The Complex-Scaled Half-Space Matching Method

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Joint work with

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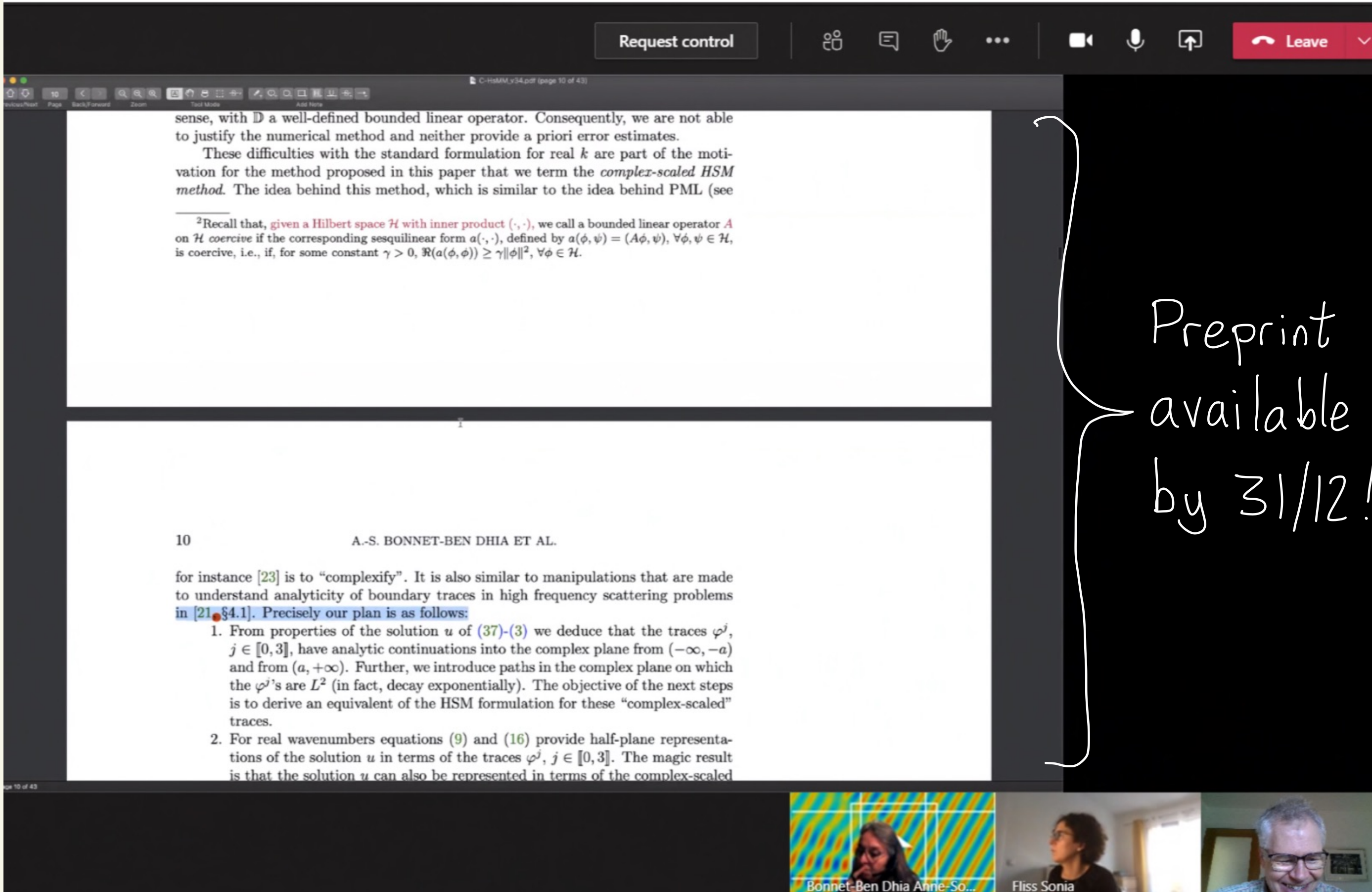


sense, with $\mathbb{D}$ a well-defined bounded linear operator. Consequently, we are not able to justify the numerical method and neither provide a priori error estimates.

These difficulties with the standard formulation for real k are part of the motivation for the method proposed in this paper that we term the *complex-scaled HSM method*. The idea behind this method, which is similar to the idea behind PML (see

[2] Recall that, given a Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$, we call a bounded linear operator A on $\mathcal{H}$ *coercive* if the corresponding sesquilinear form $a(\cdot,\cdot)$, defined by $a(\phi,\psi) = (A\phi,\psi)$, $\forall \phi,\psi \in \mathcal{H}$, is coercive, i.e., if, for some constant $\gamma > 0$, $\Re(a(\phi,\phi)) \geq \gamma\|\phi\|^2$, $\forall \phi \in \mathcal{H}$.

for instance [23] is to "complexify". It is also similar to manipulations that are made to understand analyticity of boundary traces in high frequency scattering problems in [21, §4.1]. Precisely our plan is as follows:

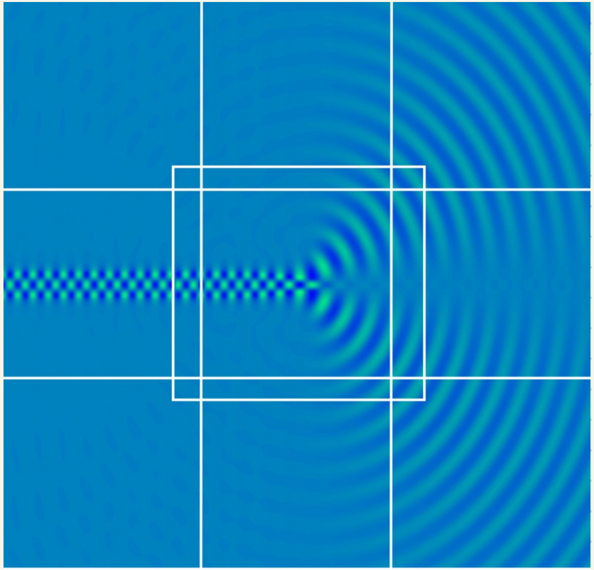
1. From properties of the solution u of (37)-(3) we deduce that the traces φ^j, $j \in [\![0,3]\!]$, have analytic continuations into the complex plane from $(-\infty,-a)$ and from $(a,+\infty)$. Further, we introduce paths in the complex plane on which the φ^j's are L^2 (in fact, decay exponentially). The objective of the next steps is to derive an equivalent of the HSM formulation for these "complex-scaled" traces.
2. For real wavenumbers equations (9) and (16) provide half-plane representations of the solution u in terms of the traces φ^j, $j \in [\![0,3]\!]$. The magic result is that the solution u can also be represented in terms of the complex-scaled

This talk is about combining:

- Half-Space Matching (HSM) Method

  Fliss & Joly (2009), Ott (2017), Bonnet-Ben Dhia et al (2018, 19)

- Complex scaling used in PML



Ott, PhD, KIT (2017)

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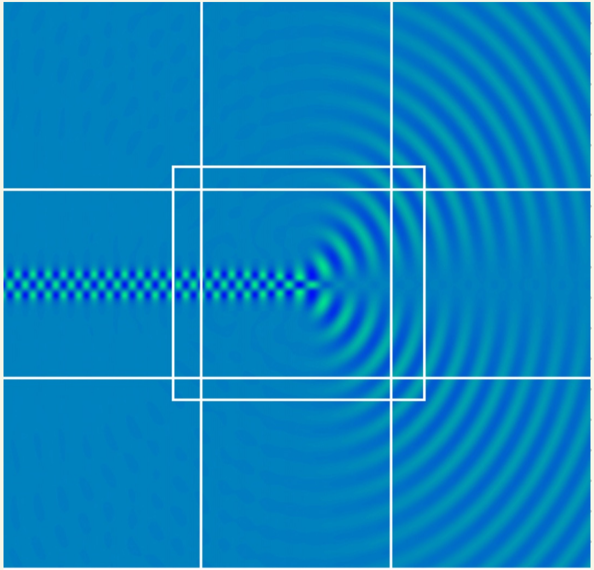

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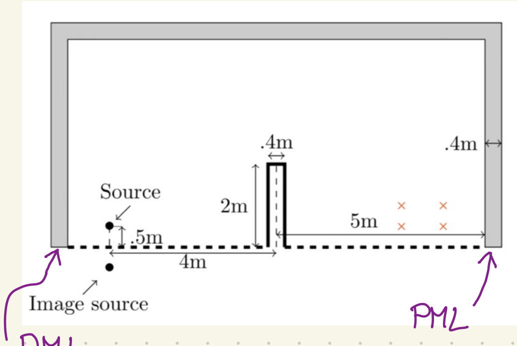

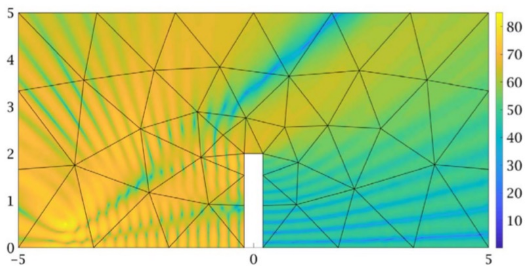

(b) PUFEM-PML pressure field (dB), mesh appears in black lines.

Diffraction around outdoor noise barrier, Langlois et al, Acta Acustica, 2020

Part I: Half-Plane Solutions: the simplest case

$$\frac{\partial u}{\partial r} - iku = o(r^{-1/2}),$$
as $r \to \infty$

$$\Delta u + k^2 u = 0$$

$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

$k = \frac{\omega}{c}$, $e^{-i\omega t}$ time dep.

$$u = g$$

for simplicity,
impedance b.c.
just as easy,
see C-W, Hothersall (1985),
Bonnet-Ben Dhia, Fliss, Tjandrawidjaja (2019)

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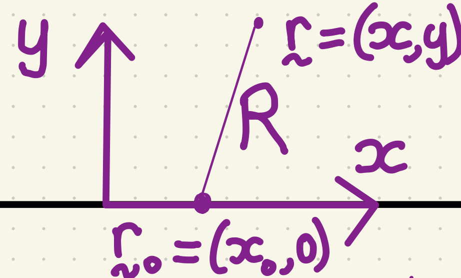

Σ

$u = g$

Soln. is $u(\mathbf{r}) = U(g)(\mathbf{r})$

$$= 2 \int_{\Sigma} \frac{\partial \Phi(\mathbf{r}, \mathbf{r}_0)}{\partial y} g(\mathbf{r}_0) \, ds(\mathbf{r}_0)$$

Double layer potential

where $\Phi(\mathbf{r}, \mathbf{r}_0) = \frac{i}{4} H_0^{(1)}(kR)$, $\quad R = |\mathbf{r} - \mathbf{r}_0| = \sqrt{y^2 + (x - x_0)^2}$.

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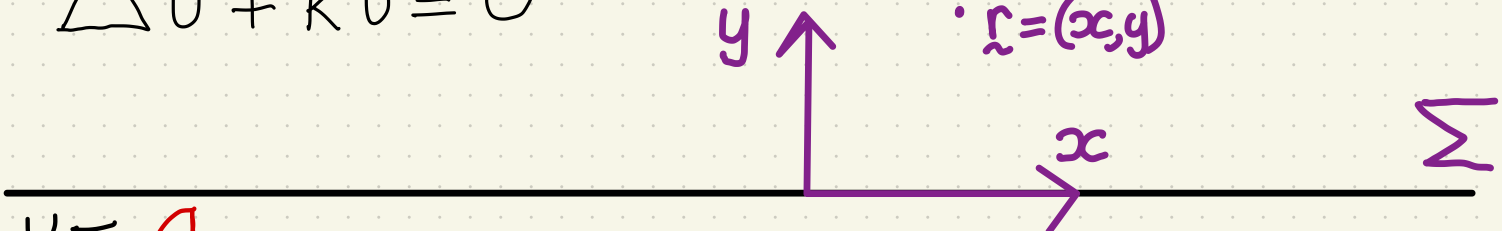

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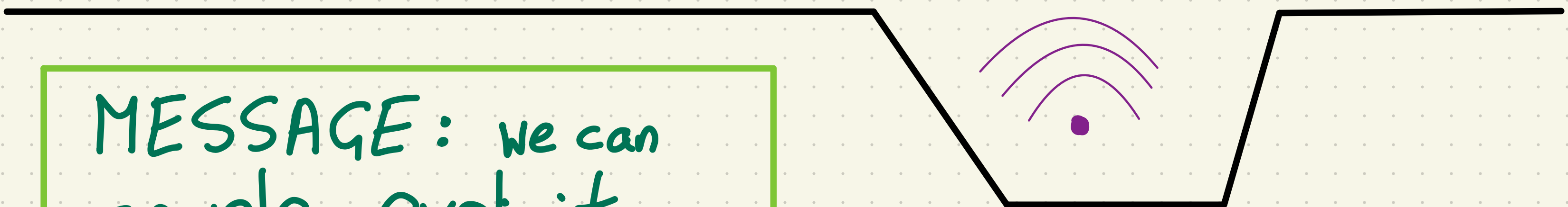
$$= 2\int_{\Sigma} \frac{\partial \Phi(\mathbf{r}, \mathbf{r}_0)}{\partial y}\, g(\mathbf{r}_0)\, ds(\mathbf{r}_0) = \boxed{\frac{iky}{2}\int_{-\infty}^{+\infty} \frac{H_1^{(1)}(kR)}{R}\, g(x_0)\, dx_0},$$

where $\Phi(\mathbf{r}, \mathbf{r}_0) = \frac{i}{4} H_0^{(1)}(kR)$, $\quad R = |\mathbf{r} - \mathbf{r}_0| = \sqrt{y^2 + (x - x_0)^2}$.

Part II: Half-Plane Solutions Enable Solution of More Complex Problems

Example 1: Propagation from Cutting

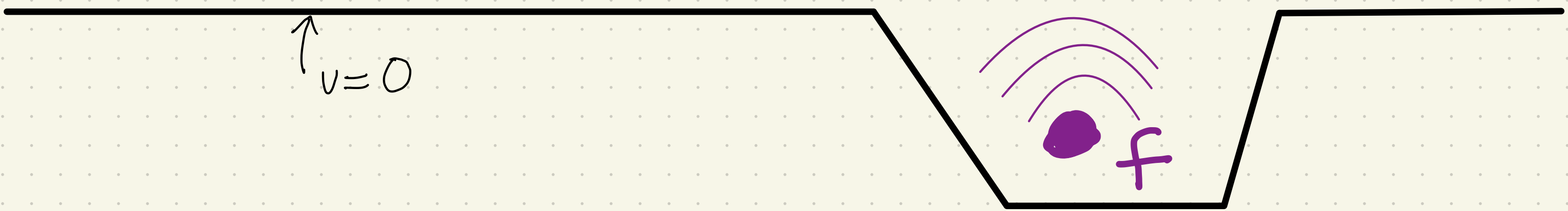


MESSAGE: we can couple explicit half-plane solutions to local FEM solves

Example 1: Propagation from Cutting

$$\frac{\partial u}{\partial r} - iku = o(r^{-1/2}), \quad \text{as } r \to \infty$$

$$\Delta u + k^2 u = f$$



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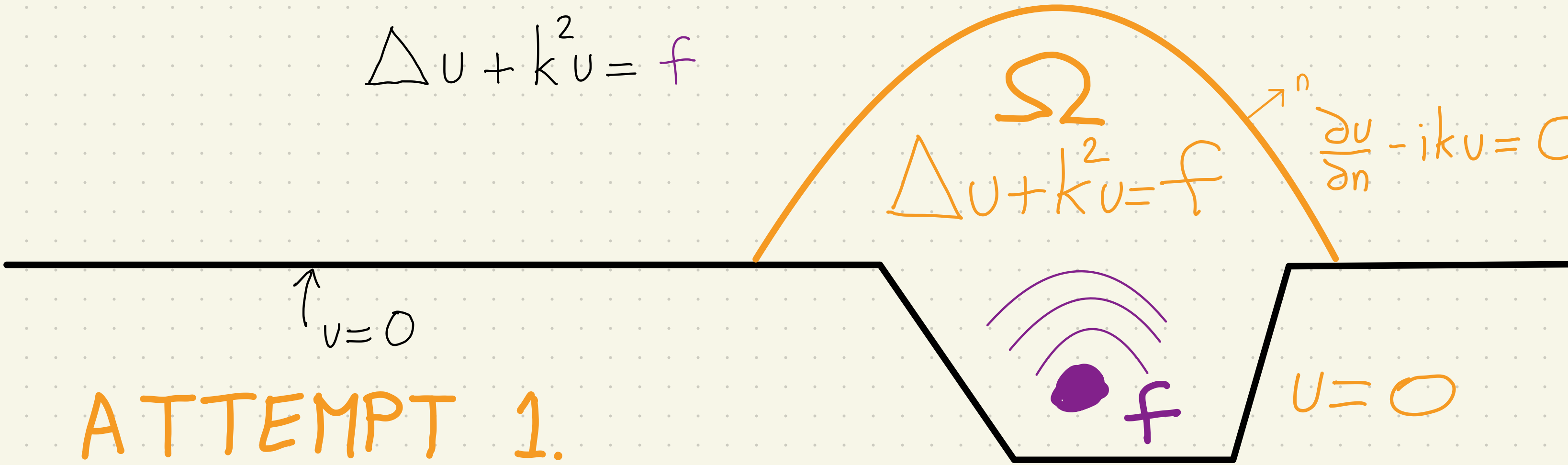

ATTEMPT 1.

Use FEM to solve approx. problem in Ω.

$$\frac{\partial u}{\partial r} - iku = o\left(r^{-1/2}\right), \quad \text{as } r \to \infty$$

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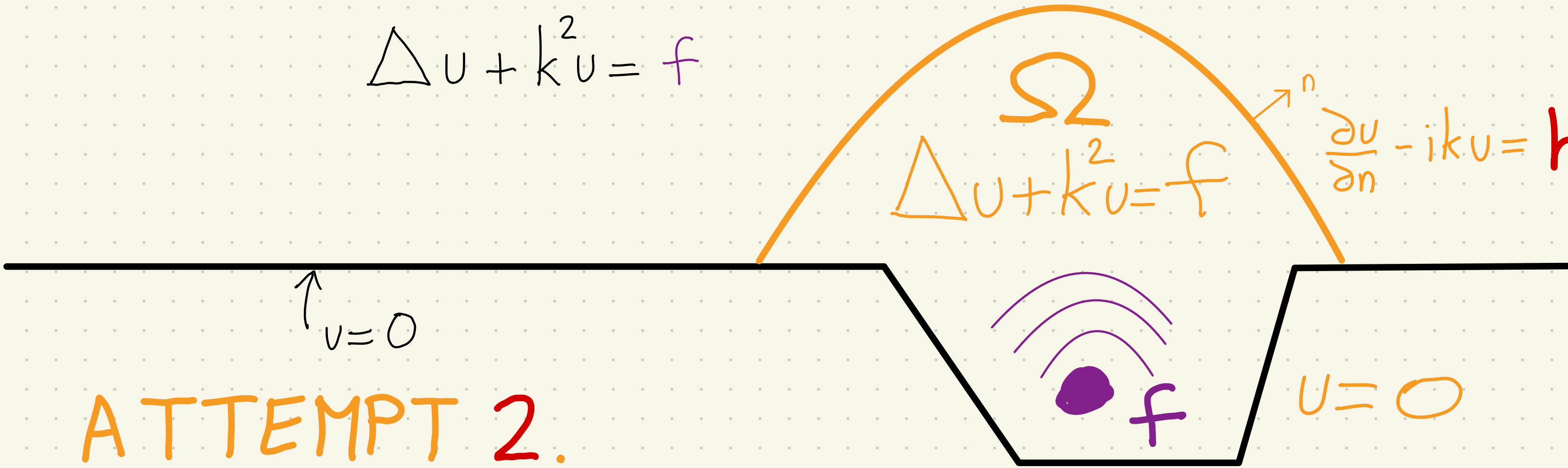
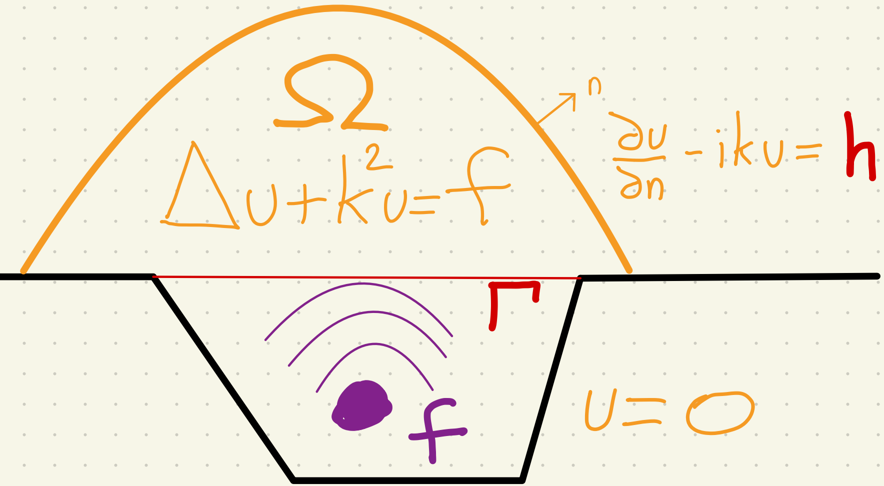


ATTEMPT 2.

Use FEM to solve exact problem in Ω.

$$\Delta u + k^2 u = f$$

$$\frac{\partial u}{\partial r} - iku = o(r^{-1/2}),$$
as $r \to \infty$



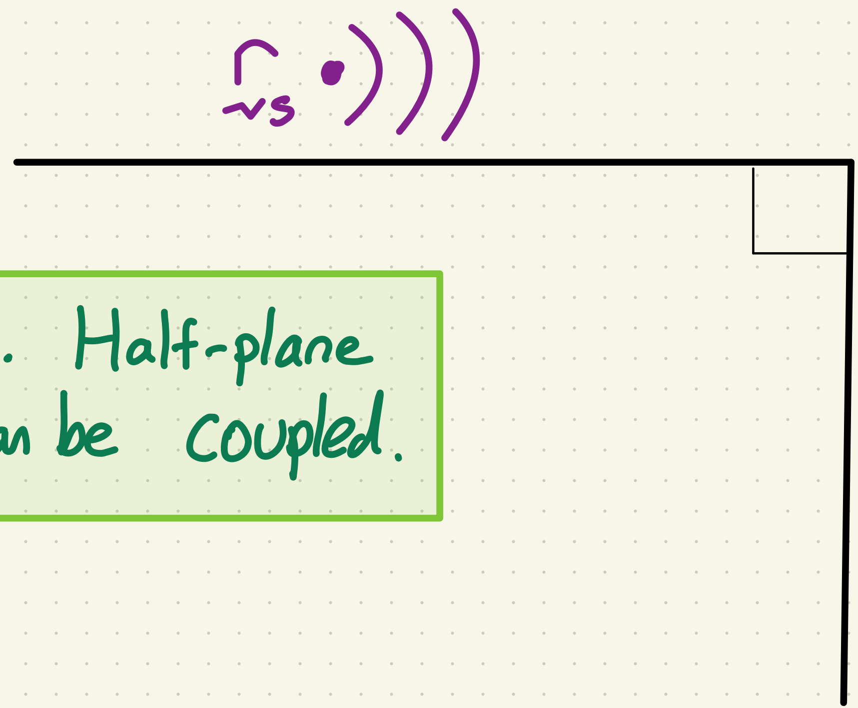
$v = 0$

ATTEMPT 2.

Use FEM to solve exact problem in Ω with

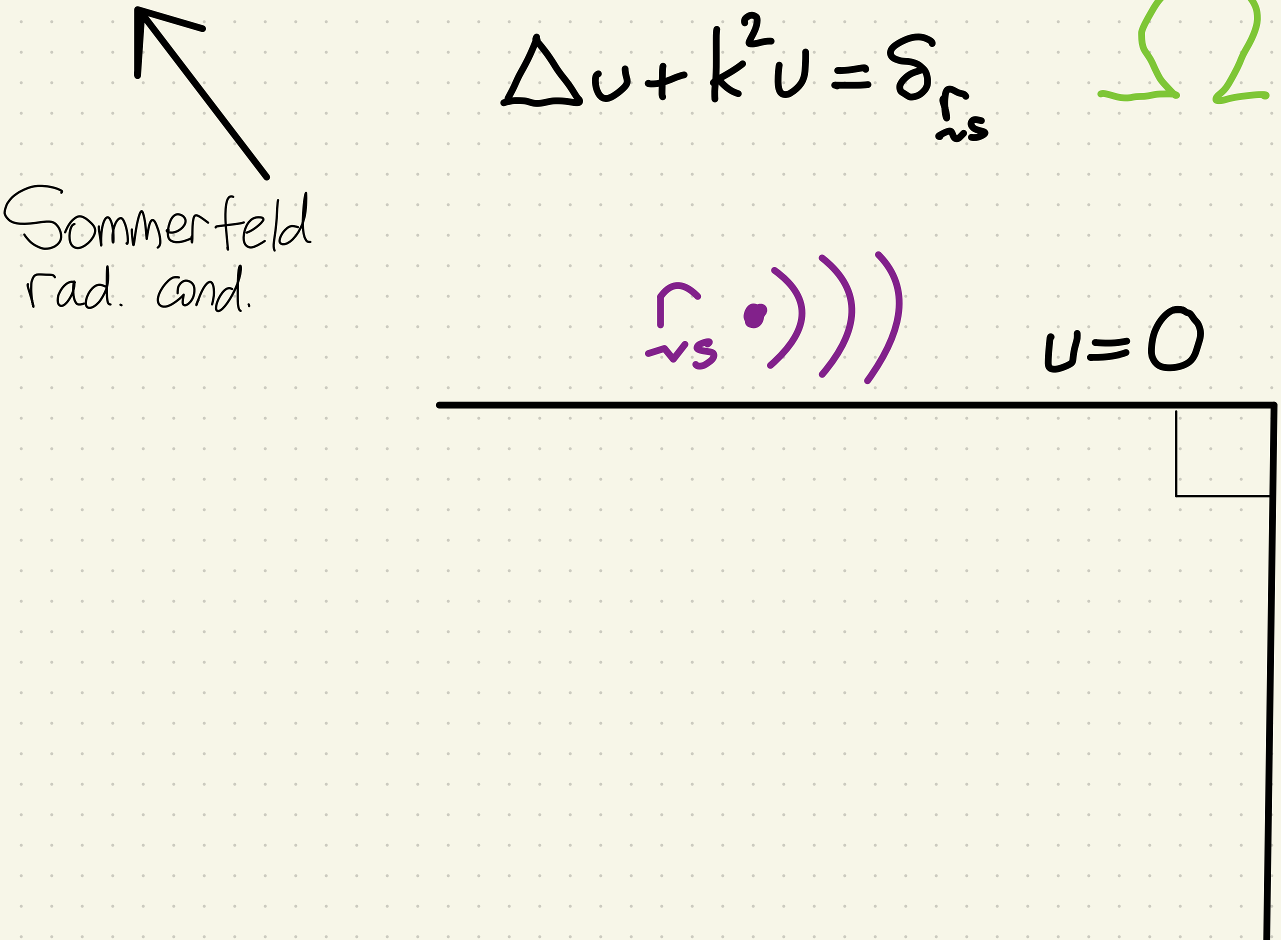
$$h := \frac{\partial v}{\partial n} - ikv, \qquad v := U(u|_\Gamma)$$

Example 2. Diffraction by 90° wedge



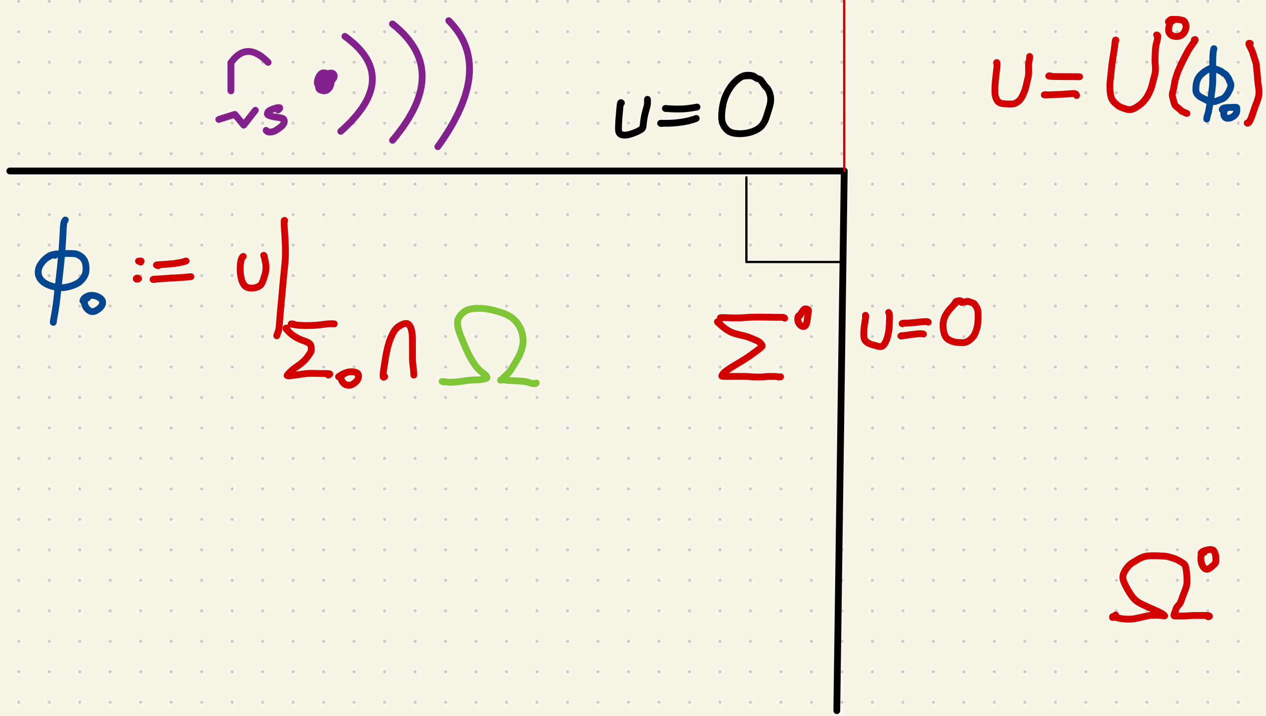
MESSAGE. Half-plane solutions can be coupled.

$$\Delta u + k^2 u = \delta_{\underset{\sim}{r}_s} \qquad \Omega$$



$$\Delta u + k^2 u = \delta_{\underset{\sim}{r}_s} \quad \Omega$$

Sommerfeld rad. cond.



Let $\phi_0 := u|_{\Sigma_0 \cap \Omega}$

Sommerfeld
rad. cond.

$$\Delta u + k^2 u = \delta_{\underset{\sim}{r}_s} \quad \Omega$$

$$U = u_s + U'(\phi_1)$$

$\underset{\sim}{r}_s$

$u = 0$

$U = \phi_1$

$\underset{\sim}{r}_s'$

Let $\phi_1 := u\big|_{\Sigma_1 \cap \Omega}$

$$u_s(\underset{\sim}{r}) := \Phi(\underset{\sim}{r}, \underset{\sim}{r}_s) - \Phi(\underset{\sim}{r}, \underset{\sim}{r}_s')$$

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Sommerfeld rad. cond.

$\underset{\sim}{r}_s$

$u = 0$

$\underset{\sim}{r}_s'$

Σ_0

Σ_1

$u = \phi_1 = U^0(\phi_0)$

Ω^0

Let $\phi_j := u\big|_{\Sigma_j \cap \Omega}$

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Σ_0 $\qquad \Omega'$

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Sommerfeld
rad. cond.

$\Sigma_0 \qquad \Omega'$

$$u = \phi_0 = u_s + U^1(\phi_1)$$

Σ_1

$$u = \phi_1 = U^0(\phi_0)$$

$$\boxed{\begin{aligned} \phi_0 &= \psi_s + D\phi_1 \\ \phi_1 &= D\phi_0 \end{aligned}}$$

Let $\phi_j := u\big|_{\Sigma_j \cap \Omega}$

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Σ_1

$$u = \phi_1 = U^0(\phi_0)$$

$$\boxed{\begin{aligned}\phi_0 &= \psi_s + D\phi_1 \\ \phi_1 &= D\phi_0\end{aligned}} \quad (*)$$

Thm. If $\operatorname{Im} k > 0$ then $(*)$ equivalent to original BVP and $\|D\|_{ess} < 1$.

Let $\phi_j := u\big|_{\Sigma_j \cap \Omega}$

$$u_s(\underset{\sim}{r}) := \Phi(\underset{\sim}{r}, \underset{\sim}{r}_s) - \Phi(\underset{\sim}{r}, \underset{\sim}{r}_s')$$

Part II: Half-Plane Solutions Enable Solution of More Complex Problems

MESSAGE: we can couple explicit half-plane solutions to local FEM solves

MESSAGE. Half-plane solutions can be coupled.

Part II : PML-type complex-scaling works for integral equations

Deforming paths of integration for numerical evaluation of oscillatory integrals

E.g. Deaño, Huybrechs, Iserles, SIAM 2017

Let

$$I(k) := \int_0^\infty \frac{e^{ikt}\,dt}{(1+t)^{1/2}}, \qquad k \gg 1$$

Deforming paths of integration for numerical evaluation of oscillatory integrals

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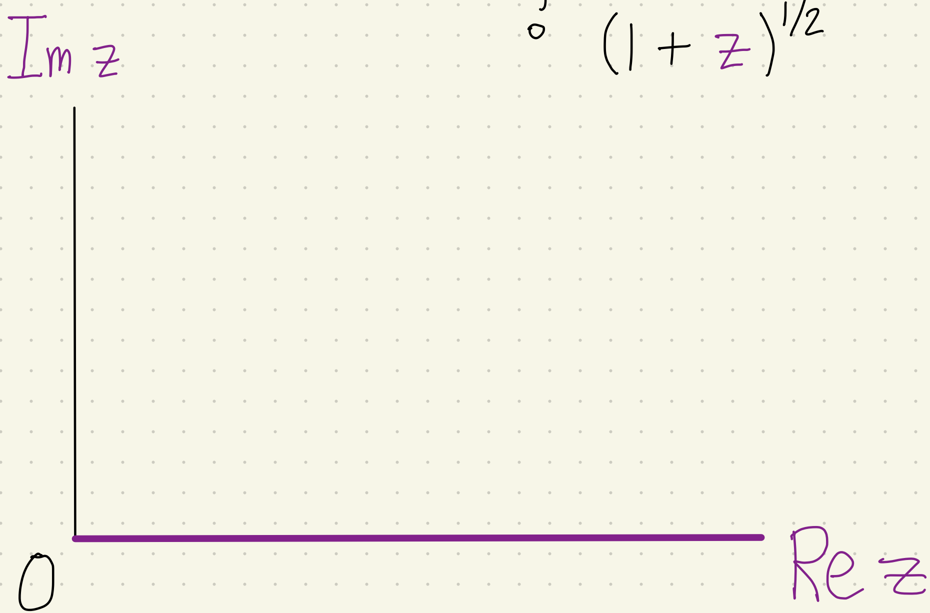
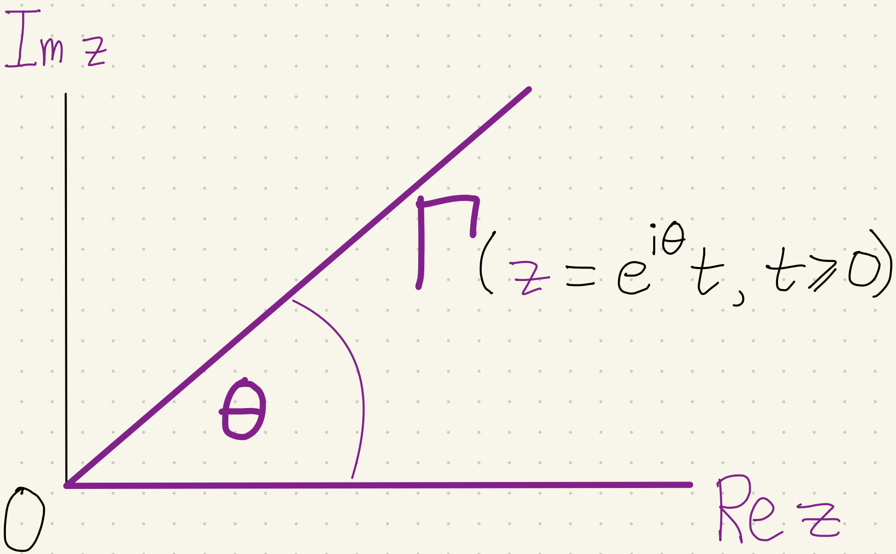
Let

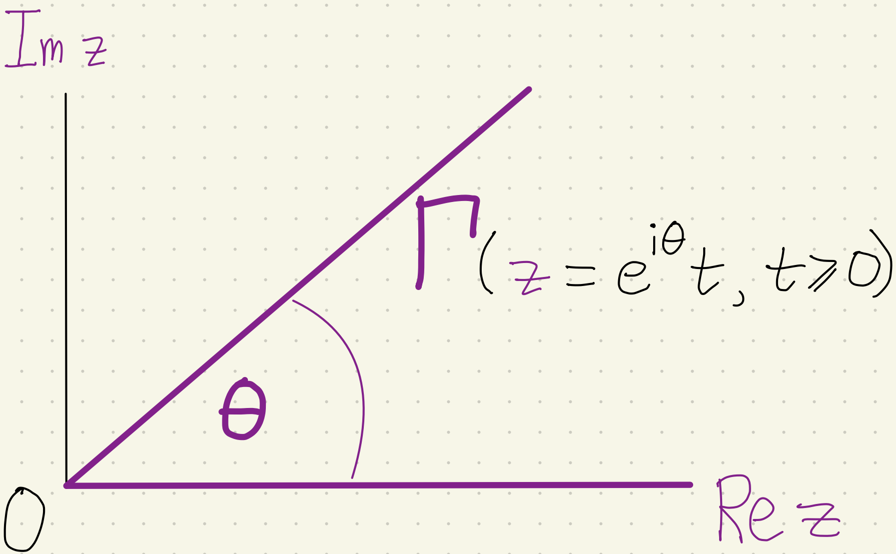
$$I(k) := \int_0^\infty \frac{e^{ikz}\, dz}{(1+z)^{1/2}}$$

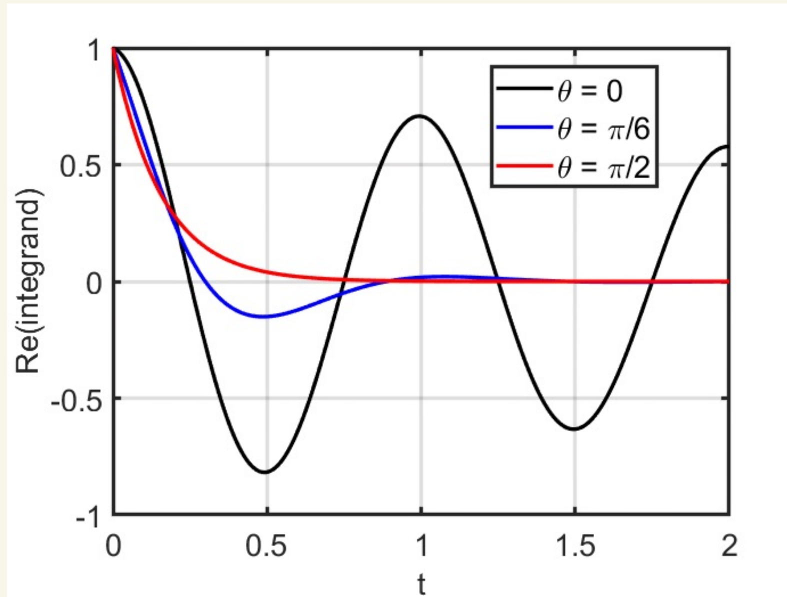


$$I(k) := \int_0^\infty \frac{e^{ikz}\,dz}{(1+z)^{1/2}} = \int_\Gamma \frac{e^{ikz}\,dz}{(1+z)^{1/2}}$$



$$I(k) := \int_0^\infty \frac{e^{ikz}\,dz}{(1+z)^{1/2}} = \int_\Gamma \frac{e^{ikz}\,dz}{(1+z)^{1/2}} = e^{i\theta}\int_0^\infty \frac{e^{-kt\sin\theta + ikt\cos\theta}}{(1+e^{i\theta}t)^{1/2}}\,dt$$





Deforming path for half-plane solution

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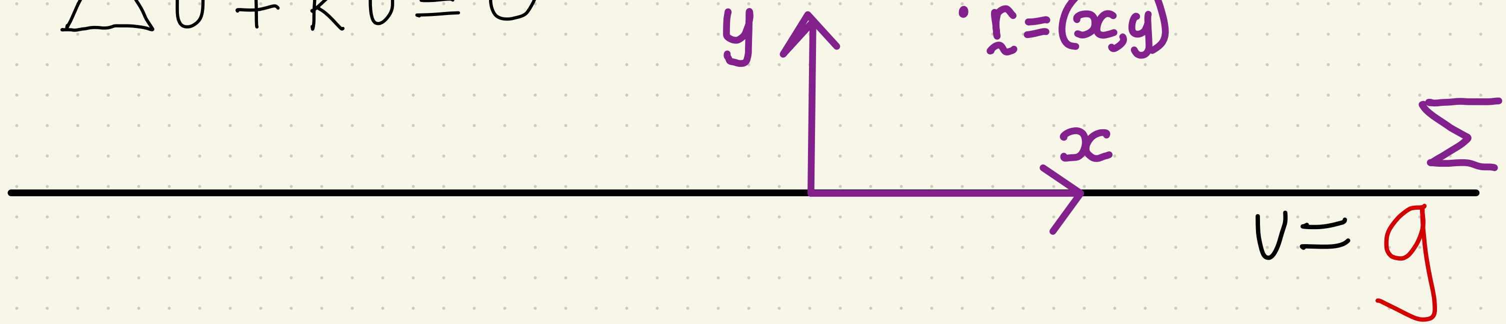
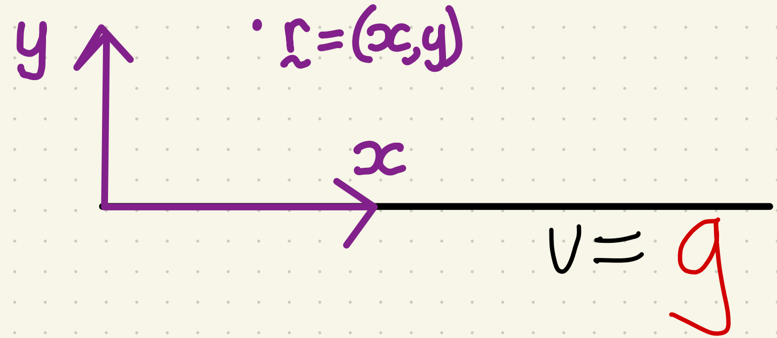


$$u(\underset{\sim}{r}) = U(g)(\underset{\sim}{r}) = \frac{iky}{2}\int_{-\infty}^{+\infty} \frac{H_1^{(1)}(kR)}{R}\, g(x_0)\, dx_0,$$

$$R = \sqrt{y^2 + (x-x_0)^2}.$$

Deforming path for half-plane solution



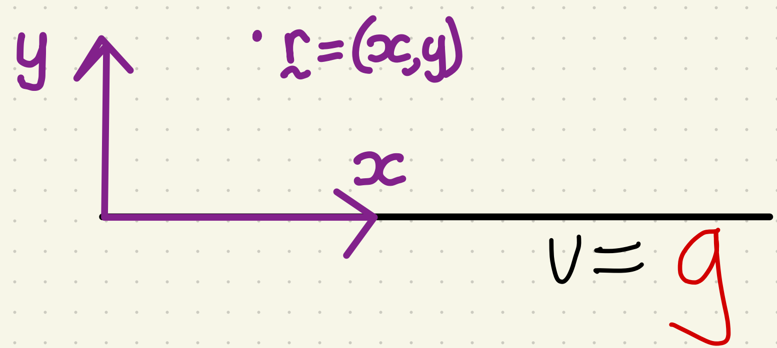
Let

$$I = \frac{iky}{2} \int_0^{+\infty} \frac{H_1^{(1)}(kR)}{R} g(x_0)\, dx_0$$

$$R = \sqrt{y^2 + (x-x_0)^2}.$$

Deforming path for half-plane solution

But is g analytic?



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Deforming path for half-plane solution

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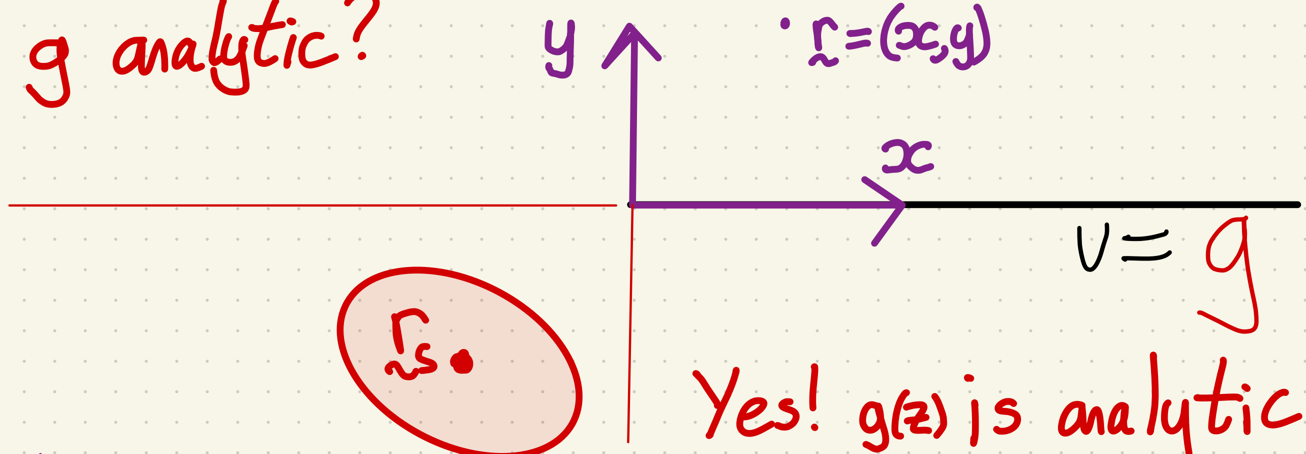
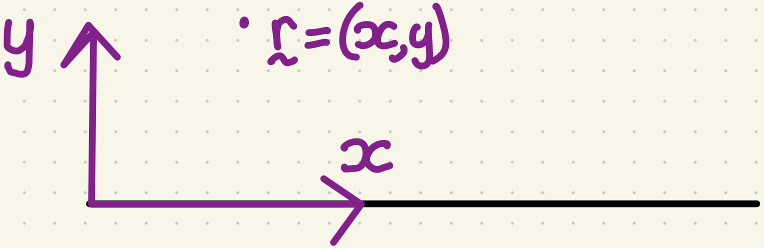


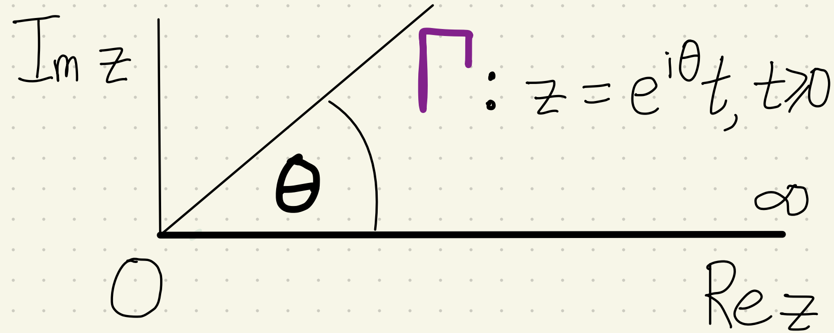
Yes! $g(z)$ is analytic and bounded in $\{z : \operatorname{Re} z \geqslant 0, \operatorname{Im} z \geqslant 0\}$ if u is field due to sources/scatterer in 3rd (physical plane) quadrant.

Let

$$I = \frac{iky}{2} \int_0^{+\infty} \frac{H_1^{(1)}(kR)}{R} \, g(x_0) \, dx_0$$

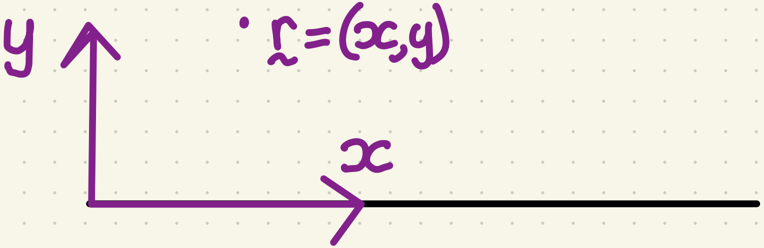
$$R = \sqrt{y^2 + (x - x_0)^2}.$$

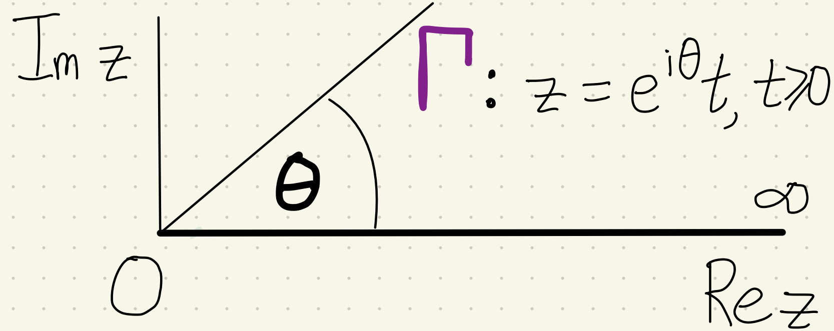




$$I = \frac{iky}{2}\int_0^\infty \frac{H_1^{(1)}(kR)}{R}\, g(z)\, dz \qquad R = \sqrt{y^2 + (x-z)^2}$$

Assume $g(z)$ analytic, bounded in $\mathrm{Re}\, z \geq 0$, $\mathrm{Im}\, z \geq 0$.

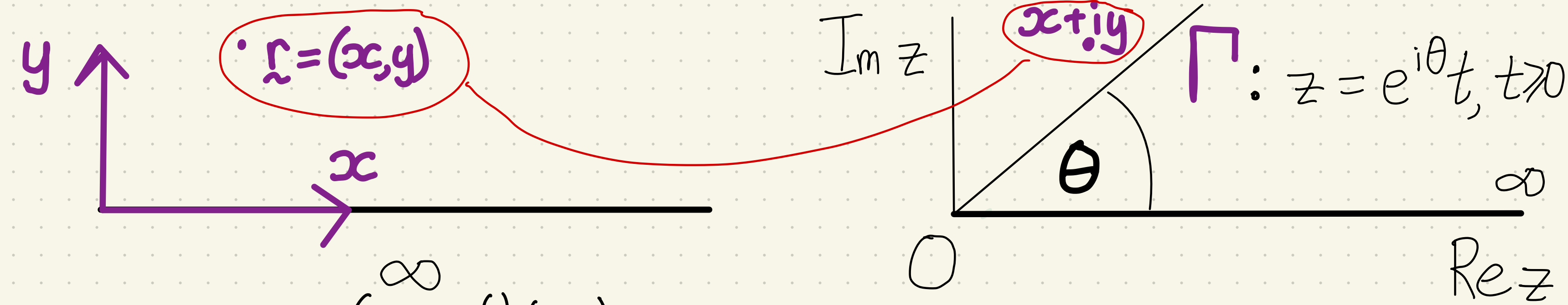




$$I = \frac{iky}{2}\int_0^\infty \frac{H_1^{(1)}(kR)}{R}\, g(z)\, dz \qquad R = \sqrt{y^2 + (x - z)^2}$$

Assume $g(z)$ analytic, bounded in $\mathrm{Re}\, z \geqslant 0$, $\mathrm{Im}\, z \geqslant 0$.

$$R \sim z, \quad \frac{H_1^{(1)}(kR)}{R} \sim \text{const.}\ \frac{e^{iz}}{z^{3/2}}, \quad z \to \infty$$



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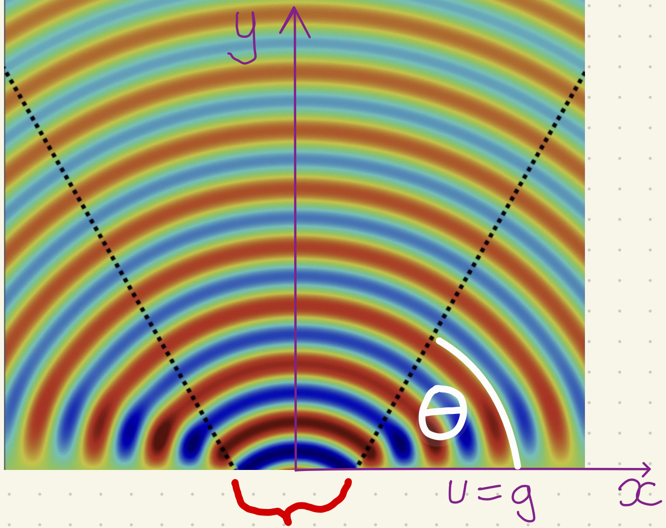
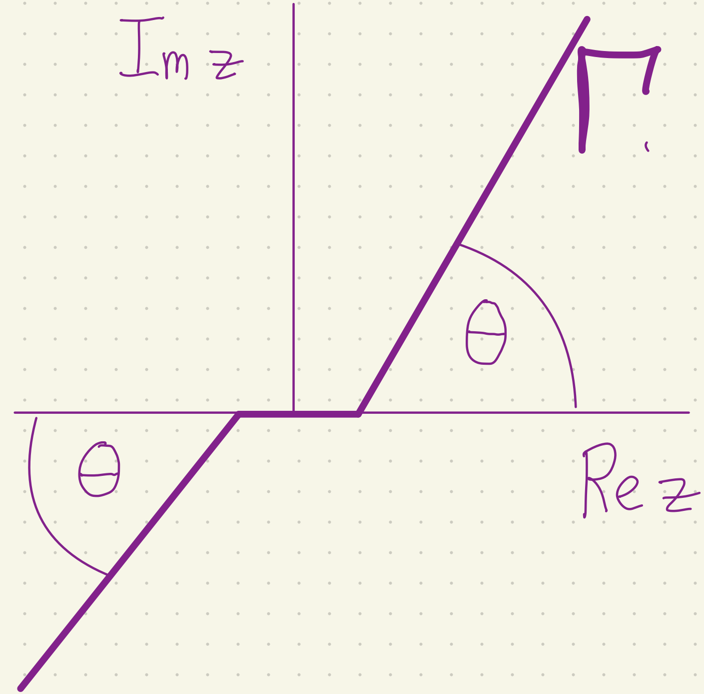
N.B. $R = 0$ if $z = x + iy$!

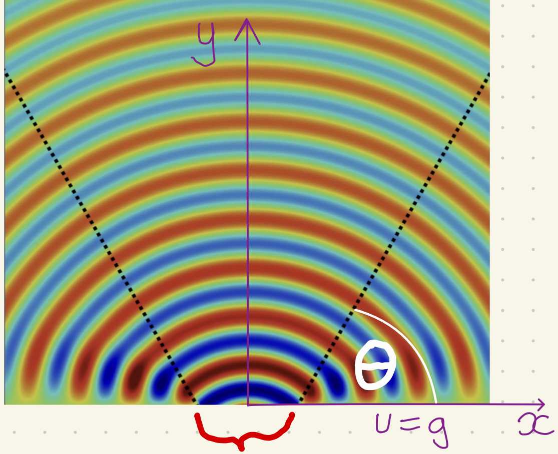


In this sector



$$U(\underset{\sim}{r}) = \frac{iky}{2}\int_{\Gamma} \frac{H_1^{(1)}(kR)}{R}\, g(z)\, dz$$

$$g(z) = \frac{i}{4} H_0^{(1)}\left(k\sqrt{z^2+1}\right)$$



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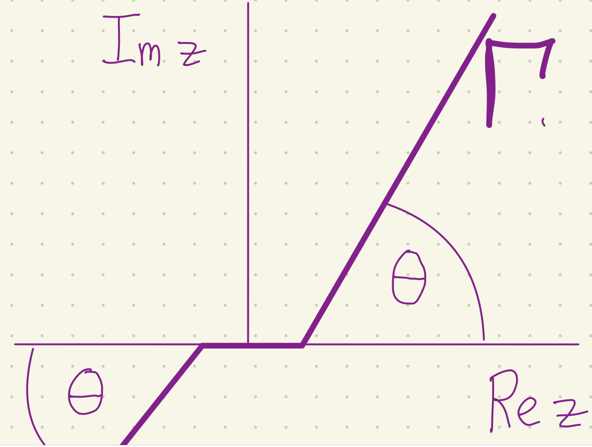

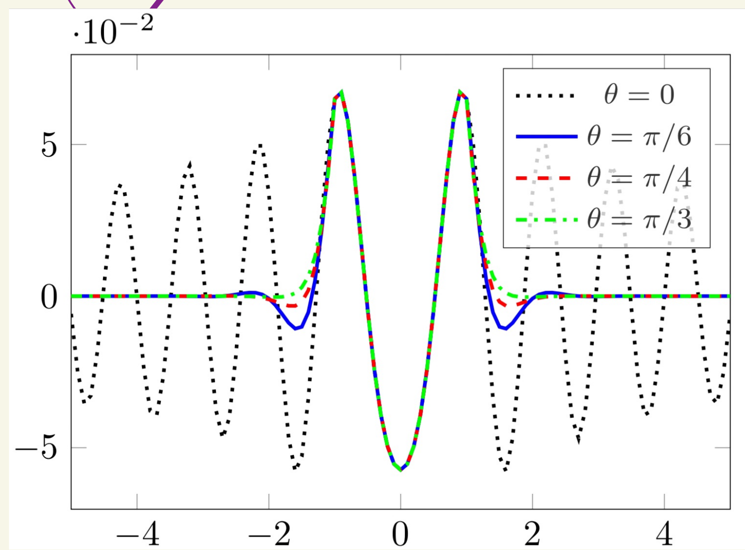

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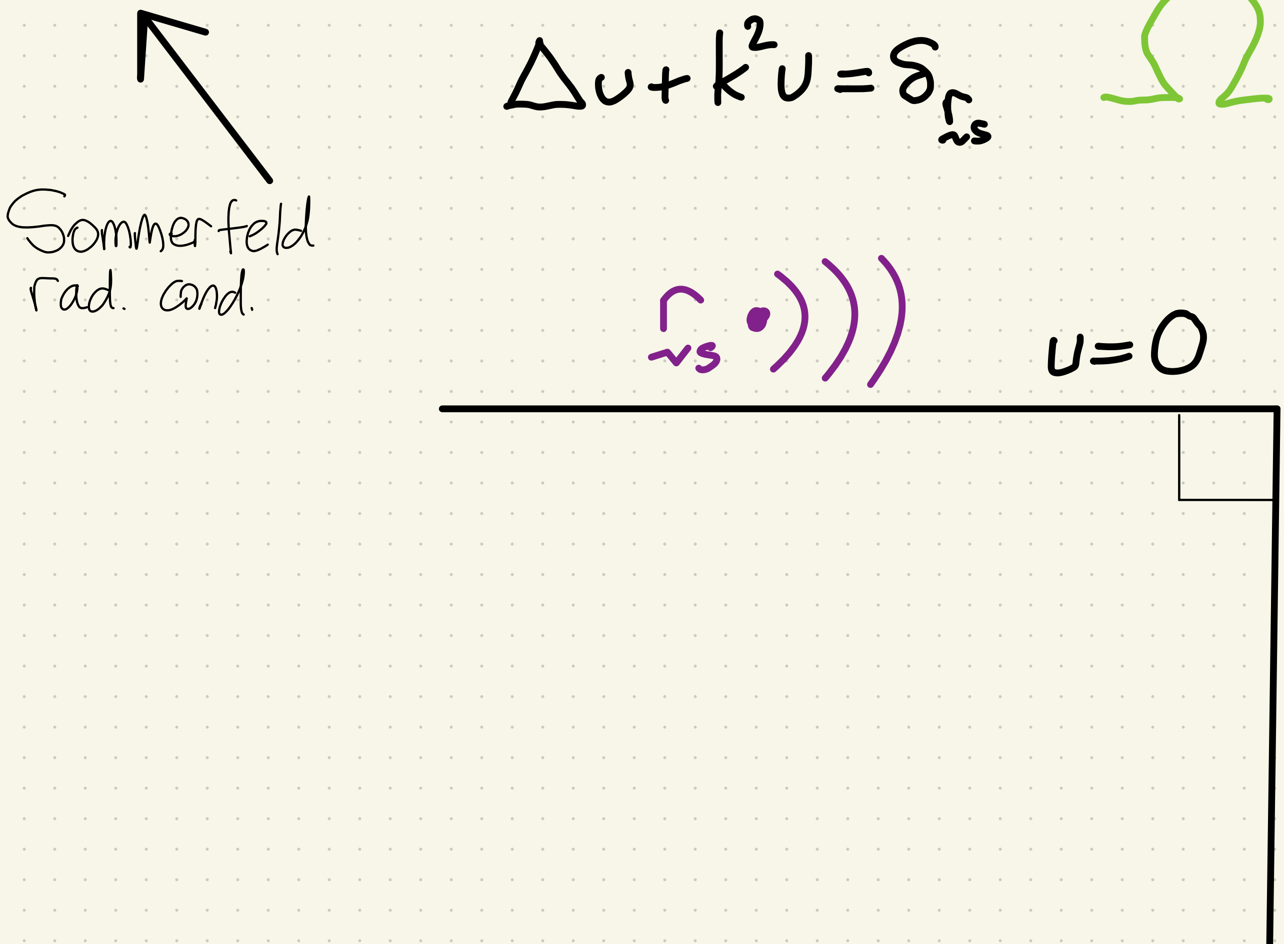
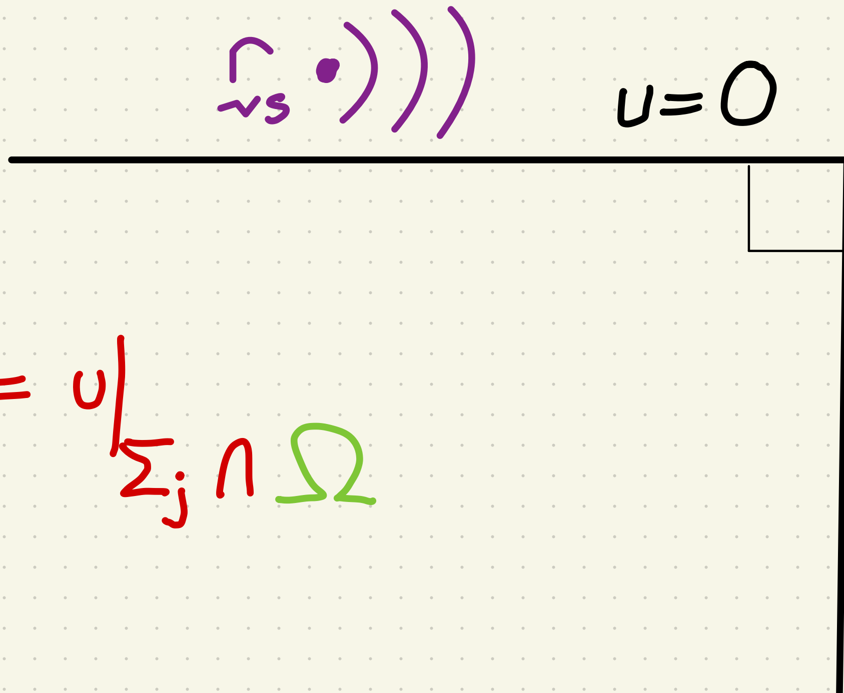
Part III: Putting things together

$$\Delta u + k^2 u = \delta_{\underset{\sim}{r}_s} \qquad \Omega$$



$$\Delta u + k^2 u = \delta_{\underset{\sim}{r}_s} \qquad \Omega$$

Sommerfeld
rad. cond.



Σ_0 $\qquad \Omega'$

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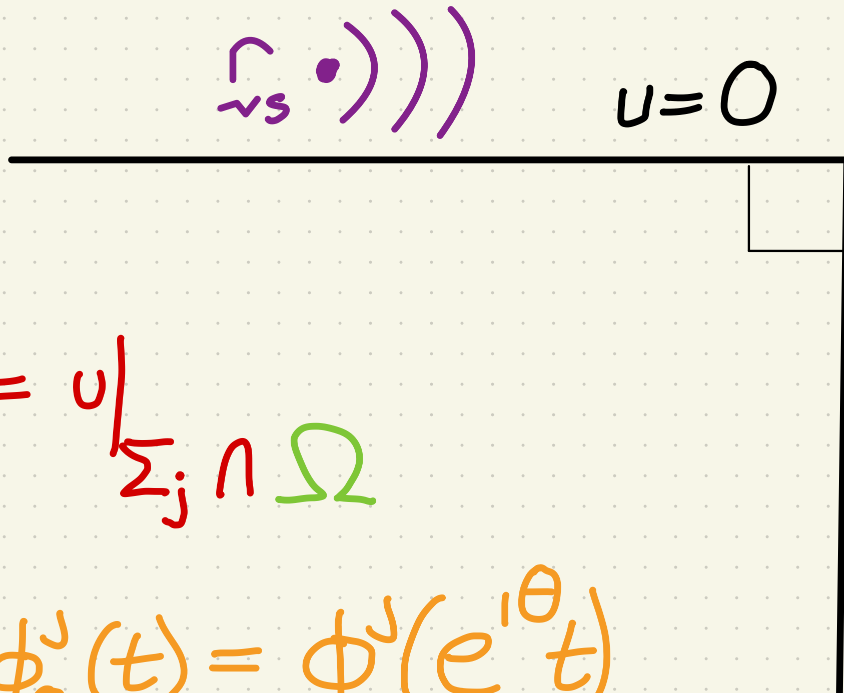
$\Delta u + k^2 u = \delta_{\underset{\sim}{r}_s}$ $\quad \Omega$

Sommerfeld rad. cond.



Σ_0 $\quad \Omega'$

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Σ_1

$u = \phi_1 = U^0(\phi_0)$

$$\boxed{\begin{aligned} \phi_0 &= \psi_s + D\phi_1 \\ \phi_1 &= D\phi_0 \end{aligned}}$$

$\phi_j := u\big|_{\Sigma_j \cap \Omega}$

Let $\phi_\theta^j(t) = \phi^j(e^{i\theta} t)$

$$\Delta u + k^2 u = \delta_{\underset{\sim}{r}_s} \qquad \Omega$$

Sommerfeld rad. cond.

$\underset{\sim}{r}_s$

$u = 0$

$\Sigma_0 \qquad \Omega'$

$$u = \phi_0 = u_s + U'_\theta(\phi'_\theta)$$

Σ_1

$$u = \phi_1 = U^0_\theta(\phi^0_\theta)$$

$$\phi_0 = \psi_s + D\phi_1$$

$$\phi_1 = D\phi_0$$

$$\phi_j := u\big|_{\Sigma_j \cap \Omega}$$

Let $\phi^j_\theta(t) = \phi^j(e^{i\theta} t)$

Sommerfeld rad. cond.

$$\Delta u + k^2 u = \delta_{\underset{\sim}{r}_s} \qquad \Omega$$

Σ_0 $\qquad \Omega'$

$$u = \phi_0 = u_s + U'_\theta(\phi'_\theta)$$

Σ_1

$\underset{\sim}{r}_s$

$u = 0$

$$u = \phi_1 = U^0_\theta(\phi^0_\theta)$$

$$\phi^0_\theta = \psi^s_\theta + D_\theta \phi'_\theta$$

$$\phi'_\theta = D_\theta \phi^0_\theta$$

$$\phi_j := u\big|_{\Sigma_j \cap \Omega}$$

Let $\phi^j_\theta(t) = \phi^j(e^{i\theta} t)$

$$\Delta u + k^2 u = \delta_{\underset{\sim}{r}_s} \qquad \Omega$$

Sommerfeld rad. cond.

$\underset{\sim}{r}_s$

$u = 0$

$\Sigma_0 \qquad \Omega'$

$$u = \phi_0 = u_s + U'_\theta(\phi'_\theta)$$

Σ_1

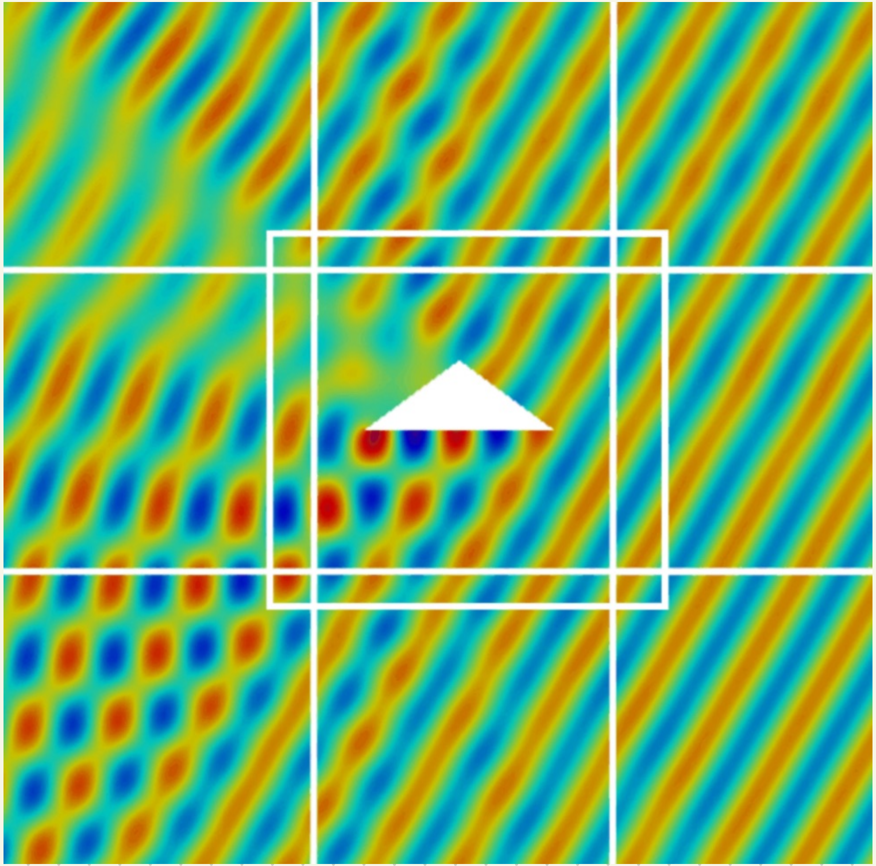
$$u = \phi_1 = U^0_\theta(\phi^0_\theta)$$

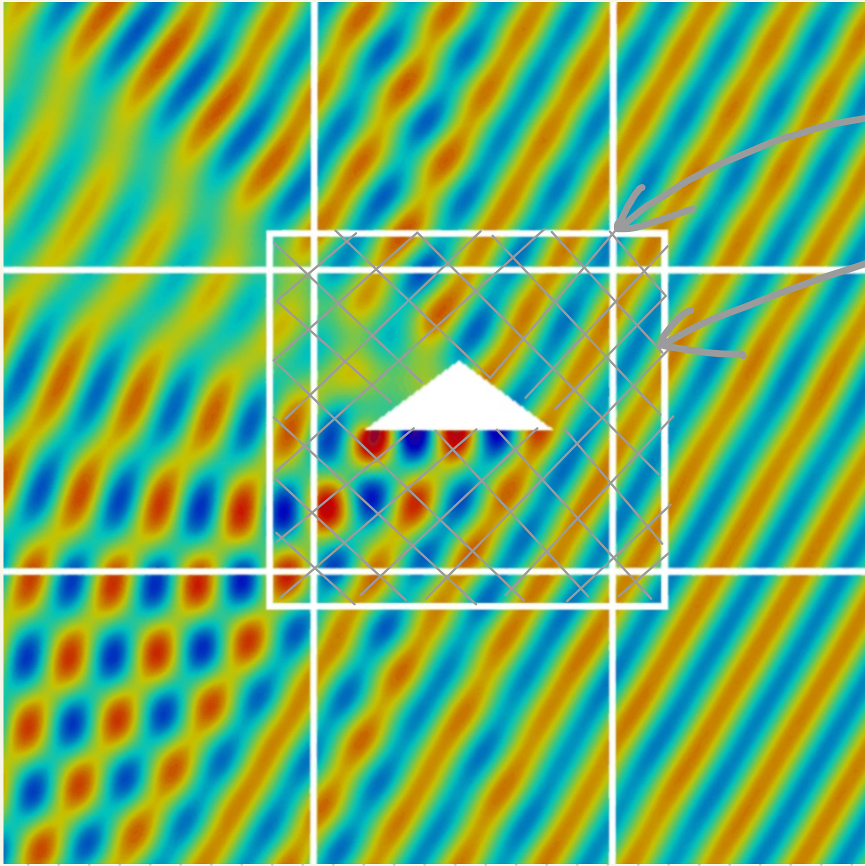
$$\phi^0_\theta = \psi^s_\theta + D_\theta \phi'_\theta$$
$$\phi'_\theta = D_\theta \phi^0_\theta \qquad (*)$$

$$\phi_j := u\big|_{\Sigma_j \cap \Omega}$$

Let $\phi^j_\theta(t) = \phi^j(e^{i\theta} t)$

Thm. If $0 < \theta < \frac{\pi}{2}$ then (*) equivalent to original BVP and $\|D_\theta\|_{ess} < 1$.

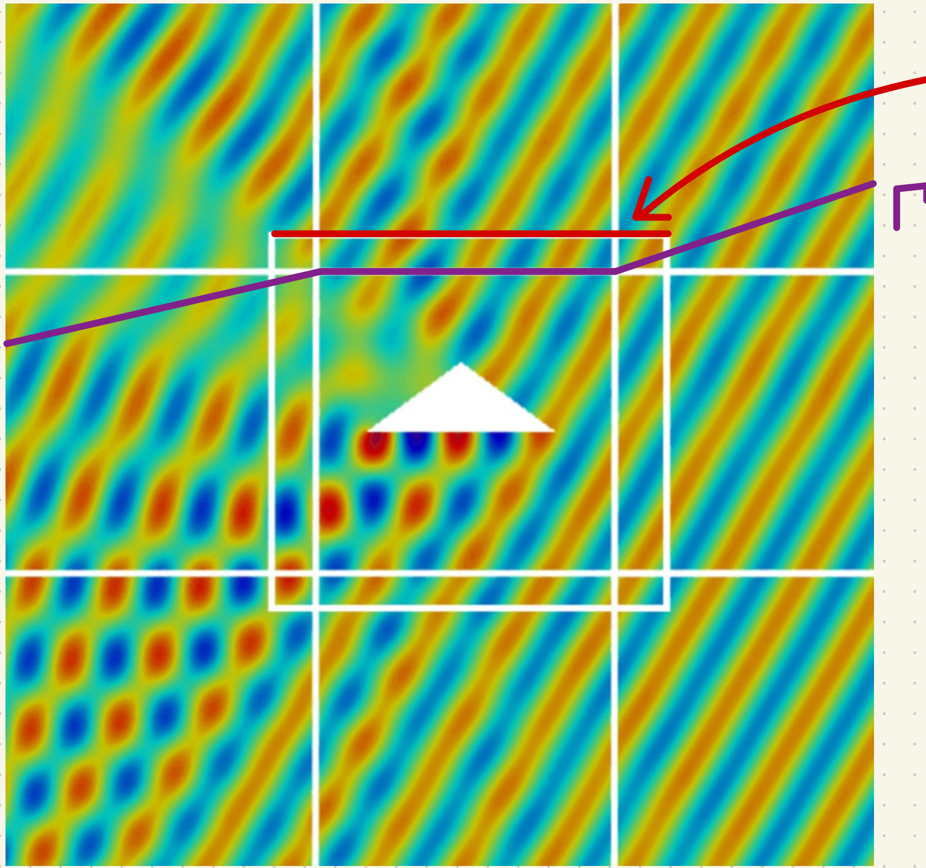




$$\frac{\partial u}{\partial n} - iku = h$$

FEM solve

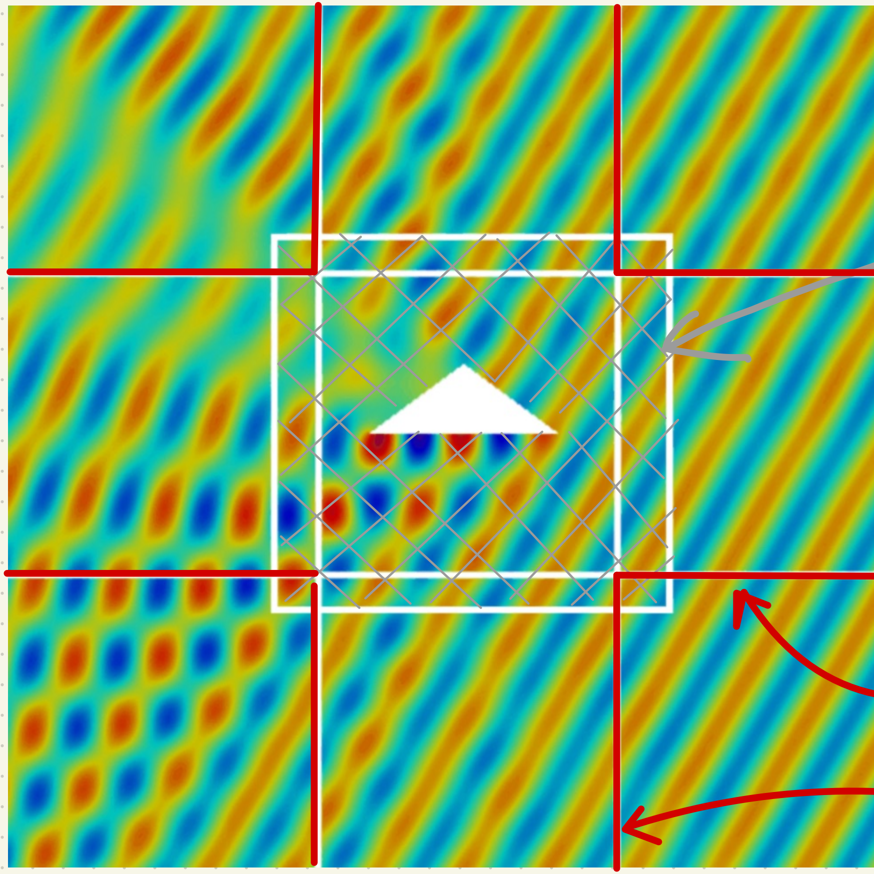
h comes from complexified half-plane formula!



$$h = \frac{\partial u}{\partial n} - iku$$

comes from

$$u(\underline{r}) = \int_{\Gamma}$$



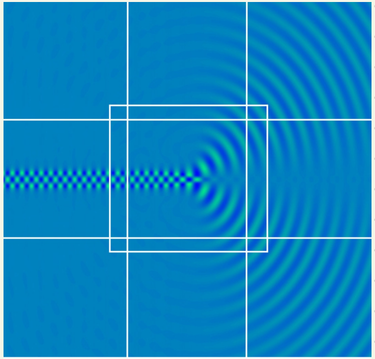
FEM unknowns

coupled to

complexified half-line traces

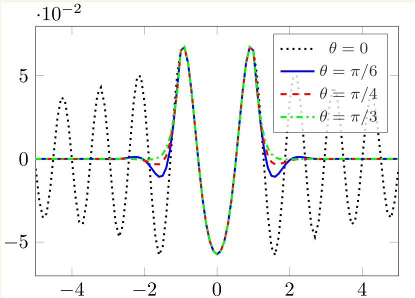
Take-Home Messages

- Half-plane solutions coupled with local FEM can solve scattering in complex backgrounds



Ott (2017)

- We can deform paths of integration in integral equations into the complex plane giving less oscillation and exponential decay



BB et al (2020)