Do Galerkin methods converge for the classical 2nd kind boundary integral equations in polyhedra and Lipschitz domains?

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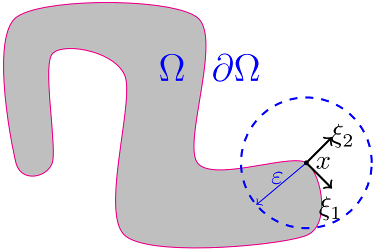
Joint work with:
Euan Spence (Bath)

Analysis and PDE Seminar, Tbilisi, May 2021

Overview of the talk

1. **Lipschitz domains**
   - What are they? An example we will meet later.
2. **Potential theory** and **2nd kind boundary integral equations** (BIEs)
   - A Dirichlet problem and 2nd kind BIE formulation
   - The Galerkin approximation to the BIE
   - **A long-standing open problem: do Galerkin methods converge?**
3. The **Hilbert space** theory of **Galerkin methods**
   - Definitions of bounded, compact, coercive
   - Galerkin methods and their convergence
4. **Do all sensible Galerkin methods (i.e., based on V convergent to $L^2(\Gamma)$) converge for the standard 2nd kind BIEs?**
   - Previous results
   - **Solving the open problem**: Constructing Ω for which $A = I - D$ is not coercive + compact



A bounded domain $\Omega \subset \mathbb{R}^2$ is **Lipschitz** if, in a neighbourhood of each point $x \in \partial\Omega$, $\partial\Omega$ is the graph of a **Lipschitz continuous function** f, with respect to some rotated coordinate system $0\xi_1\xi_2$, with Ω on precisely one side of $\partial\Omega$.

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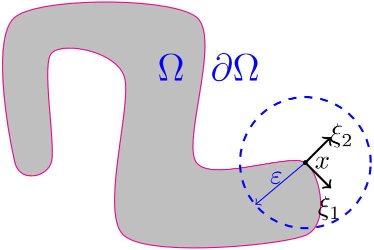
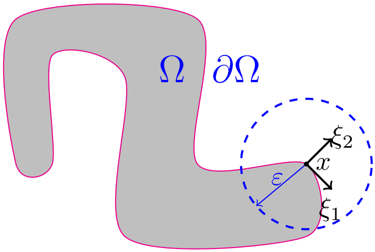


In equations,

$$\partial\Omega \cap B_\epsilon(x) = \{(\xi_1, f(\xi_1)) : \xi_1 \in \mathbb{R}\} \cap B_\epsilon(x),$$

for some f that satisfies, for some $L > 0$ (the **Lipschitz constant**)

$$|f(s) - f(t)| \leq L|s - t|, \quad \text{for } s, t \in \mathbb{R}.$$



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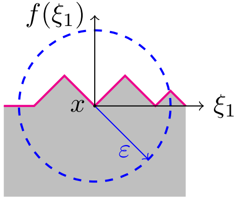
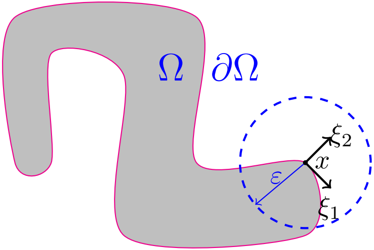
This **allows corners**, e.g. this f has $L = 1$ ...





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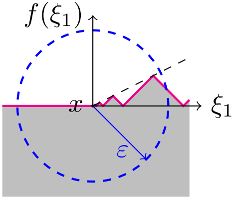
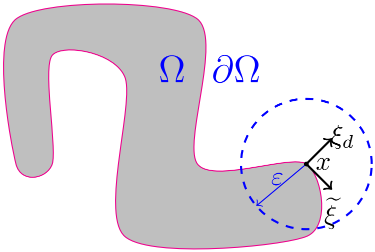
Indeed it allows **infinitely many corners**, e.g. this f also has $L = 1$ ...





A bounded domain $\Omega \subset \mathbb{R}^d$ is **Lipschitz** if, in a neighbourhood of each point $x \in \partial\Omega$, $\partial\Omega$ is the graph of a **Lipschitz continuous function** f, with respect to some rotated coordinate system $0\xi_1 ... \xi_d$, with Ω on precisely one side of $\partial\Omega$.

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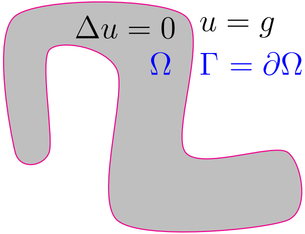
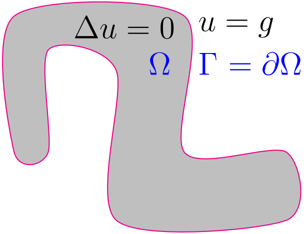


Assume that $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) is **bounded** and **Lipschitz**, and $g \in L^2(\Gamma)$.



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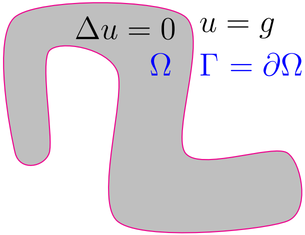
BVP: Find $u \in C^2(\Omega)$ such that $\Delta u = 0$ in Ω and $u = g$ on Γ.



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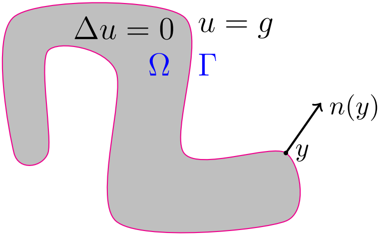
Define the **fundamental solution**

$$G(x,y) := \begin{cases} -\dfrac{1}{\pi} \log|x-y|, & d = 2, \\ (2\pi|x-y|)^{-1}, & d = 3, \end{cases}$$



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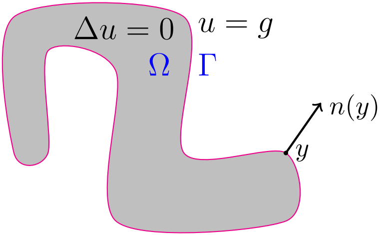
Look for a solution as the **double-layer potential** with density $\phi \in L^2(\Gamma)$ (which satisfies $\Delta u = 0$ in Ω):

$$\begin{aligned} u(x) &= \int_\Gamma \frac{\partial G(x,y)}{\partial n(y)} \phi(y) \, \mathrm{d}s(y) \\ &= \frac{1}{2^{d-2}\pi} \int_\Gamma \frac{(x-y)\cdot n(y)}{|x-y|^d} \phi(y) \, \mathrm{d}s(y). \end{aligned}$$

for $x \in \Omega$.



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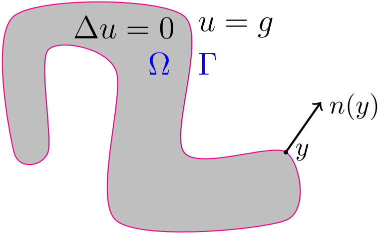
for $x \in \Omega$. This idea (with $\phi \in C(\Gamma)$) dates back to Gauss.



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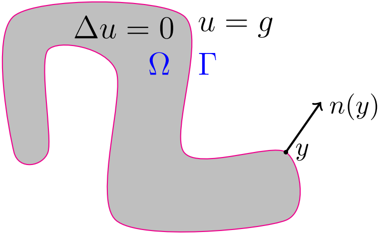

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This satisfies the BVP iff ϕ satisfies the **boundary integral equation (BIE)**

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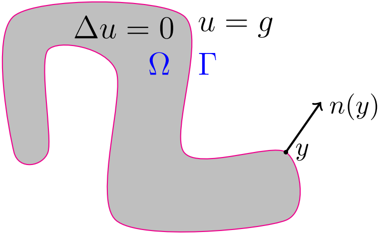

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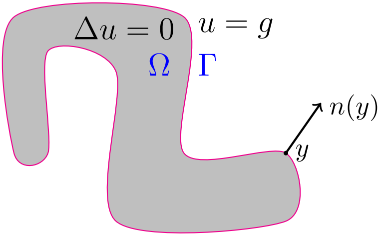
in operator form

$$\phi - D\phi = -g \quad \text{or } A\phi = -g,$$

where $A = I - D$, I is the identity operator, and D is the **double-layer potential operator** given by

$$D\phi(x) = \int_\Gamma \frac{\partial G(x,y)}{\partial n(y)} \phi(y)\, \mathrm{d}s(y), \quad x \in \Gamma,\ \phi \in L^2(\Gamma).$$



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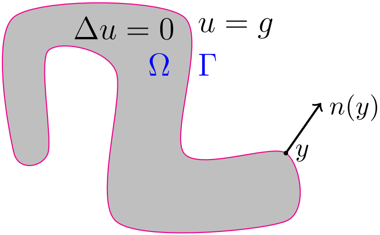

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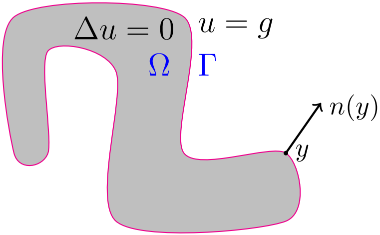
$$\phi \approx \phi_N := \sum_{n=1}^{N} \alpha_n v_n,$$

choosing the coefficients $\alpha_1, ..., \alpha_N \in \mathbb{C}$ so that

$$(A\phi_N, v_m) = -(g, v_m), \quad m = 1, ..., N, \quad \text{where } (u, v) := \int_\Gamma u\bar{v}\, \mathrm{d}s.$$



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Long-standing open problem. "For a general Lipschitz boundary Γ, however, stability and convergence of Galerkin's method in $L^2(\Gamma)$ is not yet known." Wendland (2009)

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H is a **complex Hilbert space** with inner product (u, v) and norm $\|u\| = \sqrt{(u, u)}$, e.g.

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A is a **bounded linear operator** on H if

$$A(\lambda u) = \lambda A u, \quad A(u+v) = Au + Av, \quad \forall \lambda \in C,\ u, v \in H,$$

and, for some $C \geq 0$,

$$\|Au\| \leq C\|u\|, \quad \forall u \in H.$$

The **norm** of A is

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A is **compact** if, for some sequence of finite rank operators $A_1, A_2, ...$, it holds that $\|A - A_n\| \to 0$ as $n \to \infty$.

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So $A = I - B$ is coercive if $\|B\| < 1$, with $\gamma = 1 - \|B\|$.

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Indeed A is coercive iff $A = \theta(I - B)$ with $\theta \in \mathbb{C} \setminus 0$ and $\|B\| < 1$.

Suppose that A is a **bounded linear operator** on H.

A is **invertible** if

$$Au = g$$

has exactly one solution $u \in H$ for every $g \in H$, i.e. if $A : H \to H$ is **bijective**, in which case (the **Banach theorem**) A has a **bounded inverse** A^{-1}.

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$$(Au_N, v) = (g, v), \quad \forall v \in V_N \quad (G).$$

In the case that A is invertible, we will say that the **Galerkin method is convergent for the sequence** V if, for every $g \in H$, (G) has a unique solution for all sufficiently large N and $u_N \to u = A^{-1}g$ as $N \to \infty$.

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It is clear that a **necessary condition** for the convergence of the Galerkin method is that V converges to H.

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Part a) (Markus, 1974). If A is invertible then there exists a sequence $V = (V_1, V_2, ...)$ for which the Galerkin method converges.

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This is almost as strong a result as Part b), with weaker requirements on A.

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What is known about the double-layer potential operator D and $A = I - D$ when Ω is Lipschitz? Remember the BIE in operator form is $A\phi = -g$.

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The answer is **NO** in each case (C-W & Spence, 2021).

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Here $W_{\text{ess}}(A)$ denotes the **essential numerical range** of A, defined by

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where, for a bounded linear operator B, $W(B)$ denotes the **numerical range** or **field of values** of B, given by

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If $A = I - D$ and D is the double-layer potential operator, is $0 \in W_{\mathrm{ess}}(A)$? Equivalently, is $1 \in W_{\mathrm{ess}}(D)$?

What is $W_{\mathrm{ess}}(D)$ for the double-layer potential operator?

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A couple of simple lemmas.

Lemma A. If $\Gamma' \subset \Gamma$ and D' is the DLP operator on Γ', then, since $L^2(\Gamma') \subset L^2(\Gamma)$,

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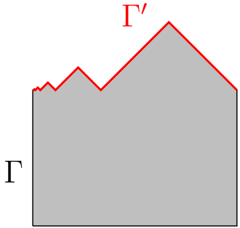

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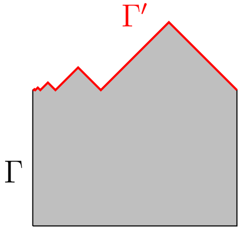

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Lemma B. If f is Lipschitz continuous and $\Gamma = \{(s, f(s)) : 0 \leq s \leq 1\}$ and, for some $0 < \alpha < 1$,

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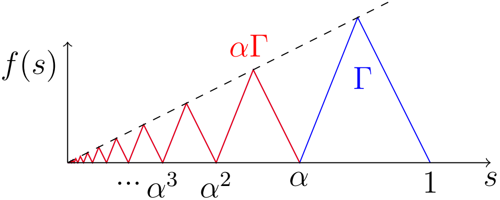

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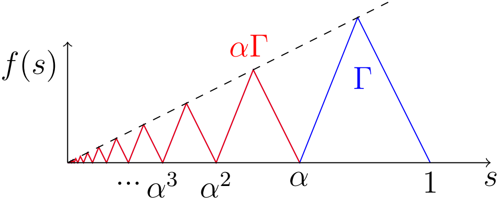

The above holds because $\alpha\Gamma \subset \Gamma \Rightarrow TD = DT$, where $T\phi(x) = \alpha^{-1/2}\phi(\alpha^{-1}x)$ is an isometry on $L^2(\Gamma)$, and $T^n\phi \rightharpoonup 0$ as $n \to \infty$, $\forall \phi \in L^2(\Gamma)$.

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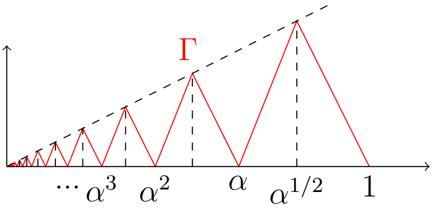

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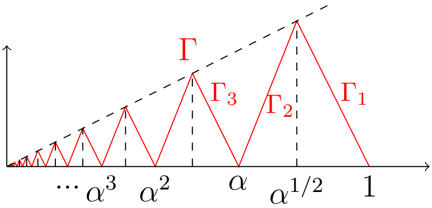

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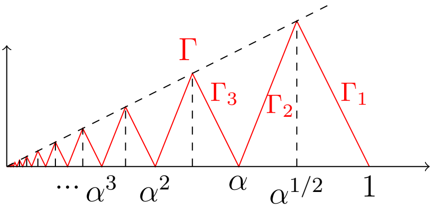

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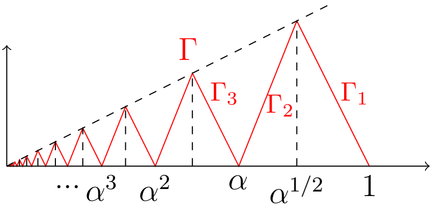

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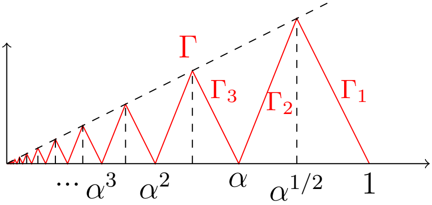
$$A_N := \left[\text{sign}(n-m)(-1)^{n+1}\right]_{m,n=1}^{N}, \quad \text{it holds that} \quad \frac{(D\phi, \phi)}{\|\phi\|^2} \to \frac{(A_N\underline{\phi}, \underline{\phi})}{\|\underline{\phi}\|^2}$$

as $\alpha \to 1^-$.

What is $W_{\text{ess}}(D) = \overline{W(D)}$ for the double-layer potential operator on this particular Γ?

$$W(D) = \{(D\phi, \phi) : \phi \in L^2(\Gamma),\ \|\phi\| = 1\} = \left\{ \frac{(D\phi, \phi)}{\|\phi\|^2} : \phi \in L^2(\Gamma) \setminus \{0\} \right\}$$



Choose $N \in \mathbb{N}$ and define $\phi \in L^2(\Gamma)$ by

$$\phi(x) := \phi_m \quad \text{on} \quad \Gamma_m, \quad \text{for } m = 1, ..., N,$$

$\phi(x) := 0$, otherwise. Then, where $\underline{\phi} = (\phi_1, ..., \phi_N)^T$, and

$$A_N := \left[\text{sign}(n-m)(-1)^{n+1}\right]_{m,n=1}^{N}, \quad \text{it holds that} \quad \frac{(D\phi, \phi)}{\|\phi\|^2} \to \frac{(A_N\underline{\phi}, \underline{\phi})}{\|\underline{\phi}\|^2}$$

as $\alpha \to 1^-$. **So every neighbourhood of $W(D)$ contains $W(A_N)$ if α is close enough to 1.**

$$A_N := \left[\operatorname{sign}(n-m)(-1)^{n+1}\right]_{m,n=1}^{N}, \quad \text{e.g.} \quad A_4 = \begin{pmatrix} 0 & -1 & 1 & -1 \\ -1 & 0 & 1 & -1 \\ -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 \end{pmatrix}.$$

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Lemma. $\operatorname{spec}(A_N) \subset \{-1, 0, 1\}$ for all N, but, for every $R > 0$, if N is large enough,

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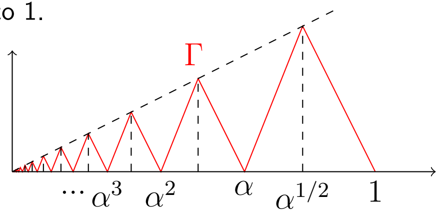
Corollary. For this particular Γ and for every $R > 0$,

$$W_{\text{ess}}(D) = \overline{W(D)} \supset \{z \in \mathbb{C} : |z| < R\}$$

if α is close enough to 1.



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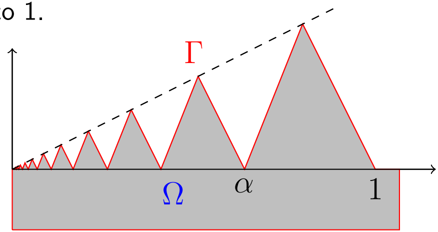

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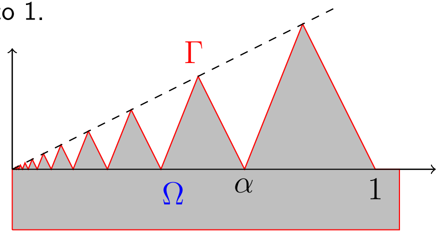

Corollary. For this domain Ω, $A = I - D$ is not coercive + compact if α is close enough to 1.

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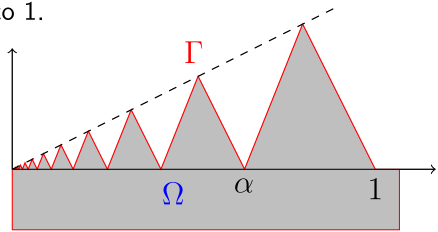
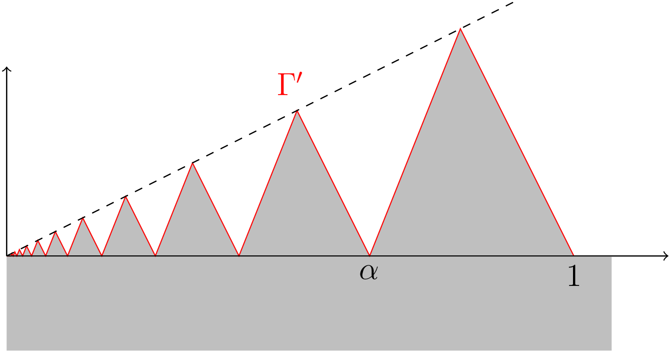


Corollary. For this domain Ω, $A = I - D$ is not coercive + compact if α is close enough to 1. **This counterexample solves the long-standing open problem.**

3D Counterexamples that are Polyhedra

The ingredients we needed for the 2D counterexample were:

- A subset Γ' of the boundary Γ that has the dilation invariance $\alpha\Gamma' \subset \Gamma'$, for some $0 < \alpha < 1$, so that $W_{\mathrm{ess}}(D) \supset W_{\mathrm{ess}}(D|_{L^2(\Gamma')}) = \overline{W(D|_{L^2(\Gamma')})}$
- Flat sides of Γ' that we can push arbitrarily close together by adjusting a parameter, reducing calculation of $W(D|_{L^2(\Gamma')})$ to calculation of $W(A_N)$



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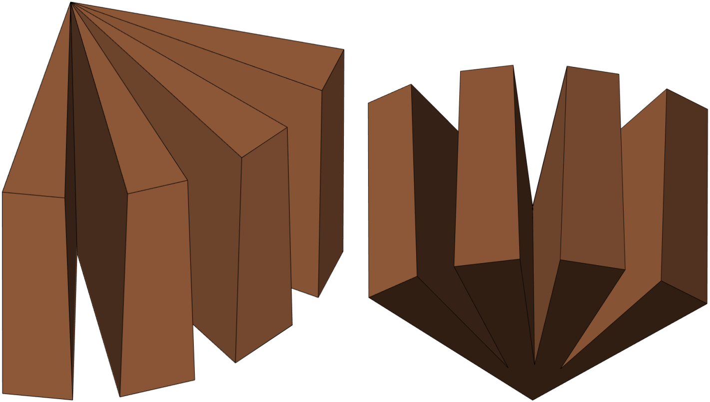
The "open book" polyhedron with four pages and opening angle $\theta = \pi/4$.



On arXiv from tomorrow ...

Coercivity, essential norms, and the Galerkin method for second-kind integral equations on polyhedral and Lipschitz domains

S. N. Chandler-Wilde*, E. A. Spence†

Dedicated to Wolfgang Wendland on the occasion of his 85th birthday

Abstract

It is well known that, with a particular choice of norm, the classical double-layer potential operator D has essential norm $< 1/2$ as an operator on the natural trace space $H^{1/2}(\Gamma)$ whenever Γ is the boundary of a bounded Lipschitz domain. This implies, for the standard second-kind boundary integral equations for the interior and exterior Dirichlet and Neumann problems in potential theory, convergence of the Galerkin method in $H^{1/2}(\Gamma)$ for any sequence of finite-dimensional subspaces $(\mathcal{H}_N)_{N=1}^{\infty}$ that is asymptotically dense in $H^{1/2}(\Gamma)$. Long-standing open questions are whether the essential norm is also $< 1/2$ for D as an operator on $L^2(\Gamma)$ for all Lipschitz Γ in 2-d; or whether, for all Lipschitz Γ in 2-d and 3-d, or at least for the smaller class of Lipschitz polyhedra in 3-d, the weaker condition holds that the operators $\pm\frac{1}{2}I + D$ are compact perturbations of coercive operators – this a necessary and sufficient condition for the convergence of the Galerkin method for every sequence of subspaces $(\mathcal{H}_N)_{N=1}^{\infty}$ that is asymptotically dense in $L^2(\Gamma)$. We settle these open questions negatively. We give examples of 2-d and 3-d Lipschitz domains with Lipschitz constant equal to one for which the essential norm of D is $\geq 1/2$, and examples with Lipschitz constant two for which the operators $\pm\frac{1}{2}I + D$ are not coercive plus compact. We also give, for every $C > 0$, examples of Lipschitz polyhedra for which the essential norm is $\geq C$ and for which $\lambda I + D$ is not a compact perturbation of a coercive operator for any real or complex λ with $|\lambda| \leq C$. We then, via a new result on the Galerkin method in Hilbert spaces, explore the implications of these results for the convergence of Galerkin boundary element methods in the $L^2(\Gamma)$ setting. Finally, we resolve negatively a related open question in the convergence theory for collocation methods, showing that, for our polyhedral examples, there is no weighted norm on $C(\Gamma)$, equivalent to the standard supremum norm, for which the essential norm of D on $C(\Gamma)$ is $< 1/2$.

Summary of the talk

1. **Lipschitz domains**
   - What are they? An example we will meet later.
2. **Potential theory** and **2nd kind boundary integral equations** (BIEs)
   - A Dirichlet problem and 2nd kind BIE formulation
   - The Galerkin approximation to the BIE
   - **A long-standing open problem: do Galerkin methods converge?**
3. The **Hilbert space** theory of **Galerkin methods**
   - Definitions of bounded, compact, coercive
   - Galerkin methods and their convergence
4. **Do all sensible Galerkin methods (i.e., based on V convergent to $L^2(\Gamma)$) converge for the standard 2nd kind BIEs?**
   - Previous results
   - **Solving the open problem**: Constructing Ω for which $A = I - D$ is not coercive + compact