) F |u|^2 \, ds = \operatorname{Im} \left\{ \bar{k} \int_U \bar{u} \, \nabla F \cdot \nabla u \, dx \right\}.$$

Since $\operatorname{Re} \beta \geqslant 0$ and $|\nabla F| \leqslant 2\varepsilon F < 2 \operatorname{Im} kF$ we have, applying the Cauchy–Schwarz inequality, that $p^2 + q^2 < 2pq$ so that $p = q = 0$ and $u \equiv 0$. □

We now consider the more subtle case $k > 0$. Our uniqueness proof for this case also depends on an application of Green's first theorem, contained in the next lemma.

Lemma 4.6. *If u satisfies BVP2 with $f = 0$ and $k > 0$, then, for some constant $C > 0$ and all $0 < h \leqslant \frac{1}{2}$ and $A > 0$,*

$$k \int_{-A}^{A} \operatorname{Re}(\beta) |u_0|^2 + \operatorname{Im} \int_{-A}^{A} \bar{u}_h u_h^* \leqslant Ch \ln(1/h). \tag{60}$$

Proof. Suppose that u satisfies BVP2 with $f = 0$ and $k > 0$. Choose $A > 0$ and $0 < h_1 < h \leqslant \frac{1}{2}$, and apply Green's first theorem [12] to u and $\bar{u}$ in $S := \{x : |x_1| < A, h_1 < x_2 < h\}$. Recall that $\Delta u = -k^2 u$ and take the imaginary part of the resulting equation, to obtain that

$$\operatorname{Im} \int_{\partial S} \bar{u} \frac{\partial u}{\partial n} \, ds = 0, \tag{61}$$

where n is the outward normal to ∂S.

For $B > 0$ and $H \geqslant 0$, let $\Gamma_H^B := \{x: |x_1| < B, x_2 = H\}$. Then $\partial S = \Gamma_{h_1}^A \cup \Gamma_h^A \cup \gamma$, where $\gamma := \partial S \backslash (\Gamma_{h_1}^A \cup \Gamma_h^A)$ consists of the two vertical sides of ∂S. By condition (ii) of BVP2 and Lemma 4.4,

$$\left| \int_\gamma \bar{u} \frac{\partial u}{\partial n} \, ds \right| \leqslant C \int_{h_1}^{h} \ln(1/s) \, ds \leqslant Ch \ln(1/h).$$

Also,

$$\int_{\Gamma_{h_1}^A} \bar{u} \frac{\partial u}{\partial n} \, ds = -\int_{-A}^{A} \bar{u}_{h_1} u_{h_1}^* \to ik \int_{-A}^{A} \beta |u_0|^2$$

as $h_1 \to 0$, by condition (iii) of BVP2, and noting that $\| \chi_{[-A,A]} (u_{h_1} - u_0) \|_1 \to 0$ as $h_1 \to 0$ since $u \in C(\bar{U})$. Thus, letting $h_1 \to 0$ in (61), we obtain the required result. □

The other tool in our uniqueness proof for $k > 0$ is that we have representations for u_h and u_h^* in terms of u_0. Specifically, if u satisfies BVP2 then, from (55),

$$u_h = \kappa_h * u_0, \quad h > 0, \tag{62}$$

from which it follows (see the end of section 3) that

$$u_h^* = \tilde{\kappa}_h * u_0, \quad h > 0, \tag{63}$$

with κ_h and $\tilde{\kappa}_h$ given in terms of the function w, defined by (32), by

$$\kappa_{x_2}(x_1) = w(x), \quad \tilde{\kappa}_{x_2}(x_1) = \frac{\partial w(x)}{\partial x_2}, \quad x \in U. \tag{64}$$

Since w and $\partial w / \partial x_2$ satisfy the bound (33), it follows that for every $h > 0$ there exists $C_h > 0$ such that

$$|\kappa_h(t)|, |\tilde{\kappa}_h(t)| \leqslant C_h(1 + |t|)^{-3/2}, \quad t \in \mathbb{R}, \tag{65}$$

so that $\kappa_h, \tilde{\kappa}_h \in L_1(\mathbb{R})$, $h > 0$.

We would like to argue at this point by applying Lemma 3.4. Specifically, if we knew that $u_0 \in L_2(\mathbb{R})$ then, from (62) and (63), $u_h, u_h^* \in L_2(\mathbb{R})$. It would then follow that $\int_{-\infty}^{+\infty} \bar{u}_h u_h^*$ exists, and that

$$\operatorname{Im} \int_{-A}^{A} \bar{u}_h u_h^* = \operatorname{Im} \int_{-\infty}^{+\infty} \bar{u}_h u_h^* + o(1) \quad \text{as } A \to \infty .$$

Then, applying Lemma 3.4 it would follow that the integral on the right is non-negative, and we would obtain, from (60), an upper bound on $\int_{-A}^{A} \operatorname{Re} \beta |u_0|^2$.

Of course, we do not know *a priori* that $u_0 \in L_2(\mathbb{R})$ but only that $u_0 \in BC(\mathbb{R})$, so that Lemma 3.4 cannot be applied in this way.

To recover the situation to some extent, define v, a solution of BVP1 for boundary data $f = u_0 \chi_{[-A,A]}$, by

$$v_h = \kappa_h * (u_0 \chi_{[-A,A]}), \quad h > 0. \tag{66}$$

Since $u_0 \chi_{[-A,A]} \in L_2(\mathbb{R}) \cap L_\infty(\mathbb{R})$, Lemma 3.4 can be applied to give that $v_h, v_h^* \in L_2(\mathbb{R}) \cap L_\infty(\mathbb{R})$, for every $h > 0$, and that

$$I_A'' := \operatorname{Im} \int_{-\infty}^{+\infty} \bar{v}_h v_h^* \geqslant 0. \tag{67}$$

Note also that (see above or end of section 3), since v satisfies (66),

$$v_h^* = \tilde{\kappa}_h * (u_0 \chi_{[-A,A]}), \quad h > 0. \tag{68}$$

Define

$$I_A := \operatorname{Im} \int_{-A}^{A} \bar{u}_h u_h^*, \quad I_A' := \operatorname{Im} \int_{-A}^{A} \bar{v}_h v_h^*, \tag{69}$$

$$J_A := \int_{-A}^{A} |u_0|^2. \tag{70}$$

The final stage of our uniqueness proof is contained in the next lemma. We require the following assumption on the admittance β, which ensures, in physical terms, that the boundary is everywhere energy-absorbing.

Assumption A1. *For some* $\eta > 0$ *and almost all* $s \in \mathbb{R}$, $\operatorname{Re}\beta(s) \geqslant \eta$.

Lemma 4.7. *If* u *satisfies* $BVP2$ *with* $f = 0$ *and* $k > 0$, *Assumption* $A1$ *is satisfied, and, for some sequence* $\{A_m\} \subset \mathbb{R}^+$ *such that* $A_m \to \infty$ *as* $m \to \infty$,

$$I_{A_m} = I''_{A_m} + o(1) \quad \text{as } m \to \infty, \tag{71}$$

then $u \equiv 0$.

Proof. From Lemma 4.6, (67), and applying Assumption A1,

$$k\eta J_A \leqslant k \int_{-A}^{A} \operatorname{Re}\beta |u_0|^2 \leqslant Ch\ln(1/h) - I_A \leqslant Ch\ln(1/h) - (I_A - I''_A). \tag{72}$$

Thus, if (71) holds,

$$\int_{-A_m}^{A_m} |u_0|^2 \leqslant \frac{C}{k\eta} h\ln(1/h) + o(1)$$

as $m \to \infty$, so that $u_0 \in L_2(\mathbb{R})$ and

$$\int_{-\infty}^{+\infty} |u_0|^2 \leqslant \frac{C}{k\eta} h\ln(1/h).$$

Since this equation holds for $0 < h \leqslant \frac{1}{2}$, it follows that $u_0 = 0$ and, from (62), that $u \equiv 0$. □

Of course, application of Lemma 4.7 depends on showing first that (71) is satisfied. If $u_0 \in L_2(\mathbb{R})$ then (71) is clear, but given only that $u_0 \in BC(\mathbb{R})$ we shall see below that the most we can show immediately is that

$$I_A - I''_A = O(A^q) \quad \text{as } A \to \infty, \tag{73}$$

for every $q > 1/2$. However, we can use (73), via (72), to bound the growth of J_A as $A \to \infty$ and, in turn, we shall see that, using (62), (63), (66), and (68), we are able to obtain a sharper bound than (73) on $I_A - I''_A$.

It proves more convenient to bound $I'_A - I''_A$ and $I_A - I'_A$ rather than $I_A - I''_A$ directly. In our next lemma we immediately obtain quite a strong bound on $I'_A - I''_A$.

Throughout the remainder of this section c denotes a positive constant, independent of A, not necessarily the same at each occurrence.

Lemma 4.8. *If* $u_0 \in L_\infty(\mathbb{R})$ *then* $I'_A = I''_A + O(\ln A)$ *as* $A \to \infty$.

Proof. If $u_0 \in L_\infty(\mathbb{R})$ then, from (66) and (65),

$$\begin{aligned} |v_h(s)| &\leqslant c \int_{-A}^{A} \frac{|u_0(t)|\,\mathrm{d}t}{(1 + |s-t|)^{3/2}}, \quad s \in \mathbb{R}, \\ &\leqslant c \int_{-A}^{A} \frac{\mathrm{d}t}{(1 + |s| - t)^{3/2}}, \quad |s| \geqslant A. \end{aligned} \tag{74}$$

Thus, for $|s| \geqslant A$,

$$|v_h(s)| \leqslant c\{(1+|s|-A)^{-1/2} - (1+|s|+A)^{-1/2}\}$$
$$\leqslant cA(1+|s|-A)^{-1/2}(1+|s|+A)^{-1}. \tag{75}$$

From (68) and (65) the same bounds, (74) and (75), apply to $|v_h^*(s)|$. Thus,

$$|I'_A - I''_A| \leqslant \int_{\mathbb{R}\setminus[-A,A]} |v_h|\,|v_h^*|$$

$$\leqslant cA^2 \int_A^\infty \frac{\mathrm{d}s}{(1+s-A)\,(1+s+A)^2}$$

$$= c\int_{1/A}^\infty \frac{\mathrm{d}u}{u(2+u)^2}, \tag{76}$$

substituting $1+s = A(1+u)$. The result follows. □

Application of this next lemma repeatedly (first with $p = 1$ to give (73)) enables us to prove (see Corollary 4.11 below) that $I_A - I''_A$ and J_A increase at most slower than any positive power of A as $A \to \infty$.

Lemma 4.9. *If $u_0 \in L_\infty(\mathbb{R})$ and, for some p in the range $0 < p \leqslant 1$, $J_A = O(A^p)$ as $A \to \infty$, then, for all $q > p/(p+1)$, $I_A = I'_A + O(A^q)$ as $A \to \infty$.*

Proof. From (62), (66), and (65),

$$|u_h(s) - v_h(s)| \leqslant c\int_{\mathbb{R}\setminus[-A,A]} \frac{|u_0(t)|\,\mathrm{d}t}{(1+|s-t|)^{3/2}}, \quad s \in \mathbb{R}. \tag{77}$$

From (63), (68), and (65), the same bound applies to $|u_h^*(s) - v_h^*(s)|$. Also,

$$|u_h(s)|,\ |v_h(s)| \leqslant \|\kappa_h\|_1\,\|u_0\|_\infty, \qquad |u_h^*(s)|,\ |v_h^*(s)| \leqslant \|\tilde{\kappa}_h\|_1\,\|u_0\|_\infty. \tag{78}$$

Thus,

$$|I_A - I'_A| \leqslant \int_{-A}^{A} (|u_h|\,|u_h^* - v_h^*| + |v_h^*|\,|u_h - v_h|) \tag{79}$$

$$\leqslant c\int_{-A}^{A} \left\{\int_{\mathbb{R}\setminus[-A,A]} \frac{|u_0(t)|\,\mathrm{d}t}{(1+|s-t|)^{3/2}}\right\} \mathrm{d}s. \tag{80}$$

Now, suppose that $J_A = O(A^p)$ and let $\alpha := 2/(p+1)$. Then, for $s \in [-A, A]$, by the Cauchy–Schwarz inequality,

$$\int_{[-A^\alpha,A^\alpha]\setminus[-A,A]} \frac{|u_0(t)|\,\mathrm{d}t}{(1+|s-t|)^{3/2}} \leqslant \left\{\int_{[-A^\alpha,A^\alpha]\setminus[-A,A]} \frac{\mathrm{d}t}{(1+|s-t|)^3}\right\}^{1/2} J_{A^\alpha}^{1/2}$$

$$\leqslant cA^{\alpha p/2} \left\{\int_A^\infty \frac{\mathrm{d}t}{(1+t-|s|)^3}\right\}^{1/2}$$

$$\leqslant \frac{cA^{p/p+1}}{1+A-|s|}. \tag{81}$$

Also, for $s \in [-A, A]$,

$$\int_{\mathbb{R}\setminus[-A^\alpha, A^\alpha]} \frac{|u_0(t)|\,\mathrm{d}t}{(1+|s-t|)^{3/2}} \leqslant c \int_{A^\alpha}^{\infty} \frac{\mathrm{d}t}{(1+t-|s|)^{3/2}}$$
$$\leqslant c(1+A^\alpha-|s|)^{-1/2}. \tag{82}$$

Now

$$\int_{-A}^{A} \frac{\mathrm{d}s}{1+A-|s|} = 2\ln(1+A) \tag{83}$$

and

$$\int_{-A}^{A} \frac{\mathrm{d}s}{(1+A^\alpha-|s|)^{1/2}} = 4\{(1+A^\alpha)^{1/2} - (1+A^\alpha-A)^{1/2}\}$$
$$\leqslant 4A^{1-\alpha/2} = 4A^{p/(p+1)}. \tag{84}$$

Combining (80)–(84) the result follows. □

Lemma 4.10. *If u satisfies BVP2 with $f = 0$ and $k > 0$, Assumption A1 is satisfied and, for some $n \geqslant 2$, $J_A = O(A^{2/n})$ as $A \to \infty$, then $J_A = O(A^{2/(n+1)})$ as $A \to \infty$.*

Proof. Suppose the hypotheses are satisfied. Then $J_A = O(A^p)$ as $A \to \infty$ with $p = 2/n$. It follows from Lemma 4.9, with $q = 2/(n+1) = 2p/(p+2) > p/(p+1)$, that $I_A = I'_A + O(A^q)$ as $A \to \infty$ and then, from Lemma 4.8, that $I_A = I''_A + O(A^q)$ as $A \to \infty$. From (72) we conclude that $J_A = O(A^q)$ as $A \to \infty$. □

Corollary 4.11. *If u satisfies BVP2 with $f = 0$ and $k > 0$ and Assumption A1 is satisfied, then, for all $\varepsilon > 0$, $J_A = O(A^\varepsilon)$ as $A \to \infty$ and $u_0 \in L'_{2,\varepsilon}(\mathbb{R})$, $u_0 \in L_{2,\varepsilon}(\mathbb{R})$. Further, for $0 \leqslant \eta < \frac{1}{2}$, $u_h, u_h^*, v_h, v_h^* \in L_{2,\eta}(\mathbb{R})$ with*

$$\|u_h\|_{2,\eta}, \|u_h^*\|_{2,\eta}, \|v_h\|_{2,\eta}, \|v_h^*\|_{2,\eta} \leqslant c\,\|u_0\|_{2,\eta}. \tag{85}$$

Proof. If the hypotheses are satisfied than $J_A = O(A)$ as $A \to \infty$ and the conditions of Lemma 4.10 are satisfied with $n = 2$. It follows by induction, applying Lemma 4.10, that $J_A = O(A^{2/n})$ as $A \to \infty$, for all $n \in \mathbf{N}$. Thus, for every $\varepsilon > 0$, $J_A = O(A^\varepsilon)$ as $A \to \infty$ so that $u_0 \in L'_{2,\varepsilon}(\mathbb{R})$. It follows from Lemma 2.1 that $u_0 \in L_{2,\varepsilon}(\mathbb{R})$ for every $\varepsilon > 0$. Further, from (62), (63), (66), (68), (65), and Lemma 2.2 it follows that $u_h, u_h^*, v_h, v_h^* \in L_{2,\eta}(\mathbb{R})$, for $0 \leqslant \eta < \frac{1}{2}$, and that the bound (85) holds. □

From the above corollary and Lemmas 4.8 and 4.9, it follows that $I_A - I''_A$ increases at most very slowly ($O(A^\varepsilon)$ for arbitrarily small $\varepsilon > 0$) as $A \to \infty$. To show, finally, that $I_A - I''_A$ actually decreases we bound, in Lemmas 4.13 and 4.14, the differences $I_A - I'_A$ and $I'_A - I''_A$ in a different way. We require first the following preliminary lemma.

Lemma 4.12. *Suppose that $\varphi \in L'_{2,\varepsilon}(\mathbb{R})$, for some $\varepsilon > 0$. Then, for every α in the range $0 \leqslant \alpha < \frac{1}{2} - \varepsilon$, these exists a constant $c > 0$ and a sequence $\{A_m : m \in \mathbf{N}\} \subset [1, \infty)$ such that $A_m \to \infty$ as $m \to \infty$ and*

$$\int_{\Omega_{A_m}} |\varphi|^2 \leqslant c A_m^{-\alpha}, \quad m \in \mathbf{N},$$

where $\Omega_A := (-A - A^{1/2}, A + A^{1/2}) \backslash (-A + A^{1/2}, A - A^{1/2})$, $A \in [1, \infty)$.

Proof. Note that, if $n, m \in \mathbf{N}$ and $n \neq m$, then $\Omega_{n^2} \cap \Omega_{m^2} = \emptyset$.

Suppose that the result is false. Then there exists some α in the range $0 \leqslant \alpha < \frac{1}{2} - \varepsilon$ and some $N \in \mathbf{N}$ such that, for $n = N, N+1, \dots,$

$$\int_{\Omega_{n^2}} |\varphi|^2 \geqslant (n^2)^{-\alpha}.$$

Thus, for $n = N, N+1, \dots,$

$$\int_{-n^2-n}^{n^2+n} |\varphi|^2 \geqslant \sum_{m=N}^{n} m^{-2\alpha} > cn^{1-2\alpha} > c(n^2+n)^{1/2-\alpha}.$$

But this contradicts that $\varphi \in L'_{2,\varepsilon}(\mathbb{R})$. □

Suppose now that the conditions of Corollary 4.11 are satisfied. Then Lemma 4.12 can be applied with $\varphi = u_0$ and any $\varepsilon > 0$. In particular, we can obtain a sequence $\{A_m \colon m \in \mathbf{N}\} \subset [1, \infty)$ such that $A_m \to \infty$ as $m \to \infty$ and

$$\int_{\omega_m} |u_0|^2 \leqslant cA_m^{-1/3}, \quad m \in \mathbf{N}, \tag{86}$$

where $\omega_m := \Omega_{A_m}$. For $m \in \mathbf{N}$ let

$$A_m^- := A_m - A_m^{1/2}, \quad A_m^+ := A_m + A_m^{1/2}, \tag{87}$$

and define $\omega_m^{\pm} \subset \omega_m$ by

$$\omega_m^+ := (-A_m^+, A_m^+) \backslash (-A_m, A_m), \quad \omega_m^- := (-A_m, A_m) \backslash (-A_m^-, A_m^-).$$

Lemma 4.13. *If u satisfies BVP2 with* $f = 0$ *and* $k > 0$ *and Assumption A1 is satisfied, then* $I'_{A_m} = I''_{A_m} + o(1)$ *as* $m \to \infty$.

Proof. From (74), if $A = A_m$,

$$\begin{aligned} |v_h(s)| &\leqslant c \int_{-A_m}^{A_m} \frac{|u_0(t)|\,\mathrm{d}t}{(1+|s-t|)^{3/2}}, \\ &\leqslant c \int_{-A_m^-}^{A_m^-} \frac{|u_0(t)|\,\mathrm{d}t}{(1+|s-t|)^{3/2}} + c \int_{\omega_m^-} \frac{|u_0(t)|\,\mathrm{d}t}{(1+|s-t|)^{3/2}}. \end{aligned} \tag{88}$$

Now, by the Cauchy–Schwarz inequality,

$$\int_{-A_m^-}^{A_m^-} \frac{|u_0(t)|\,\mathrm{d}t}{(1+|s-t|)^{3/2}} \leqslant J_{A_m^-}^{1/2} \left\{ \int_{-A_m^-}^{A_m^-} \frac{\mathrm{d}t}{(1+|s-t|)^3} \right\}^{1/2}$$

and, for $|s| > A_m$,

$$\int_{-A_m^-}^{A_m^-} \frac{\mathrm{d}t}{(1+|s-t|)^3} = \int_{-A_m^-}^{A_m^-} \frac{\mathrm{d}t}{(1+|s|-t)^3} \leqslant \frac{1}{2}(1+|s|-A_m^-)^{-2}.$$

Applying Corollary 4.11 with $\varepsilon = \frac{1}{6}$,

$$\int_{-A_m^-}^{A_m^-} \frac{|u_0(t)|\,dt}{(1+|s-t|)^{3/2}} \leqslant cA_m^{1/12}(1+|s|-A_m^-)^{-1}, \quad |s| > A_m. \tag{89}$$

Similarly, and using (86),

$$\begin{aligned}\int_{\omega_m^-} \frac{|u_0(t)|\,dt}{(1+|s-t|)^{3/2}} &\leqslant \left\{\int_{\omega_m^-} |u_0(t)|^2\,dt\right\}^{1/2} \left\{\int_{\omega_m^-} \frac{dt}{(1+|s-t|)^3}\right\}^{1/2} \\ &\leqslant cA_m^{-1/6}(1+|s|-A_m)^{-1}, \quad |s| > A_m. \end{aligned}\tag{90}$$

Combining (88), (89), and (90),

$$|v_h(s)| \leqslant cA_m^{1/12}(1+|s|-A_m^-)^{-1} + cA_m^{-1/6}(1+|s|-A_m)^{-1}, \quad |s| > A_m. \tag{91}$$

Since the bound (74) holds also for $|v_h^*(s)|$, the bound (91) also holds for $|v_h^*(s)|$. Thus, for $|s| > A_m$,

$$\begin{aligned}|v_h(s)||v_h^*(s)| &\leqslant (cA_m^{1/12}(1+|s|-A_m^-)^{-1} + cA_m^{-1/6}(1+|s|-A_m)^{-1})^2 \\ &\leqslant cA_m^{1/6}(1+|s|-A_m^-)^{-2} + cA_m^{-1/12}(1+|s|-A_m)^{-2}.\end{aligned}$$

Thus, from (76),

$$\begin{aligned}|I'_{A_m} - I''_{A_m}| &\leqslant cA_m^{1/6}\int_{A_m}^{\infty} \frac{ds}{(1+s-A_m^-)^2} + cA_m^{-1/12}\int_{A_m}^{\infty} \frac{ds}{(1+s-A_m)^2} \\ &= \frac{cA_m^{1/6}}{1+A_m^{1/2}} + cA_m^{-1/12} \\ &\to 0\end{aligned}$$

as $m \to \infty$. □

Lemma 4.14. *If u satisfies BVP2 with $f = 0$ and $k > 0$ and Assumption A1 is satisfied, then $I_{A_m} = I'_{A_m} + o(1)$ as $m \to \infty$.*

Proof. From (77), for $A = A_m$,

$$|u_h(s) - v_h(s)| \leqslant c\int_{\mathbb{R}\setminus[-A_m, A_m]} \frac{|u_0(t)|\,dt}{(1+|s-t|)^{3/2}}, \quad s \in \mathbb{R}, \tag{92}$$

the same bound applying also to $|u_h^*(s) - v_h^*(s)|$.

Let $\varepsilon = \frac{1}{24}$. Then $u_0 \in L_{2,\varepsilon}(\mathbb{R})$ by Corollary 4.11, and, applying the Cauchy–Schwarz inequality,

$$\begin{aligned}&\int_{\mathbb{R}\setminus[-A_m^+, A_m^+]} \frac{|u_0(t)|\,dt}{(1+|s-t|)^{3/2}} \\ &\quad\leqslant \left\{\int_{\mathbb{R}\setminus[-A_m^+, A_m^+]} (1+t^2)^{-\varepsilon}|u_0(t)|^2\,dt\right\}^{1/2} \left\{\int_{\mathbb{R}\setminus[-A_m^+, A_m^+]} \frac{(1+t^2)^{\varepsilon}\,dt}{(1+|s-t|)^3}\right\}^{1/2} \\ &\quad\leqslant 2\|u_0\|_{2,\varepsilon}\left\{\int_{A_m^+}^{\infty} \frac{(1+t^2)^{\varepsilon}\,dt}{(1+t-|s|)^3}\right\}^{1/2}, \quad |s| \leqslant A_m.\end{aligned}$$

Now, for $|s| \leqslant A_m$, $t \geqslant A_m^+$,

$$\frac{1+t^2}{(1+t-|s|)^2} \leqslant \frac{1+t^2}{(1+t-A_m^+)^2} \leqslant 1 + A_m^{+\,2},$$

so that, for $|s| \leqslant A_m$,

$$\int_{\mathbb{R}\backslash[-A_m^+, A_m^+]} \frac{|u_0(t)|\,\mathrm{d}t}{(1+|s-t|)^{3/2}} \leqslant c A_m^{+\,\varepsilon} \left\{ \int_{A_m^+}^{\infty} \frac{\mathrm{d}t}{(1+t-|s|)^{3-2\varepsilon}} \right\}^{1/2}$$

$$\leqslant c A_A^{\varepsilon} (1 + A_m^+ - |s|)^{\varepsilon - 1}. \tag{93}$$

Similarly, and using (86),

$$\int_{\omega_m^+} \frac{|u_0(t)|\,\mathrm{d}t}{(1+|s-t|)^{3/2}} \leqslant \left\{ \int_{\omega_m^+} |u_0|^2 \right\}^{1/2} \left\{ \int_{\omega_m^+} \frac{\mathrm{d}t}{(1+|s-t|)^3} \right\}^{1/2}$$

$$\leqslant c A_m^{-1/6} \left\{ \int_{A_m}^{\infty} \frac{\mathrm{d}t}{(1+t-|s|)^3} \right\}^{1/2}, \quad |s| \leqslant A_m,$$

$$\leqslant c A_m^{-1/6} (1 + A_m - |s|)^{-1}, \quad |s| \leqslant A_m. \tag{94}$$

Combining (92)–(94),

$$|u_h(s) - v_h(s)| \leqslant c A_m^{\varepsilon} (1 + A_m^+ - |s|)^{\varepsilon - 1} + c A_m^{-1/6} (1 + A_m - |s|)^{-1}, \quad |s| \leqslant A_m,$$

so that

$$|u_h(s) - v_h(s)|^2 \leqslant \frac{c A_m^{2\varepsilon}}{(1 + A_m^+ - |s|)^{2-2\varepsilon}} + \frac{c A_m^{\varepsilon - 1/6}}{(1 + A_m - |s|)^{2-\varepsilon}}, \quad |s| \leqslant A_m. \tag{95}$$

The same bound applies to $|u_h^*(s) - v_h^*(s)|^2$.

By Corollary 4.11 and Lemma 2.1, $u_h, v_h^* \in L'_{2,\varepsilon}(\mathbb{R})$ with

$$\int_{-A_m}^{A_m} |u_h|^2, \quad \int_{-A_m}^{A_m} |v_h^*|^2 \leqslant c A_m^{2\varepsilon}, \quad m \in \mathbf{N}. \tag{96}$$

Thus, applying the Cauchy–Schwarz inequality to (79) and utilizing (95),

$$|I_{A_m} - I'_{A_m}| \leqslant c A_m^{\varepsilon} \left\{ A_m^{2\varepsilon} \int_{-A_m}^{A_m} \frac{\mathrm{d}s}{(1 + A_m^+ - |s|)^{2-2\varepsilon}} + A_m^{\varepsilon - 1/6} \int_{-A_m}^{A_m} \frac{\mathrm{d}s}{(1 + A_m - |s|)^{2-\varepsilon}} \right\}^{1/2}$$

$$\leqslant c A_m^{\varepsilon} \{ A_m^{2\varepsilon} (1 + A_m^+ - A_m)^{2\varepsilon - 1} + A_m^{\varepsilon - 1/6} \}^{1/2}$$

$$\leqslant c A_m^{-1/48}. \qquad \square$$

We have now established enough to apply Lemma 4.7 to obtain the following uniqueness result:

Theorem 4.15. *If, for some* $\eta \geqslant 0$, $\operatorname{Re} \beta(s) \geqslant \eta$ *for almost all* $s \in \mathbb{R}$ *and either* $\operatorname{Im} k > 0$ *or* $\eta > 0$, *then BVP2 has at most one solution.*

Proof. If BVP2 has two solutions, $u^{(1)}$ and $u^{(2)}$, their difference, $u = u^{(1)} - u^{(2)}$, satisfies BVP2 with $f = 0$. In the case $k > 0$, $\eta > 0$, if follows from Lemmas 4.13 and 4.14 that there exists a sequence $\{A_m\} \subset \mathbb{R}^+$ such that $A_m \to \infty$ as $m \to \infty$ and such that $I_{A_m} = I''_{A_m} + o(1)$ as $m \to \infty$. Applying Lemma 4.7 it follows that $u \equiv 0$. In the case $\operatorname{Im} k > 0$ that $u \equiv 0$ follows from Theorem 4.5. □

Remark 4.4. Note that Theorem 4.15 no longer holds if we require only that $\operatorname{Re}\beta \geqslant 0$. For if β is constant on Γ, with $\operatorname{Re}\beta = 0$, $\operatorname{Im}\beta \leqslant 0$, and $k > 0$, then it is easy to see, using (53), that, if

$$u_0(s) := \exp(\pm \mathrm{i}k\sqrt{1-\beta^2}s), \quad s \in \mathbb{R},$$

then

$$u_0 = \mathrm{i}k(\beta - 1)\lambda_0 * u_0,$$

so that the integral equation (52) with $f = 0$ has a non-trivial solution. It follows from Theorem 4.2 that BVP2 with $f = 0$ also has a non-trivial solution. (This solution is $u(x) = \exp(\mathrm{i}k(\pm\sqrt{1-\beta^2}x_1 - \beta x_2))$.)

4.3. *Existence of solution*

To prove existence of solution we note that from Theorems 4.1 and 4.2, BVP2 and (51) are equivalent, and that if u satisfies (51) then u_0 satisfies the boundary integral equation, in operator form,

$$u_0 = F + K_\beta u_0, \tag{97}$$

where $F \in BC(\mathbb{R})$ is defined by $F := -\lambda_0 * f$, and K_β: $BC(\mathbb{R}) \to BC(\mathbb{R})$ is defined by

$$K_\beta \psi = \mathrm{i}k\lambda_0 * ((\beta - 1)\psi), \quad \psi \in BC(\mathbb{R}).$$

If $\|\beta - 1\|_\infty < C^*$, where $C^* := 1/(|k|\,\|\lambda_0\|_1) \approx 0.509$ when $k > 0$, then $\|K_\beta\| \leqslant |k|\,\|\lambda_0\|_1\,\|\beta - 1\|_\infty < 1$, and existence (and uniqueness) of solution of BVP2 is guaranteed by a Neumann series argument. Uniqueness of solution has been shown in Theorem 4.15 under more general conditions, and existence of solution can be deduced from the following result on convolution integral equations with coefficients:

Theorem 4.16 (*Chandler–Wilde* [5, *Corollary* 4.7]). *Suppose that* $\kappa \in L_1(\mathbb{R})$ *and, for* $z \in L_\infty(\mathbb{R})$, *define* $\mathscr{K}_z : BC(\mathbb{R}) \to BC(\mathbb{R})$ *by*

$$\mathscr{K}_z \psi = \kappa * (z\psi), \quad \psi \in BC(\mathbb{R}).$$

If $Q \subset \mathbb{C}$ *is compact and convex and* $I - \mathscr{K}_z$ *is injective for every* $z \in L^Q := \{z \in L_\infty(\mathbb{R})$: ess. range $z \subset Q\}$, *then* $I - \mathscr{K}_z$ *is surjective for every* $z \in L^Q$, *so that* $(I - \mathscr{K}_z)^{-1}$ *exists as an operator on* $BC(\mathbb{R})$. *Further,*

$$\sup_{z \in L^Q} \|(I - \mathscr{K}_z)^{-1}\| < \infty.$$

The proof of Theorem 4.16 uses Lemma 2.3 and that the set L^Q is translation invariant and also weak$*$ sequentially compact if Q is compact and convex.

Applying the above result to the integral equation (97) we obtain the following result which establishes unique existence of solution for BVP2 and continuous dependence of the solution on the boundary data f, uniformly in β.

Theorem 4.17. *Suppose that, for some* $\eta \geqslant 0$, $\mathrm{Re}\,\beta(s) \geqslant \eta$ *for almost all* $s \in \mathbb{R}$ *and that either* $\mathrm{Im}\,k > 0$ *or* $\eta > 0$. *Then BVP2 has exactly one solution and, given any bounded set* $P \subset \mathbb{C}_\eta := \{w \in \mathbb{C}\colon \mathrm{Re}\,w \geqslant \eta\}$, *there exists a constant* C_P, *dependent on* P *but independent of* β *and* f, *such that, if* ess. range $\beta \subset P$, *the solution* u *of BVP2 satisfies* (3).

Proof. Suppose that $P \subset \mathbb{C}_\eta$ is bounded and let P' be the closure of the convex hull of P. Then P' is compact and convex and $P \subset P' \subset \mathbb{C}_\eta$. By Theorems 4.2 and 4.15, $I - K_\beta$ is injective for every $\beta \in L^{P'}$. Since P' is compact and convex, so is the set

$$Q := \{\mathrm{i}k(w-1)\colon w \in P'\}. \tag{98}$$

Applying Theorem 4.16, with $\kappa := \lambda_0$ and Q defined by (98), we deduce that $I - K_\beta$ is surjective for every $\beta \in L^{P'}$ and that

$$C' := \sup_{\beta \in L^{P'}} \|(I - K_\beta)^{-1}\| < \infty.$$

Thus, for every $\beta \in L^{P'}$ and $f \in L_\infty(\mathbb{R})$, the integral equation (97) has a unique solution $u_0 \in BC(\mathbb{R})$, which satisfies

$$\|u_0\|_\infty \leqslant C' \|F\|_\infty \leqslant C' \|\lambda_0\|_1 \|f\|_\infty.$$

From Theorems 4.1 and 4.2 it follows that BVP2 has a unique solution, defined in terms of u_0 by (51). Further, by Corollary 4.3, u satisfies (3), with $C_P := C'\tilde{C}\,\|\lambda_0\|_1$. $\square$

4.4. *Limiting absorption principle*

We show in this section, by establishing a limiting absorption principle, that the radiation condition in BVP2, which is superfluous by Remark 4.3 if $\mathrm{Im}\,k > 0$, is selecting in the case $\mathrm{Re}\,\beta \geqslant \eta > 0$ the physically correct solution for $k > 0$.

Write $u, \kappa_h, \lambda_0, K_\beta$, and $F := -\lambda_0 * f$ as $u^{(k)}$, $\kappa_h^{(k)}$, $\lambda_0^{(k)}$, $K_\beta^{(k)}$, and $F^{(k)}$, respectively, to denote their dependence on k. Explicitly,

$$\lambda_0^{(k)}(s) = \frac{\mathrm{i}}{2} H_0^{(1)}(k|s|) + \frac{\mathrm{e}^{\mathrm{i}k|s|}}{\pi} \int_0^\infty \frac{t^{-1/2}\mathrm{e}^{-k|s|t}}{\sqrt{t-2i}\,(t-i)^2}\,\mathrm{d}t, \quad s \in \mathbb{R}. \tag{99}$$

It is shown in [11] that (42) holds uniformly not only in x and y but also in $\mathrm{Im}\,k$, for $0 \leqslant \mathrm{Im}\,k \leqslant C$. From this and (99), applying the dominated convergence theorem, it follows that, for every $k > 0$, $\|\lambda_0^{(k+\mathrm{i}\varepsilon)} - \lambda_0^{(k)}\|_1 \to 0$ as $\varepsilon \to 0^+$. Thus

$$\|K_\beta^{(k+\mathrm{i}\varepsilon)} - K_\beta^{(k)}\| \leqslant \{|k+\mathrm{i}\varepsilon|\,\|\lambda_0^{(k+\mathrm{i}\varepsilon)} - \lambda_0^{(k)}\|_1 + \varepsilon \|\lambda_0^{(k)}\|_1\}\,\|\beta - 1\|_\infty \to 0$$

as $\varepsilon \to 0^+$.

Suppose that, for some $\eta > 0$, $\mathrm{Re}\,\beta(s) \geqslant \eta$ for almost all $s \in \mathbb{R}$. Then, by Theorem 4.17, BVP2 has exactly one solution and, for $k > 0$ and $\varepsilon \geqslant 0$, $I - K_\beta^{(k+\mathrm{i}\varepsilon)}$ is invertible.

By standard operator perturbation results, for all ε sufficiently small such that $C(\varepsilon) := \|(I - K_\beta^{(k)})^{-1}\| \, \|K_\beta^{(k+i\varepsilon)} - K_\beta^{(k)}\| < 1$, $\|(I - K_\beta^{(k+i\varepsilon)})^{-1}\| \leqslant \|(I - K_\beta^{(k)})^{-1}\|/(1 - C(\varepsilon))$. Thus,

$$\begin{aligned}\|u_0^{(k+i\varepsilon)} - u_0^{(k)}\|_\infty &= \|(I - K_\beta^{(k+i\varepsilon)})^{-1}\{(K_\beta^{(k+i\varepsilon)} - K_\beta^{(k)})u_0^{(k)} + F^{(k+i\varepsilon)} - F^{(k)}\}\|_\infty \\ &\leqslant \|(I - K_\beta^{(k+i\varepsilon)})^{-1}\| \, \{\|K_\beta^{(k+i\varepsilon)} - K_\beta^{(k)}\| \, \|u_0^{(k)}\|_\infty \\ &\quad + \|\lambda_0^{(k+i\varepsilon)} - \lambda_0^{(k)}\|_1 \|f\|_\infty\} \\ &\to 0\end{aligned}$$

as $\varepsilon \to 0^+$. Further,

$$u_h^{(k+i\varepsilon)} - u_h^{(k)} = \kappa_h^{(k+i\varepsilon)} * u_0^{(k+i\varepsilon)} - \kappa_h^{(k)} * u_0^{(k)}$$

so that (cf. proof of Theorem 3.3),

$$\begin{aligned}\|u_h^{(k+i\varepsilon)} - u_h^{(k)}\|_\infty &\leqslant \|\kappa_h^{(k+i\varepsilon)} - \kappa_h^{(k)}\|_1 \, \|u_0^{(k+i\varepsilon)}\|_\infty + \|\kappa_h^{(k)}\|_1 \, \|u_0^{(k+i\varepsilon)} - u_0^{(k)}\|_\infty \\ &\to 0\end{aligned}$$

as $\varepsilon \to 0^+$. We have shown the following result.

Theorem 4.18. *Suppose that, for some $\eta > 0$, $\operatorname{Re}\beta(s) \geqslant \eta$ for almost all $s \in \mathbb{R}$ and that $k > 0$. Then, for every $x \in \bar{U}$, $u^{(k+i\varepsilon)}(x) \to u^{(k)}(x)$ as $\varepsilon \to 0^+$, where $u^{(k+i\varepsilon)}$ and $u^{(k)}$ denote the unique solutions of BVP2 for wave numbers $k + i\varepsilon$ and k, respectively.*

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