On the convergence of Galerkin BEM for classical 2nd kind boundary integral equations in Lipschitz domains

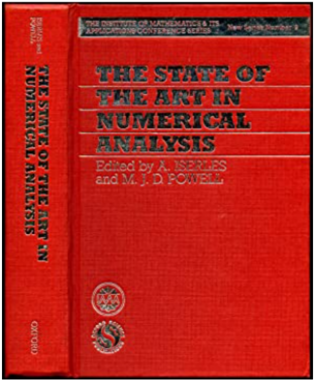
Simon Chandler-Wilde

Department of Mathematics and Statistics
University of Reading
s.n.chandler-wilde@reading.ac.uk

Joint work with:
Euan Spence (Bath)

Söllerhaus Workshop 2021, dedicated to Wolfgang L. Wendland in celebration of his 85th birthday.

Wolfgang Wendland (and me)



Our first meeting: IMA/SIAM Joint conference on the State of the Art in Numerical Analysis, University of Birmingham, April 1986.

Wolfgang is a plenary speaker on **"Strongly Elliptic Boundary Integral Equations"**. I'm a PhD student, School of Civil and Structural Engineering, University of Bradford, UK. This is my first maths conference.

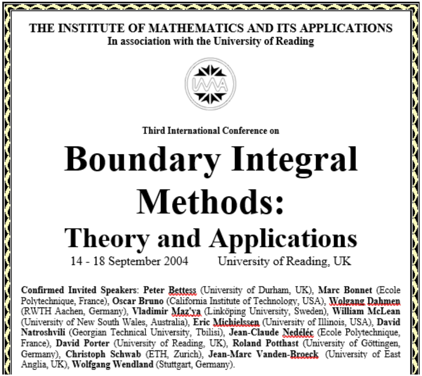
Wolfgang Wendland (and me)

Occasion: Randelementmethoden: Anwendungen und Fehleranalysis

1st MFO workshop focussed on BEM, Oberwolfach, October 1994.

Wolfgang is organiser with Ernst Stephan. I'm recently appointed Lecturer in Mathematics, Brunel University. I'm an even more recent father. This is my first Oberwolfach.

Wolfgang Wendland (and me)



THE INSTITUTE OF MATHEMATICS AND ITS APPLICATIONS
In association with the University of Reading

Third International Conference on

Boundary Integral Methods: Theory and Applications

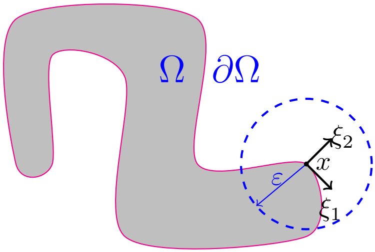
14 - 18 September 2004 University of Reading, UK

Confirmed Invited Speakers: **Peter Bettess** (University of Durham, UK), **Marc Bonnet** (Ecole Polytechnique, France), **Oscar Bruno** (California Institute of Technology, USA), **Wolgang Dahmen** (RWTH Aachen, Germany), **Vladimir Maz'ya** (Linköping University, Sweden), **William McLean** (University of New South Wales, Australia), **Eric Michielssen** (University of Illinois, USA), **David Natroshvili** (Georgian Technical University, Tbilisi), **Jean-Claude Nedélec** (Ecole Polytechnique, France), **David Porter** (University of Reading, UK), **Roland Potthast** (University of Göttingen, Germany), **Christoph Schwab** (ETH, Zurich), **Jean-Marc Vanden-Broeck** (University of East Anglia, UK), **Wolfgang Wendland** (Stuttgart, Germany).

Wolfgang is plenary speaker (introduced by Christoph Schwab), lecture title: **"On J. Radon's Convergence Proof for C. Neumann's Method with Double Layer Potentials"**. I'm recently appointed Professor of Applied Mathematics, University of Reading, and this is my first time chairing a conference.

Overview

1. **Lipschitz domains** and an example we will meet later
2. **Potential theory, 2nd kind boundary integral equations, and a long-standing open question**
3. The **Hilbert space** theory of **Galerkin methods**
4. **Do all Galerkin BEMs, based on asymptotically dense subspace sequences and testing with L^2 inner products, converge for the standard 2nd kind BIEs on Lipschitz domains?**

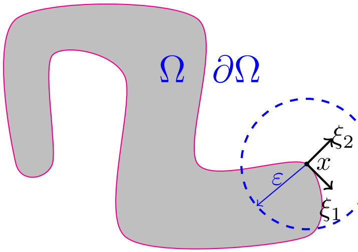


A bounded domain $\Omega \subset \mathbb{R}^2$ is **Lipschitz** if, in a neighbourhood of each point $x \in \partial\Omega$,

$$\partial\Omega \cap B_\epsilon(x) = \{(\xi_1, f(\xi_1)) : \xi_1 \in \mathbb{R}\} \cap B_\epsilon(x),$$

for some f that satisfies, for some $L > 0$ (the **Lipschitz constant**)

$$|f(s) - f(t)| \leq L|s - t|, \quad \text{for } s, t \in \mathbb{R}.$$



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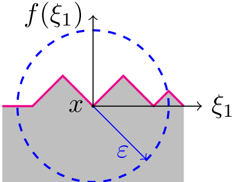
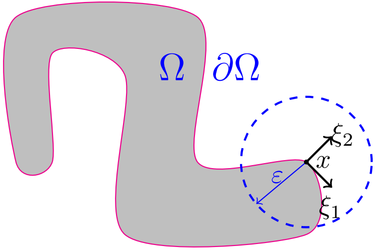
This **allows corners**, e.g. this f has $L = 1$ ...





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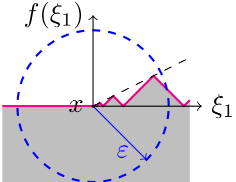
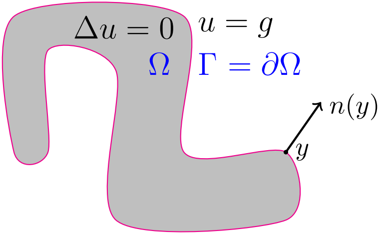
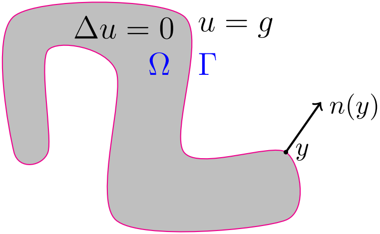
Indeed it allows **infinitely many corners**, e.g. this f also has $L = 1$ ...





Assume that $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) is **bounded** and **Lipschitz**, and $g \in L^2(\Gamma)$.



BVP: Find $u \in C^2(\Omega)$ such that $\Delta u = 0$ in Ω and $u = g$ on Γ.

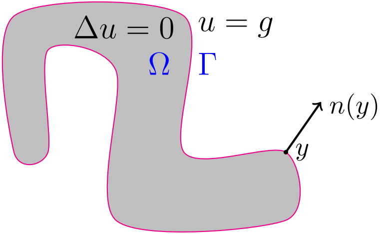
Define the **fundamental solution**

$$G(x,y) := \begin{cases} -\dfrac{1}{\pi} \log|x-y|, & d = 2, \\ (2\pi|x-y|)^{-1}, & d = 3, \end{cases}$$

Look for a solution as the **double-layer potential** with density $\phi \in L^2(\Gamma)$:

$$\begin{aligned} u(x) &= \int_\Gamma \frac{\partial G(x,y)}{\partial n(y)} \phi(y) \, \mathrm{d}s(y) \\ &= \frac{1}{2^{d-2}\pi} \int_\Gamma \frac{(x-y)\cdot n(y)}{|x-y|^d} \phi(y) \, \mathrm{d}s(y). \end{aligned}$$

for $x \in \Omega$.



Look for a solution as the **double-layer potential** with density $\phi \in L^2(\Gamma)$:

$$u(x) = \int_\Gamma \frac{\partial G(x,y)}{\partial n(y)} \phi(y)\, \mathrm{d}s(y), \quad x \in \Omega.$$

This satisfies the BVP iff ϕ satisfies the **boundary integral equation (BIE)**

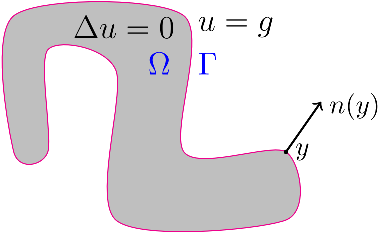
$$\phi(x) - \int_\Gamma \frac{\partial G(x,y)}{\partial n(y)} \phi(y)\, \mathrm{d}s(y) = -g(x), \quad x \in \Gamma,$$

in operator form

$$\phi - D\phi = -g \quad \text{or } A\phi = -g,$$

where $A = I - D$, I is the identity operator, and D is the **double-layer potential operator** given by

$$D\phi(x) = \int_\Gamma \frac{\partial G(x,y)}{\partial n(y)} \phi(y)\, \mathrm{d}s(y), \quad x \in \Gamma,\ \phi \in L^2(\Gamma).$$



The double-layer potential satisfies the BVP iff ϕ satisfies the **BIE** in operator form

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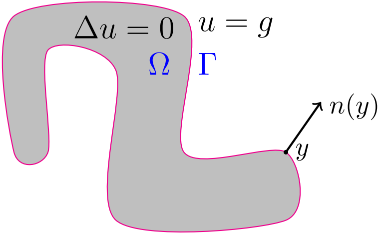
where $A = I - D$. The **Galerkin method** for solving the BIE numerically is: choose a finite-dimensional subspace $V_N \subset L^2(\Gamma)$ and approximate

$$\phi \approx \phi_N \in V_N,$$

where

$$(A\phi_N, \psi_N) = -(g, \psi_N), \quad \forall \psi_N \in V_N, \quad \text{and } (u, v) := \int_\Gamma u\bar{v}\, \mathrm{d}s.$$



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Long-standing open problem. "For a general Lipschitz boundary Γ, however, stability and convergence of Galerkin's method in $L^2(\Gamma)$ is not yet known."
Wendland (2009)

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H is a **complex Hilbert space** with norm $\|u\| = \sqrt{(u,u)}$, e.g.

$$H = L^2(\Gamma), \quad (u,v) = \int_\Gamma u\bar{v}\,\mathrm{d}s, \quad \|u\|^2 = \int_\Gamma |u|^2\,\mathrm{d}s.$$

Suppose that A is a **bounded linear operator** on H.

A is **coercive** if, for some $\gamma > 0$,

$$|(Au,u)| \geq \gamma \|u\|^2, \quad \forall u \in H.$$

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E.g. if $A = I - B$, where I is the identity operator and B is bounded,

$$(Au,u) = (u - Bu, u) \quad = \quad (u,u) - (Bu,u) \geq (1 - \|B\|)\|u\|^2.$$

So $A = I - B$ is coercive if $\|B\| < 1$, with $\gamma = 1 - \|B\|$.

Suppose that A is a **bounded linear operator** on H.

The Galerkin method. Pick a sequence $V = (V_1, V_2, ...)$ of finite-dimensional subspaces of H, and seek $u_N \in V_N$ such that

$$(Au_N, v) = (g, v), \quad \forall v \in V_N \quad (G).$$

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In the case that A is invertible, we will say that the **Galerkin method is convergent for the sequence** V if, for every $g \in H$, (G) has a unique solution for all sufficiently large N and $u_N \to u = A^{-1}g$ as $N \to \infty$.

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We will say that V **is asymptotically dense in** H if, for every $u \in H$,

$$\inf_{v_N \in V_N} \|u - v_N\| \to 0 \quad \text{as} \quad N \to \infty.$$

It is easy to see that a **necessary condition** for the convergence of the Galerkin method is that V is asymptotically dense in H.

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The Key Abstract Theorem. (Markus, 1974). If A is invertible then the following statements are equivalent:

- The Galerkin method converges for every V that is asymptotically dense in H.
- $A = A_0 + K$ where A_0 is **coercive** and K is **compact**.

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What is known about the double-layer potential operator D and $A = I - D$ when Ω is Lipschitz? Remember the BIE in operator form is $A\phi = -g$.

- A is a bounded linear operator on $L^2(\Gamma)$ if Ω is a bounded Lipschitz domain (Coifman, McIntosh, Meyer *Ann. Math.* 1982)
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Open question: is $A =$ coercive + compact on $L^2(\Gamma)$

- for every bounded Lipschitz domain Ω?
- at least for every bounded Lipschitz domain in 2D?
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The answer is **NO** in each case (C-W & Spence, 2021).

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Here $W_{\mathrm{ess}}(A)$ denotes the **essential numerical range** of A, defined by

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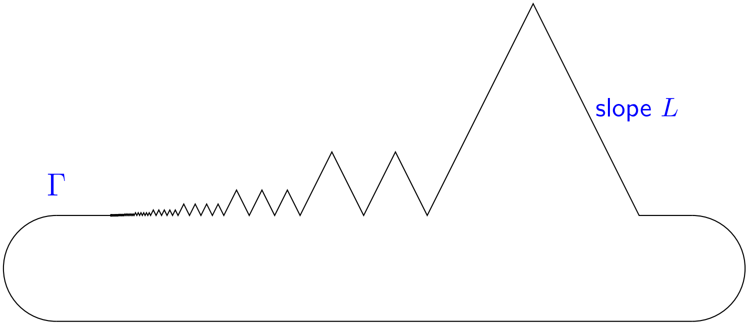
Key question: If $A = I - D$ and D is the double-layer potential operator, is $0 \in W_{\mathrm{ess}}(A)$? Equivalently, is $1 \in W_{\mathrm{ess}}(D)$?

Theorem. (C-W, Spence 2021) If Γ is the boundary of the Lipschitz domain shown below with Lipschitz constant L, then

$$W_{\mathrm{ess}}(D) \supset \{z \in \mathbb{C} : |z| \leq L/2\}.$$

Thus, if $L \geq 2$, then $1 \in W_{\mathrm{ess}}(D)$, so that $A = I - D$ is not coercive + compact.



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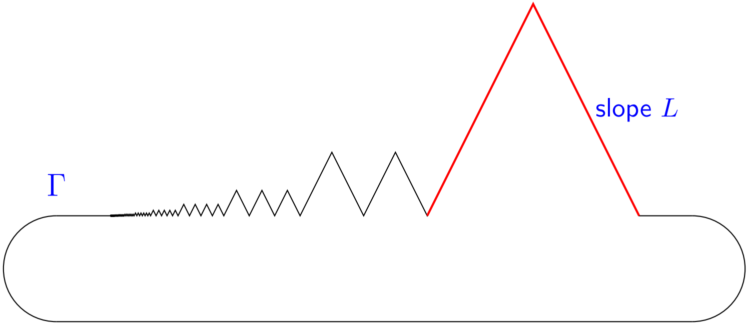

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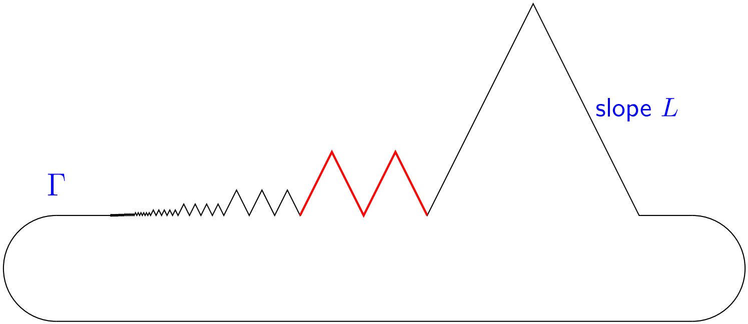

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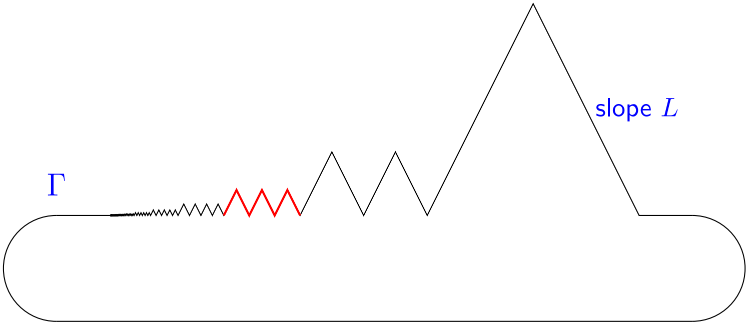
$$W_{\mathrm{ess}}(D) \supset \{z \in \mathbb{C} : |z| \leq L/2\}.$$

Thus, if $L \geq 2$, then $1 \in W_{\mathrm{ess}}(D)$, so that $A = I - D$ is not coercive + compact.



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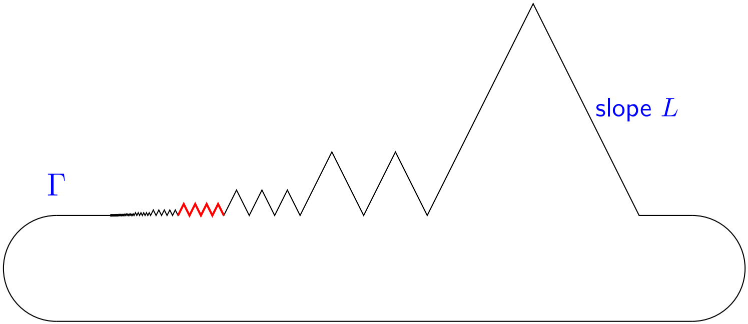

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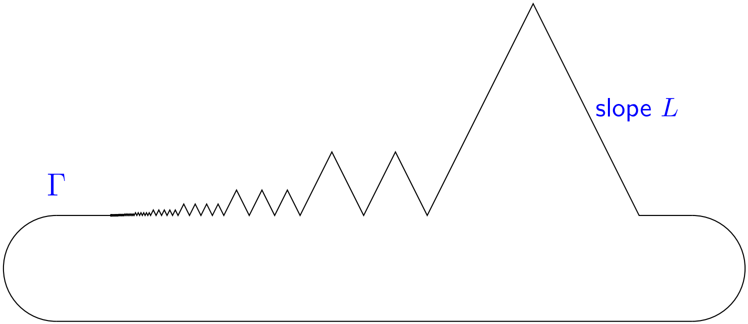
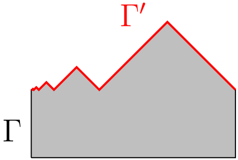


How is this proved?

Three simple lemmas.

Lemma A. If $\Gamma' \subset \Gamma$ and D' is the DLP operator on Γ', then

$$W\left(D'\right) \subset W(D).$$



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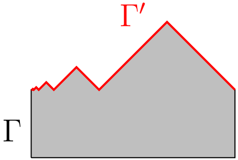
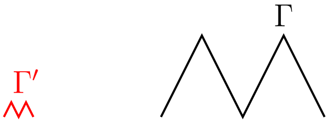


Lemma B. If Γ' and Γ are similar and D' is the DLP operator on Γ', then

$$W(D') = W(D).$$



Lemma C. If $\Gamma_1 \subset \Gamma_2 \subset \ldots \Gamma = \bigcup_{j=1}^{\infty} \Gamma_j$, and D_j denotes the DLP on Γ_j, then

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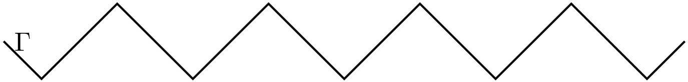

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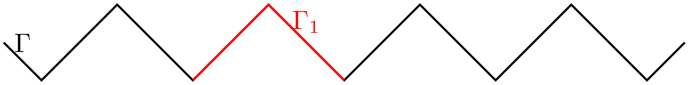

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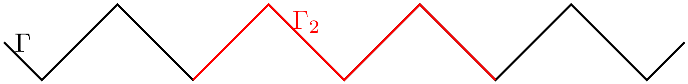

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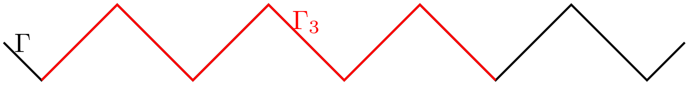

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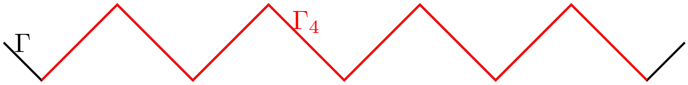
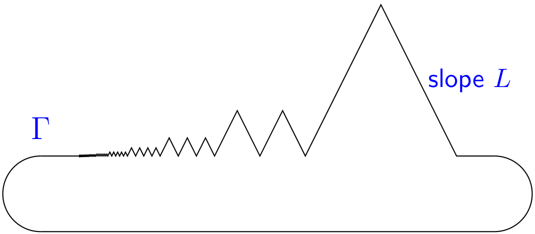


What can we say about $W(D)$ for the DLP operator D on this Γ?



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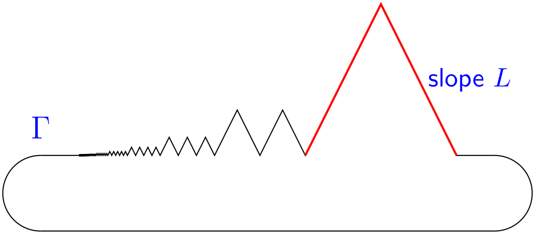

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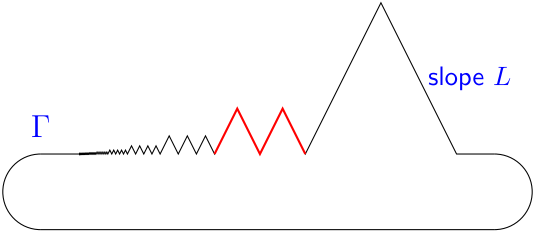

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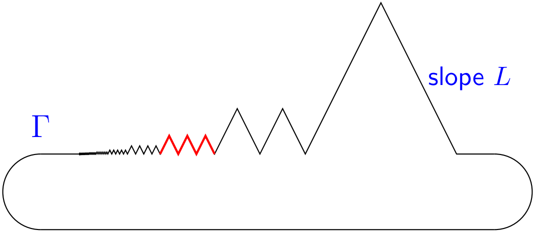

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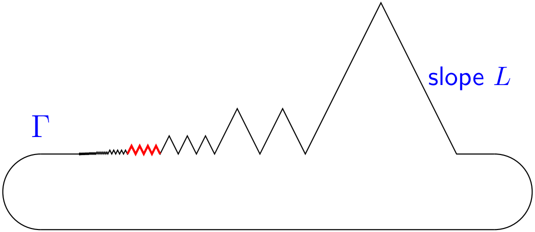

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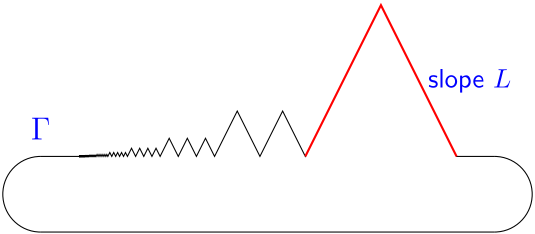

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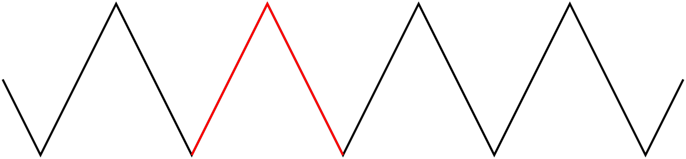

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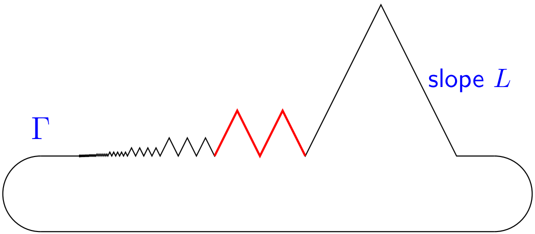

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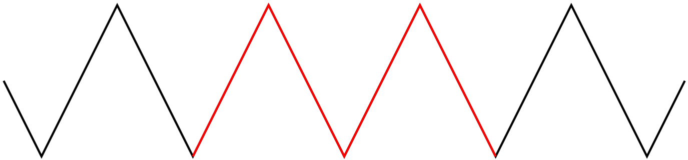

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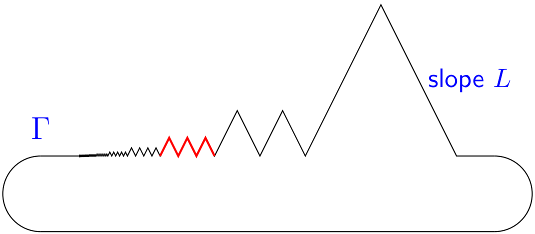

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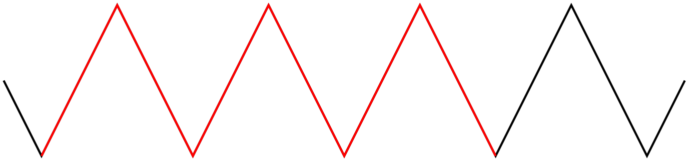

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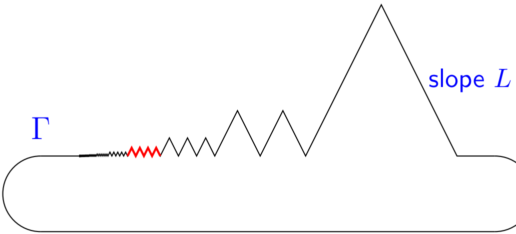

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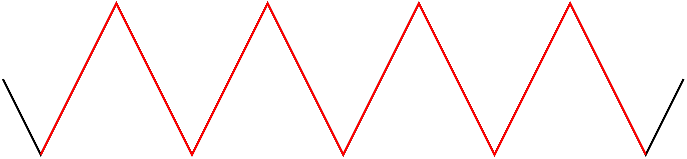

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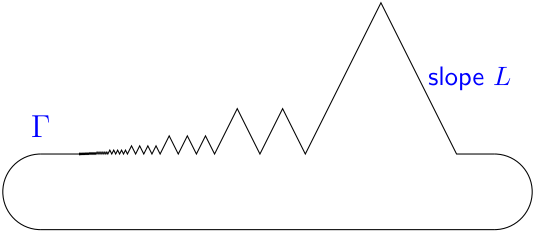

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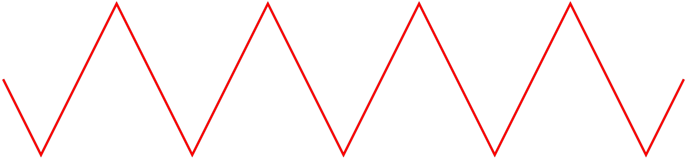
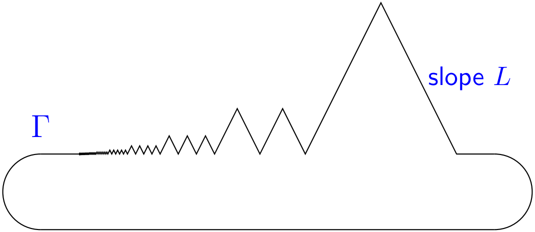


So, by **Lemma C**, also $W(D) \supset W(D^\dagger)$ where $D^\dagger$ is the DLP operator on the **infinite sawtooth**.

What can we say about $W(D)$ for the DLP operator D on this Γ?



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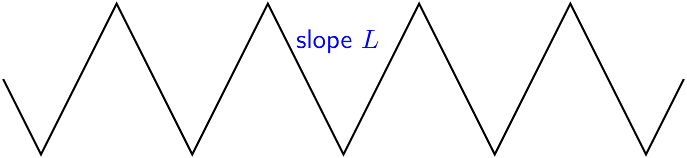
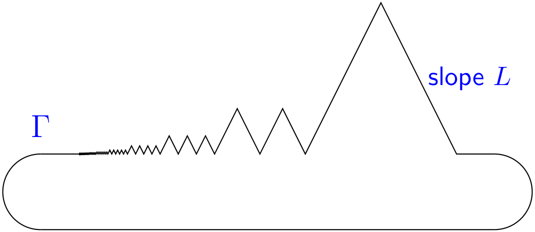


So, by **Lemma C**, also $W(D) \supset W(D^\dagger)$ where $D^\dagger$ is the DLP operator on the **infinite sawtooth**. And $W(D^\dagger)$ (by some explicit calculations) contains $\{z \in \mathbb{C} : |z| \le L/2\}$.

What can we say about $W(D)$ for the DLP operator D on this Γ?



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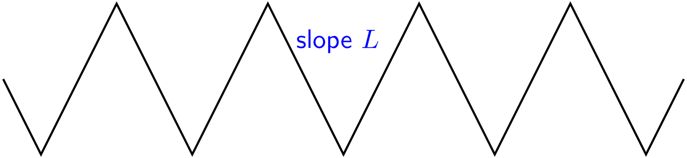

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Theorem. (C-W, Spence 2021) If Γ is the boundary of the Lipschitz domain shown below with Lipschitz constant L, then

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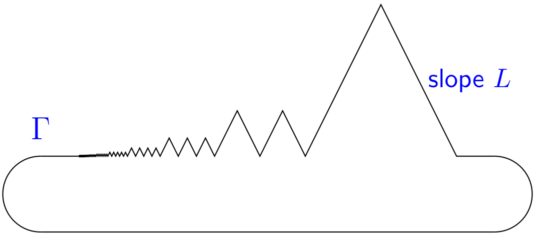

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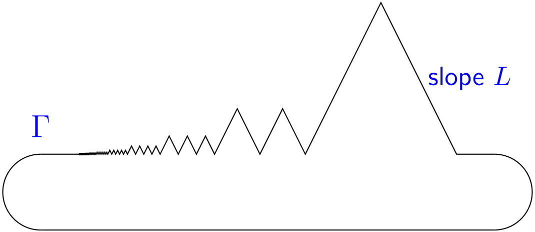

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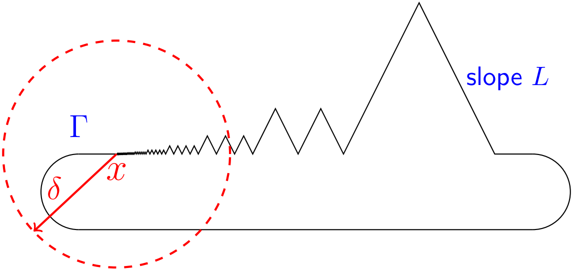

Localisation Lemma. (C-W, Spence 2021, cf. I. Mitrea, 1999)

$$W_{\mathrm{ess}}(D) \supseteq \bigcap_{\delta>0} W\left(D_{x,\delta}\right), \quad \forall x \in \Gamma,$$

with equality for at least one x, where $D_{x,\delta}$ is the DLP operator on $\Gamma \cap B_\delta(x)$.

In conclusion we have proved ...

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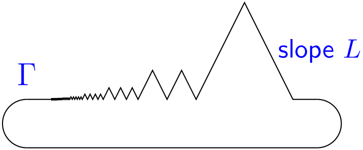

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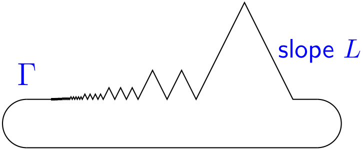
Thus, if $L \geq 2$, then $1 \in W_{\mathrm{ess}}(D)$, so that $A = I - D$ is not coercive + compact, so that there exists a sequence of subspaces that is asymptotically dense in $L^2(\Gamma)$ for which the Galerkin method does not converge.



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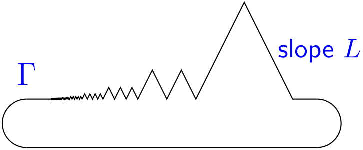

Theorem. (C-W, Spence 2021) If A is not coercive + compact then for every asymptotically dense $V = (V_1, V_2, \ldots)$ there exists a sequence $V^* = (V_1^*, V_2^*, \ldots)$ for which the Galerkin method does not converge which is **sandwiched by** V, meaning that, for each N,

$$V_N \subset V_N^* \subset V_{M_N}, \quad \text{for some } M_N \ge N.$$

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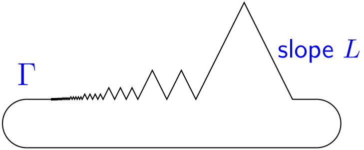

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$$V_N \subset V_N^* \subset V_{M_N}, \quad \text{for some } M_N \geq N.$$

Choose V to be any asymptotically dense sequence of **BEM** spaces. Then V^* is a BEM space sequence ($V_N^* \subset V_{M_N}$) that is asymptotically dense ($V_N \subset V_N^*$) for which **the Galerkin method does not converge**.

On arXiv ...

Coercivity, essential norms, and the Galerkin method for second-kind integral equations on polyhedral and Lipschitz domains

S. N. Chandler-Wilde*, E. A. Spence†

Dedicated to Wolfgang Wendland on the occasion of his 85th birthday

Abstract

It is well known that, with a particular choice of norm, the classical double-layer potential operator D has essential norm $< 1/2$ as an operator on the natural trace space $H^{1/2}(\Gamma)$ whenever Γ is the boundary of a bounded Lipschitz domain. This implies, for the standard second-kind boundary integral equations for the interior and exterior Dirichlet and Neumann problems in potential theory, convergence of the Galerkin method in $H^{1/2}(\Gamma)$ for any sequence of finite-dimensional subspaces $(\mathcal{H}_N)_{N=1}^{\infty}$ that is asymptotically dense in $H^{1/2}(\Gamma)$. Long-standing open questions are whether the essential norm is also $< 1/2$ for D as an operator on $L^2(\Gamma)$ for all Lipschitz Γ in 2-d; or whether, for all Lipschitz Γ in 2-d and 3-d, or at least for the smaller class of Lipschitz polyhedra in 3-d, the weaker condition holds that the operators $\pm\frac{1}{2}I + D$ are compact perturbations of coercive operators – this a necessary and sufficient condition for the convergence of the Galerkin method for every sequence of subspaces $(\mathcal{H}_N)_{N=1}^{\infty}$ that is asymptotically dense in $L^2(\Gamma)$. We settle these open questions negatively. We give examples of 2-d and 3-d Lipschitz domains with Lipschitz constant equal to one for which the essential norm of D is $\geq 1/2$, and examples with Lipschitz constant two for which the operators $\pm\frac{1}{2}I + D$ are not coercive plus compact. We also give, for every $C > 0$, examples of Lipschitz polyhedra for which the essential norm is $\geq C$ and for which $\lambda I + D$ is not a compact perturbation of a coercive operator for any real or complex λ with $|\lambda| \leq C$. We then, via a new result on the Galerkin method in Hilbert spaces, explore the implications of these results for the convergence of Galerkin boundary element methods in the $L^2(\Gamma)$ setting. Finally, we resolve negatively a related open question in the convergence theory for collocation methods, showing that, for our polyhedral examples, there is no weighted norm on $C(\Gamma)$, equivalent to the standard supremum norm, for which the essential norm of D on $C(\Gamma)$ is $< 1/2$.

Summary of the talk

1. **Lipschitz domains** and an example we will meet later
2. **Potential theory, 2nd kind boundary integral equations, and a long-standing open question**
3. The **Hilbert space** theory of **Galerkin methods**
4. **Do all Galerkin BEMs, based on asymptotically dense subspace sequences and testing with L^2 inner products, converge for the standard 2nd kind BIEs on Lipschitz domains?**

3D Polyhedra for which $A = I - D$ is not coercive + compact.

The "open book" polyhedron with four pages and opening angle $\theta = \pi/4$.