An Integral Equation Method for a Boundary Value Problem arising in Unsteady Water Wave Problems

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Abstract

In this paper we consider the 2D Dirichlet boundary value problem for Laplace's equation in a non-locally perturbed half-plane, with data in the space of bounded and continuous functions. We show uniqueness of solution, using standard Phragmén-Lindelöf arguments. The main result is to propose a boundary integral equation formulation, to prove equivalence with the boundary value problem, and to show that the integral equation is well posed by applying a recent partial generalisation of the Fredholm alternative in Arens et al [J Int. Equ. Appl. 15 (2003) pp1-35]. This then leads to an existence proof for the boundary value problem.
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This paper is concerned with the boundary integral equation method for a problem in potential theory, namely the Dirichlet boundary value problem in a non-locally perturbed half-plane. The main aim of the paper is to discuss the well-posedness of this problem and of a novel second kind boundary integral equation formulation.

Our motivation for studying this problem is that it arises in the theory of classical free surface water wave problems, for which boundary integral equation methods are well-established as a computational and theoretical tool [B1, B2, HZ, FD]. In particular, accurate numerical schemes, based on boundary integral equation formulations, for the time dependent water wave problem have been proposed and fully analysed in [B1, B2, HZ], these papers providing a full nonlinear stability analysis for the spatial discretisations they propose. A significant component in this analysis is the well-posedness of the boundary integral equation formulation.

A key restriction in the analysis in the above papers is the requirement that the free surface be periodic (in 2D) or doubly-periodic (in 3D). This restriction is helpful theoretically and computationally. It enables the boundary integral equation on the free surface to be reduced to one on a (bounded) single periodic cell, which can then be discretised with a finite size mesh. Moreover, the boundary integral formulation is of second kind with a compact integral operator, and therefore standard Riesz/Fredholm theory gives well-posedness via the Fredholm Alternative.

As a step towards a broader extension of the results of [B1, B2, HZ], this paper is concerned with studying the Dirichlet boundary value problem for Laplace's equation in a perturbed half-plane Ω without the requirement that the boundary $\partial\Omega$ of Ω be periodic. To simplify our task somewhat, we impose other conditions on the boundary, namely: the boundary surface is the graph of a bounded continuous function (this excludes configurations relevant to breaking waves); the surface is sufficiently smooth (at least Lyapunov, that is the unit normal direction is Hölder continuous).

We note that there exists a well-developed L^2 theory of the boundary integral equation method for the Dirichlet problem when the boundary is the graph of an arbitrary Lipschitz function and the Dirichlet data is in $L^2(\partial\Omega)$ (see e.g. Verchota [V1], Meyer and Coifman [MC] and the references therein). However, this theory does not extend to the case of data in $L^\infty(\partial\Omega)$. In particular, even if the boundary is smoother than Lipschitz, the standard double layer potential operator is not well-defined on $L^\infty(\partial\Omega)$.

Addressing this difficulty, our aim in this paper is to develop a theory which includes the case when neither the surface elevation nor the Dirichlet data exhibit decay at infinity. To obtain a boundary integral equation formulation appropriate to this case we make the *ansatz* that the solution can be represented as a double layer potential supported on the (infinite) boundary of the domain, with the twist that we replace the standard fundamental solution of Laplace's equation in 2D with the Dirichlet Green's function for a half-plane Ω_H that strictly includes the domain Ω. We note that the case of periodic surface elevation and Dirichlet data will be included as a special case in the theory we develop.

The boundary value problem we solve and the double-layer potential are specified precisely in the next section. In §2 we describe the properties of the half-plane Green's function and the double-layer potential; the main issue is the unboundedness of $\partial\Omega$ and how this affects the double-layer potential and its mapping properties.

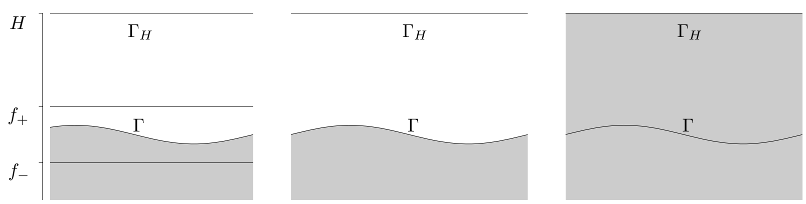
In §3 we state the boundary integral equation and we prove results on its well-posedness, and this enables us to establish the well-posedness of the boundary value problem. In general terms the argument follows the usual pattern: we first show uniqueness for the boundary value problem and next establish that the integral equation formulation has at most one solution. Finally, we deduce existence of a solution to the integral equation, and thus to the boundary value problem. However, the unboundedness of $\partial\Omega$ adds significant difficulties to the analysis. First of all, uniqueness of solution for the boundary value problem requires a Phragmén-Lindelöf argument. Establishing uniqueness of solution for the integral equation requires study of an additional mixed Dirichlet-Neumann boundary value problem in the infinite domain $\Omega_H \setminus \overline{\Omega}$. To deduce surjectivity from injectivity for the integral equation we cannot use the Fredholm alternative as the integral operator is not compact. Instead we use a partial generalisation of the Fredholm alternative, due to Arens et al. [A2], which applies when the operator is only locally compact.

Notation: Throughout this paper x and y will denote points in the plane $\mathbb{R}^2$ such that $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Given $G \subseteq \mathbb{R}^2$, let $BC(G)$ denote the space of (real-valued) bounded and continuous functions on G. For $0 < \alpha \leqslant 1$, let $BC^{0,\alpha}(G) \subset BC(G)$ denote the space of functions that are bounded and Hölder continuous with index α. Let $BC^{1,\alpha}(\mathbb{R})$ denote the space of functions $\psi : \mathbb{R} \to \mathbb{R}$ that are bounded and continuously differentiable with $\psi' \in BC^{0,\alpha}(\mathbb{R})$. Similarly, for a domain $G \subset \mathbb{R}^2$, let $BC^{1,\alpha}(\overline{G})$ denote the space of bounded functions $\psi \in C^1(\overline{G})$ for which $\nabla\psi \in BC^{0,\alpha}(\overline{G})$. All of these function spaces are Banach spaces with their respective norms. In particular, the standard norm on $BC(G)$ is $\|\cdot\|_{BC(G)}$, defined by $\|\psi\|_{BC(G)} := \sup_{x\in G} |\psi(x)|$ and that on $BC^{0,\alpha}(G)$ is $\|\cdot\|_{BC^{0,\alpha}(G)}$, defined by

$$\|\psi\|_{BC^{0,\alpha}(G)} := \|\psi\|_{BC(G)} + \sup_{x,y\in G, x\neq y} \frac{|\psi(x) - \psi(y)|}{|x-y|^\alpha}.$$

1 The Boundary Value Problem

Throughout we suppose that $f \in BC^{1,\alpha}(\mathbb{R})$. The problem we wish to solve will be set in the region $\Omega := \{x : x_2 < f(x_1)\}$ with boundary $\Gamma = \{(x_1, f(x_1)) : x_1 \in \mathbb{R})\}$. Since f is bounded there exist $f_-, f_+ \in \mathbb{R}$ such that $f_- \leqslant f(x_1) \leqslant f_+$ for all $x_1 \in \mathbb{R}$ and since f' is Hölder continuous, Γ is a fairly smooth (Lyapunov) surface.



(a) The lines Γ_H, $x_2 = f_+$, Γ and $x_2 = f_-$ (b) The domain Ω (c) The domain Ω_H

Figure 1: The domain for the boundary value problem

This paper is concerned with the solution, via a boundary integral equation method, of the following Dirichlet boundary value problem: Given $\phi_0 \in BC(\Gamma)$, find $\phi \in BC(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$\phi = \phi_0 \text{ on } \Gamma \qquad \text{and} \qquad \Delta\phi = 0 \text{ in } \Omega. \tag{1}$$

We choose $H > f_+$. Then $\Omega \subset \Omega_H$ where $\Omega_H := \{x \in \mathbb{R}^2 : x_2 < H\}$ and $\Gamma_H := \{x \in \mathbb{R}^2 : x_2 = H\}$. Now let

$$\Phi_H(x,y) := \Phi(x,y) - \Phi(x,y^r) = \Phi(x,y) - \Phi(x^r,y), \qquad x,y \in \overline{\Omega_H},\ x \neq y,$$

where $y^r := (y_1, 2H - y_2)$ is the reflection of y in Γ_H and

$$\Phi(x,y) \quad := \quad \frac{1}{2\pi} \ln \frac{1}{|x-y|}, \qquad x,y \in \mathbb{R}^2, x \neq y,$$

is the standard fundamental solution of Laplace's equation in two dimensions. Then Φ_H is the Dirichlet Green's function for the half-plane Ω_H.

To solve the boundary value problem we will look for a solution in the form of the *double layer potential*

$$\phi(x) := \int_\Gamma \frac{\partial \Phi_H(x,y)}{\partial \mathbf{n}(y)} \mu(y) \mathrm{d}\mathbf{s}(y), \qquad x \in \Omega, \tag{2}$$

for some *density* $\mu \in BC(\Gamma)$. Throughout $\mathbf{n}(y) = (-f'(y_1), 1)/\sqrt{1+f'(y_1)^2}$ will denote the unit normal vector at $y \in \Gamma$ directed out of Ω. We will abbreviate the norm on $BC(\Gamma)$ by $\|\cdot\|_\infty$, so that $\|\mu\|_\infty = \|\mu\|_{BC(\Gamma)} = \sup_{x\in\Gamma} |\mu(x)|$. Explicitly, for $y \in \Gamma$, $x \in \overline{\Omega_H}$,

$$\frac{\partial \Phi_H(x,y)}{\partial \mathbf{n}(y)} = \frac{1}{2\pi} \frac{(x-y).\mathbf{n}(y)}{|x-y|^2} - \frac{1}{2\pi} \frac{(x^r - y).\mathbf{n}(y)}{|x-y^r|^2}.$$

We will see shortly that $\phi(x)$ is well-defined by (2) for all $x \in \Omega_H$ (as a Lebesgue or Riemann improper integral). This statement is no longer true if we replace Φ_H by Φ in (2) unless μ decays sufficiently rapidly at infinity (e.g. $\mu(x) = O(|x|^{-b})$ as $|x| \to \infty$ for some $b > 1$). It is this fact that makes the choice of Φ_H instead of Φ in (2) essential to the analysis that is to follow.

As $f \in BC^{1,\alpha}(\mathbb{R})$, by definition, there exist constants $C_{f'}$ and C_α such that

$$|f'(x_1)| \leqslant C_{f'} \qquad \text{and} \qquad |f'(x_1) - f'(y_1)| \leqslant C_\alpha |x_1 - y_1|^\alpha, \tag{3}$$

for $x_1, y_1 \in \mathbb{R}$. We also define the constant $C_{\mathrm{int}} := \sqrt{1 + {C_{f'}}^2}$ that will be used later. Further, defining $\delta_\pm := 2(H - f_\mp)$, it holds that

$$0 < \delta_- \leqslant |y - y^r| \leqslant \delta_+, \qquad y \in \Gamma.$$

We begin by establishing some straightforward bounds.

Lemma 1.1
For $x, y \in \Gamma$, $|(x-y).\mathbf{n}(y)| \leqslant C_\alpha |x_1 - y_1|^{1+\alpha}$ and $|\mathbf{n}(x) - \mathbf{n}(y)| \leqslant \sqrt{5} C_\alpha |x_1 - y_1|^\alpha$.

Proof
Let $g(y_1) := \sqrt{1 + f'(y_1)^2}$, $y_1 \in \mathbb{R}$. By (3) and applying the mean value theorem, we have, for some ξ between x_1 and y_1,

$$|(x-y).\mathbf{n}(y)| = \left| \frac{(x_1 - y_1)(f'(y_1) - f'(\xi))}{g(y_1)} \right| \leqslant C_\alpha |x_1 - y_1|^{1+\alpha}.$$

Further,

$$|g(x_1) - g(y_1)| = \frac{|f'(x_1) + f'(y_1)|\, |f'(x_1) - f'(y_1)|}{g(x_1) + g(y_1)} \leqslant C_\alpha |x_1 - y_1|^\alpha$$

and so

$$\begin{aligned} &\frac{|f'(x_1) g(y_1) - f'(y_1) g(x_1)|}{g(x_1) g(y_1)} \\ &\qquad = \left| \frac{(f'(x_1) - f'(y_1)) g(y_1)}{g(x_1) g(y_1)} - \frac{f'(y_1)(g(x_1) - g(y_1))}{g(x_1) g(y_1)} \right| \\ &\qquad \leqslant 2 C_\alpha |x_1 - y_1|^\alpha. \end{aligned}$$

Thus

$$
\begin{aligned}
|\mathbf{n}(x) - \mathbf{n}(y)|^2 &= \left(\frac{f'(x_1)g(y_1) - f'(y_1)g(x_1)}{g(x_1)g(y_1)}\right)^2 + \left(\frac{g(x_1) - g(y_1)}{g(x_1)g(y_1)}\right)^2 \\
&\leqslant 5{C_\alpha}^2 |x_1 - y_1|^{2\alpha}.
\end{aligned}
$$

□

2 Properties of the Double-Layer Potential

In this section we establish the behaviour of the double layer potential given by (2) when $\mu \in BC(\Gamma)$. Throughout this section let ϕ denote the double layer potential defined by

$$
\phi(x) := \int_\Gamma \frac{\partial \Phi_H(x,y)}{\partial \mathbf{n}(y)} \mu(y) \mathrm{d}\mathbf{s}(y), \qquad x \in \overline{\Omega_H}, \tag{4}
$$

for some $\mu \in BC(\Gamma)$.

We first derive a key bound on the integrand, given by (5) below. This requires a simple preliminary result.

Lemma 2.1
If $x \in \Omega_H$, $y \in \Gamma$ and $x \neq y$, then

$$
|\nabla_y \Phi_H(x,y)| \leqslant \frac{3\,|x - x^r|}{2\pi\,|x - y|^2}.
$$

Proof
We have $0 < |x - y| < |x - y^r|$ and $0 < |x - y^r|^2 - |x - y|^2 = |x - x^r|\,|y - y^r|$. So

$$
\begin{aligned}
|\nabla_y \Phi_H(x,y)| &= \frac{1}{2\pi}\left|\frac{x - x^r}{|x - y|^2} + \frac{(x^r - y)\left(|x - y^r|^2 - |x - y|^2\right)}{|x - y|^2\,|x - y^r|^2}\right| \\
&\leqslant \frac{1}{2\pi}\left[\frac{|x - x^r|}{|x - y|^2} + \frac{|x^r - y|\,|x - x^r|\,|y - y^r|}{|x - y|^2\,|x - y^r|^2}\right] \\
&\leqslant \frac{|x - x^r|}{2\pi\,|x - y|^2}\left[1 + \frac{|x - y^r| + |x - y|}{|x - y^r|}\right] \\
&\leqslant \frac{3\,|x - x^r|}{2\pi\,|x - y|^2},
\end{aligned}
$$

as required. □

It now follows from Lemma 2.1 that

$$
\left|\frac{\partial \Phi_H(x,y)}{\partial \mathbf{n}(y)}\right| \leqslant \frac{3\,|H - x_2|}{\pi\,|x - y|^2} \leqslant \frac{3\delta_+}{2\pi\,|x - y|^2}, \qquad x \in \Omega_H, y \in \Gamma. \tag{5}
$$

This bound, together with Lemma 1.1, implies that the double-layer potential $\phi(x)$, given by (4), is well-defined for all $x \in \overline{\Omega_H}$ and $\mu \in BC(\Gamma)$. Further, it follows from the bound in Lemma 2.1 together with standard local elliptic regularity results [GT, Lemma 3.9 and Theorem 4.1] and standard results on the differentiation of functions defined as integrals, that the following lemma holds. Here and subsequently $\Gamma' := \{x^r : x \in \Gamma\}$ denotes the image of Γ in Γ_H.

Lemma 2.2
$\phi \in C^2(\mathbb{R}^2 \setminus (\Gamma \cup \Gamma'))$ and is harmonic in $\mathbb{R}^2 \setminus (\Gamma \cup \Gamma')$.

Having established that ϕ is harmonic above and below Γ, we now prove that it can be continuously extended onto Γ, specifically that the standard jump relations for the double-layer potential remains valid in this case.

Theorem 2.3
ϕ can be continuously extended from Ω to $\overline{\Omega}$ and from $\Omega_H \setminus \overline{\Omega}$ to $\Omega_H \setminus \Omega$ with limiting values

$$\phi_\pm(x) = \int_\Gamma \frac{\partial \Phi_H(x,y)}{\partial \mathbf{n}(y)} \mu(y) \mathrm{d}\mathbf{s}(y) \pm \frac{1}{2}\mu(x), \qquad x \in \Gamma, \tag{6}$$

where

$$\phi_\pm(x) := \lim_{h \to 0^+} \phi(x \pm h\mathbf{n}(x)).$$

Proof
It is clear, from the bounds in Lemma 1.1 and Lemma 2.1, that the right hand side of (6) is continuous on Γ. So we only need to show that, for every $x^* \in \Gamma$, $\phi(x) \to \phi_+(x^*)$ as $x \to x^*$ with $x \in \Omega$ and $\phi(x) \to \phi_-(x^*)$ as $x \to x^*$ with $x \in \Omega_H \setminus \overline{\Omega}$, and $\phi_\pm(x^*)$ given by (6).

Suppose $x^* \in \Gamma$ and define $\psi \in C(\Gamma)$ so that ψ is compactly supported and $\psi(y) = 1$ in a neighbourhood of x^*. Let I_1 and I_2 represent the double layer potentials with densities $\mu\psi$ and $\mu(1-\psi)$, respectively, so that $\phi = I_1 + I_2$. Then, by Lemma 2.2, I_2 is continuous in a neighbourhood of x^*. Further, from the standard jump relations for the double layer potential on bounded Lyapunov domains, [M1, Theorem 18.5.1 and Chapter 18, §13] or [G1, Chapter 2, §3, Theorem 2], it follows that $I_1(x) \to I_{1,-}(x^*)$ as $x \to x^*$ with $x \in \Omega$ and $I_1(x) \to I_{1,+}(x^*)$ as $x \to x^*$ with $x \in \Omega_H \setminus \overline{\Omega}$, where $I_{1,\pm}(x^*)$ is given by

$$I_{1,\pm} = \int_\Gamma \frac{\partial \Phi_H(x^*,y)}{\partial \mathbf{n}(y)} \mu(y)\psi(y) \mathrm{d}\mathbf{s}(y) \pm \frac{1}{2}\mu(x^*).$$

(To see that the results of [M1] and [G1] apply, note that we can consider I_1 to be a standard double layer potential supported on a closed Lyapunov curve with density which is zero except in neighbourhoods of x^* and its image point x^{*r}.) Combining these properties of I_1 and I_2 the result follows. (We note that this partition of unity technique has been used previously: for the two-dimensional case, see *e.g.* [C1], and for the three dimensional case, see *e.g.* [C2, PC]). $\square$

Corollary 2.4
With $\phi_\pm$ defined as in Theorem 2.3, it holds that $\mu = \phi_+ - \phi_-$.

Arguing similarly to Theorem 2.3 we also have the following result.

Theorem 2.5
The normal derivative of ϕ is continuous across Γ, in the sense that, for $x \in \Gamma$,

$$\mathbf{n}(x).\,(\nabla\phi(x + h\mathbf{n}(x)) - \nabla\phi(x - h\mathbf{n}(x))) \to 0, \qquad \text{as } h \to 0. \tag{7}$$

Further, (7) holds uniformly for x on every compact subset of Γ.

Proof
It is sufficient to show that, for every $x^* \in \Gamma$, (7) holds uniformly for x in a neighbourhood of x^*. So suppose $x^* \in \Gamma$ and define I_1, I_2 and ψ as in Theorem 2.3. Arguing as before we have that I_2 is continuously differentiable in a neighbourhood of x^*. To see that I_1 satisfies (7) we apply the standard jump relations [M1, Theorem 18.5.2 and Chapter 18, §13]. □

We next prove that ϕ in bounded in Ω_H (which, of course, includes Γ).

Theorem 2.6
There exists a positive constant C_Γ, dependent only on $\alpha, f_\pm, H, C_\alpha$ and $C_{f'}$, such that

$$|\phi(x)| = \left|\int_\Gamma \frac{\partial\Phi_H(x,y)}{\partial\mathbf{n}(y)}\mu(y)\mathrm{d}\mathbf{s}(y)\right| \leqslant C_\Gamma \left\|\mu\right\|_\infty, \qquad x \in \Omega_H. \tag{8}$$

Proof
Note first that, for $\mu \in BC(\Gamma)$, $x \in \Omega_H$, and defining $\Gamma_x := \{y \in \Gamma : |x_1 - y_1| < 2\delta_+\}$,

$$\left|\int_\Gamma \frac{\partial\Phi_H(x,y)}{\partial\mathbf{n}(y)}\mu(y)\mathrm{d}\mathbf{s}(y)\right| \leqslant \|\mu\|_\infty \left[I_{\mathrm{near}}(x) + I_{\mathrm{far}}(x)\right],$$

where

$$I_{\mathrm{near}}(x) := \int_{\Gamma_x} \left|\frac{\partial\Phi_H(x,y)}{\partial\mathbf{n}(y)}\right| \mathrm{d}\mathbf{s}(y), \quad I_{\mathrm{far}}(x) := \int_{\Gamma\setminus\Gamma_x} \left|\frac{\partial\Phi_H(x,y)}{\partial\mathbf{n}(y)}\right| \mathrm{d}\mathbf{s}(y).$$

We now bound I_{far} and I_{near} separately.

To bound $I_{\mathrm{far}}(x)$ note that, for $x \in \Omega_H$, $y \in \Gamma$,

$$|x - y| \geqslant |x - y_H| - |y - y_H| \quad \Rightarrow \quad |x - y|^2 \geqslant |x - y_H|\left[|x - y_H| - \delta_+\right],$$

where y_H is the projection of y onto Γ_H, that is, $y_H := (y_1, H)$. If $y \in \Gamma \setminus \Gamma_x$ then

$$\delta_+ \leqslant \frac{1}{2}|x_1 - y_1| \leqslant \frac{1}{2}|x - y_H|.$$

Thus, for $x \in \Omega_H$, $y \in \Gamma \setminus \Gamma_x$,

$$|x - y|^2 \geqslant \frac{1}{2}|x - y_H|^2 = \frac{1}{2}\left[(x_1 - y_1)^2 + (H - x_2)^2\right].$$

Using this inequality, the substitution $r = (x_1 - y_1)/(x_2 - H)$ and (5), we have

$$\begin{aligned}
I_{\mathrm{far}}(x) &\leqslant \int_{\Gamma\setminus\Gamma_x} \frac{3(H - x_2)}{\pi|x - y|^2}\mathrm{d}\mathbf{s}(y) \\
&\leqslant \frac{6C_{\mathrm{int}}}{\pi}\int_{-\infty}^{\infty} \frac{(H - x_2)}{(x_1 - y_1)^2 + (x_2 - H)^2}\mathrm{d}y_1 \\
&\leqslant \frac{6C_{\mathrm{int}}}{\pi}\int_{-\infty}^{\infty} \frac{1}{1 + r^2}\mathrm{d}r = 6C_{\mathrm{int}}.
\end{aligned}$$

To bound $I_{\text{near}}(x)$ we initially bound I_{near} by

$$I_{\text{near}}(x) \leqslant I_1(x) + I_2(x), \qquad x \in \Omega_H,$$

where

$$I_1(x) \;:=\; \int_{\Gamma_x} \left| \frac{\partial \Phi(x,y)}{\partial \mathbf{n}(y)} \right| \mathrm{d}\mathbf{s}(y), \quad I_2(x) := \int_{\Gamma_x} \left| \frac{\partial \Phi(x^r,y)}{\partial \mathbf{n}(y)} \right| \mathrm{d}\mathbf{s}(y).$$

To bound I_1 we note that, where $s = x_1 - y_1$ and $t = |x_2 - f(x_1)|$ (so that $t = 0$ if $x \in \Gamma$), it follows by the triangle inequality that

$$(s^2 + t^2)^{1/2} \leqslant |x - y| + |f(x_1) - f(y_1)| \leqslant (1 + C_{f'}) \, |x - y| \, .$$

Further, for $x \in \Omega_H$ and $y \in \Gamma$, where $x_\Gamma := (x_1, f(x_1))$, it follows from Lemma 1.1 that

$$|(x - y).\mathbf{n}(y)| \leqslant |(x_\Gamma - y).\mathbf{n}(y)| + |(x - x_\Gamma).\mathbf{n}(y)| \leqslant C_\alpha \, |s|^{1+\alpha} + t.$$

Using these bounds we have

$$\begin{aligned}
\int_{\Gamma_x} \left| \frac{\partial \Phi(x,y)}{\partial \mathbf{n}(y)} \right| \mathrm{d}\mathbf{s}(y) \;&=\; \frac{1}{2\pi} \int_{\Gamma_x} \left| \frac{(x-y).\mathbf{n}(y)}{|x-y|^2} \right| \mathrm{d}\mathbf{s}(y) \\
&\leqslant\; \frac{C_{\text{int}}(1 + C_{f'})^2}{2\pi} \int_{-2\delta_+}^{2\delta_+} \frac{C_\alpha \, |s|^{1+\alpha} + t}{s^2 + t^2} \mathrm{d}s \\
&\leqslant\; \frac{C_{\text{int}}(1 + C_{f'})^2}{2\pi} \left(\frac{C_\alpha (2\delta_+)^\alpha}{\alpha} + \pi \right).
\end{aligned}$$

To bound I_2, we have, for all $x \in \Omega_H$,

$$\left| \frac{\partial \Phi(x,y^r)}{\partial \mathbf{n}(y)} \right| \leqslant \frac{1}{2\pi} \frac{1}{|x - y^r|} \leqslant \frac{1}{\pi \delta_-}.$$

Thus

$$\int_{\Gamma_x} \left| \frac{\partial \Phi(x,y^r)}{\partial \mathbf{n}(y)} \right| \mathrm{d}\mathbf{s}(y) \leqslant \frac{C_{\text{int}}}{\pi \delta_-} \int_{\Gamma_x} \mathrm{d}\mathbf{s}(y) = \frac{4 C_{\text{int}} \delta_+}{\pi \delta_-}.$$

Putting these bounds together, we have shown that (8) holds with

$$C_\Gamma \;=\; C_{\text{int}} \left(6 + \frac{4\delta_+}{\pi \delta_-} + \frac{1 + C_{f'}}{2\pi} \left(\frac{C_\alpha (2\delta_+)^\alpha}{\alpha} + \pi \right) \right).$$

□

Properties of the double layer potential evaluated on the boundary Γ are particularly important. We define the double layer potential operator K by

$$(K\psi)(x) := 2 \int_\Gamma \frac{\partial \Phi_H(x,y)}{\partial \mathbf{n}(y)} \psi(y) \mathrm{d}\mathbf{s}(y), \qquad x \in \Gamma, \tag{9}$$

for $\psi \in BC(\Gamma)$.

We now prove three mapping properties of the operator K. These mapping properties are:

$K : BC(\Gamma) \to BC^{0,\alpha}(\Gamma)$

$K : BC^{0,\beta}(\Gamma) \to BC^{0,\alpha+\beta}(\Gamma)$, if $\alpha + \beta < 1$

$K : BC^{0,\beta}(\Gamma) \to BC^{1,\alpha+\beta-1}(\Gamma)$, if $\beta \in (0,1)$ and $\alpha + \beta > 1$.

We first prove the first two properties together. It is convenient in this proof to use the notation $BC^{0,0}(\Gamma)$ for $BC(\Gamma)$ and, for $\beta \in (0,1)$, $BC^{1,\beta}(\Gamma)$ is the set of $\phi : \Gamma \to \mathbb{R}$ for which $\tilde{\phi} \in BC^{1,\beta}(\mathbb{R})$ where $\tilde{\phi}(x_1) := \phi((x_1, f(x_1)))$.

Theorem 2.7

For $\beta \in [0, 1-\alpha)$, K is a bounded operator from $BC^{0,\beta}(\Gamma)$ to $BC^{0,\alpha+\beta}(\Gamma)$. In particular, for some $C > 0$, depending only on $\alpha, \beta, f_\pm, H, C_\alpha$ and $C_{f'}$,

$$|K\psi(x) - K\psi(z)| \leqslant C \|\psi\|_{BC^{0,\beta}(\Gamma)} |x-z|^{\alpha+\beta}, \qquad x, z \in \Gamma. \qquad (10)$$

Proof

We already know, from Theorem 2.6, that $K\psi \in BC(\Gamma)$ with $\|K\psi\|_\infty \leqslant 2C_\Gamma \|\psi\|_\infty$. In the straightforward case where $x, z \in \Gamma$ with $|x_1 - z_1| \geqslant 1$, we have

$$|K\psi(x) - K\psi(z)| \leqslant 4C_\Gamma \|\psi\|_\infty \leqslant 4C_\Gamma \|\psi\|_{BC^{0,\beta}(\Gamma)} |x-z|^{\alpha+\beta}.$$

To consider the case when $x, z \in \Gamma$ and $|x_1 - z_1| < 1$, we write

$$K\psi(x) - K\psi(z) = 2\left[I(x,z) - I(x^r, z^r)\right],$$

where

$$I(x,z) := \int_\Gamma \left[\frac{\partial\Phi(x,y)}{\partial\mathbf{n}(y)} - \frac{\partial\Phi(z,y)}{\partial\mathbf{n}(y)}\right] \psi(y) \mathrm{d}\mathbf{s}(y).$$

We examine $I(x,z)$ first. Let $\Gamma_t := \{y \in \Gamma : |x_1 - y_1| < t\}$, for $t > 0$, and let $\Gamma_\cup := \left\{y \in \overline{\Omega} : y_1 = x_1 \pm 2, y_2 \geqslant f_- - 1\right\} \cup \{y : |x_1 - y_1| \leqslant 2, y_2 = f_- - 1\}$, and note that the interior of $\Gamma_2 \cup \Gamma_\cup$ is a Lipschitz domain. By Gauss' theorem and Green's first theorem, cf. [CK, (2.42)], we know that

$$\int_{\Gamma_2 \cup \Gamma_\cup} \frac{\partial\Phi(w,y)}{\partial\mathbf{n}(y)} \mathrm{d}\mathbf{s}(y) = \frac{1}{2}, \qquad w \in \Gamma_2, \qquad (11)$$

and thus

$$\int_{\Gamma_2 \cup \Gamma_\cup} \left[\frac{\partial\Phi(x,y)}{\partial\mathbf{n}(y)} - \frac{\partial\Phi(z,y)}{\partial\mathbf{n}(y)}\right] \mathrm{d}\mathbf{s}(y) = 0.$$

Using this identity, we split $I(x,z)$ into four integrals,

$$I(x,z) = I_1(x,z) + I_2(x,z) + I_3(x,z) + I_4(x,z),$$

where

$$\begin{aligned}
I_1(x,z) &:= \int_{\Gamma_{2r}} \left[\frac{\partial\Phi(x,y)}{\partial\mathbf{n}(y)} - \frac{\partial\Phi(z,y)}{\partial\mathbf{n}(y)}\right] [\psi(y) - \psi(x)] \,\mathrm{d}\mathbf{s}(y), \\
I_2(x,z) &:= \int_{\Gamma_2 \setminus \Gamma_{2r}} \left[\frac{\partial\Phi(x,y)}{\partial\mathbf{n}(y)} - \frac{\partial\Phi(z,y)}{\partial\mathbf{n}(y)}\right] [\psi(y) - \psi(x)] \,\mathrm{d}\mathbf{s}(y), \\
I_3(x,z) &:= \int_{\Gamma \setminus \Gamma_2} \left[\frac{\partial\Phi(x,y)}{\partial\mathbf{n}(y)} - \frac{\partial\Phi(z,y)}{\partial\mathbf{n}(y)}\right] \psi(y) \mathrm{d}\mathbf{s}(y), \\
I_4(x,z) &:= -\psi(x) \int_{\Gamma_\cup} \left[\frac{\partial\Phi(x,y)}{\partial\mathbf{n}(y)} - \frac{\partial\Phi(z,y)}{\partial\mathbf{n}(y)}\right] \mathrm{d}\mathbf{s}(y),
\end{aligned}$$

in which $r = |x_1 - z_1| < 1$, see Figure 2.

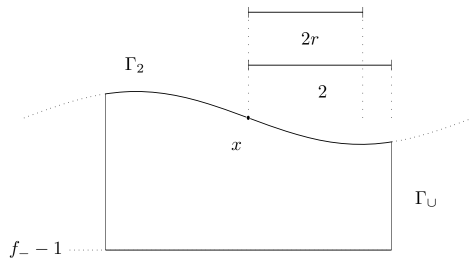


Figure 2: The point x, the surfaces Γ_2 and $\Gamma_\cup$.

By Lemma 1.1, for $x, y \in \Gamma$,

$$\left|\frac{\partial \Phi_H(x,y)}{\partial \mathbf{n}(y)}\right| \leqslant \frac{1}{2\pi} C_\alpha \left|x_1 - y_1\right|^{\alpha-1}.$$

Note also that, for $x, y \in \Gamma$, $|\psi(y) - \psi(x)| \leqslant 2 \|\psi\|_\infty = 2 \|\psi\|_{BC^{0,0}(\Gamma)}$. Thus, for $0 \leqslant \beta < 1 - \alpha$,

$$|\psi(y) - \psi(x)| \leqslant 2 \|\psi\|_{BC^{0,\beta}(\Gamma)} |x - y|^\beta \leqslant 2{C_{\mathrm{int}}}^\beta \|\psi\|_{BC^{0,\beta}(\Gamma)} |x_1 - y_1|^\beta.$$

To bound $I_1(x,z)$, we note that if $|x_1 - y_1| < 2r$ then $|z_1 - y_1| < 3r$, and therefore

$$\begin{aligned}
|I_1(x,z)| &\leqslant \frac{1}{\pi} {C_{\mathrm{int}}}^\beta C_\alpha \|\psi\|_{BC^{0,\beta}(\Gamma)} (2r)^\beta \left[\int_{\Gamma_{2r}} |x_1 - y_1|^{\alpha-1} \mathrm{d}\mathbf{s}(y)\right. \\
&\qquad\qquad \left. + \int_{\Gamma_{2r}} |z_1 - y_1|^{\alpha-1} \mathrm{d}\mathbf{s}(y)\right] \\
&\leqslant \frac{1}{\pi} {C_{\mathrm{int}}}^{1+\beta} C_\alpha \|\psi\|_{BC^{0,\beta}(\Gamma)} (2r)^\beta \left[\int_{-2r}^{2r} |s|^{\alpha-1} \mathrm{d}s + \int_{-3r}^{3r} |s|^{\alpha-1} \mathrm{d}s\right] \\
&\leqslant \frac{1}{\pi} 2^{1+\beta}(2^\alpha + 3^\alpha) {C_{\mathrm{int}}}^{1+\beta} C_\alpha \|\psi\|_{BC^{0,\beta}(\Gamma)} |x - z|^{\alpha+\beta}. \qquad (12)
\end{aligned}$$

Turning to $I_2(x,z)$, we note that if $|x_1 - y_1| \geqslant 2r$, $\xi \in \Gamma$ and ξ_1 lies in the closed interval between x_1 and z_1 then $|\xi_1 - y_1| \geqslant r \geqslant |\xi_1 - x_1|$ so $|x_1 - y_1| \leqslant |x_1 - \xi_1| + |\xi_1 - y_1| \leqslant 2|\xi_1 - y_1|$. So, by the mean value theorem, we have

$$\left|\frac{1}{|x-y|^2} - \frac{1}{|z-y|^2}\right| \leqslant 16 \frac{|x-z|}{|x_1 - y_1|^3}$$

and, by Lemma 1.1,

$$\begin{aligned}
|\mathbf{n}(y).(x-z)| &\leqslant |(\mathbf{n}(y) - \mathbf{n}(x)).(x-z)| + |\mathbf{n}(x).(x-z)| \\
&\leqslant \sqrt{5} C_\alpha |x_1 - y_1|^\alpha |x - z| + C_\alpha |x_1 - z_1|^{1+\alpha}.
\end{aligned}$$

Combining these inequalities we have, for $2r \leqslant |x_1 - y_1| \leqslant 2$, that $|z_1 - y_1| \leqslant$

$\frac{3}{2}|x_1 - y_1|$ and

$$
\begin{aligned}
&\left|\frac{\partial \Phi(x,y)}{\partial \mathbf{n}(y)} - \frac{\partial \Phi(z,y)}{\partial \mathbf{n}(y)}\right| \\
&\qquad\leqslant \quad \frac{1}{2\pi}\left|\frac{1}{|x-y|^2} - \frac{1}{|z-y|^2}\right| |\mathbf{n}(y).(z-y)| + \frac{1}{2\pi}\frac{|\mathbf{n}(y).(x-z)|}{|x-y|^2} \\
&\qquad\leqslant \quad \frac{1}{2\pi}\left(\frac{24(3^\alpha)}{2^\alpha} + \sqrt{5} + 1\right) C_\alpha |x-z| \, |x_1 - y_1|^{\alpha-2}, \qquad (13)
\end{aligned}
$$

so, letting $C_1 := \frac{1}{2\pi}\left(\frac{24(3^\alpha)}{2^\alpha} + \sqrt{5} + 1\right)$,

$$
\begin{aligned}
|I_2(x,z)| &\leqslant 2C_1 C_\alpha {C_{\mathrm{int}}}^\beta \|\psi\|_{BC^{0,\beta}(\Gamma)} |x-z| \int_{\Gamma_2 \setminus \Gamma_{2r}} |x_1 - y_1|^{\alpha+\beta-2} \, \mathrm{d}\mathbf{s}(y) \\
&\leqslant 4C_1 C_\alpha {C_{\mathrm{int}}}^{1+\beta} \|\psi\|_{BC^{0,\beta}(\Gamma)} |x-z| \int_{2r}^{2} s^{\alpha+\beta-2} \mathrm{d}s \\
&\leqslant \frac{2^{1+\alpha+\beta}}{1-\alpha-\beta} C_1 C_\alpha {C_{\mathrm{int}}}^{1+\beta} \|\psi\|_{BC^{0,\beta}(\Gamma)} |x-z|^{\alpha+\beta}.
\end{aligned}
$$

To bound $I_3(x,z)$ we note that, for $i = 1, 2$,

$$
\frac{\partial}{\partial x_i}\frac{\partial \Phi(x,y)}{\partial \mathbf{n}(y)} = \frac{1}{2\pi |x-y|^2}\left(\mathbf{n}_i(y) - 2\frac{(x-y).\mathbf{n}(y)}{|x-y|^2}(x_i - y_i)\right)
$$

so that

$$
\left|\nabla_x \frac{\partial \Phi(x,y)}{\partial \mathbf{n}(y)}\right| \leqslant \frac{3\sqrt{2}}{2\pi} |x-y|^{-2}. \qquad (14)
$$

Hence, by the mean value theorem,

$$
\left|\frac{\partial \Phi(x,y)}{\partial \mathbf{n}(y)} - \frac{\partial \Phi(x,y)}{\partial \mathbf{n}(y)}\right| \leqslant \frac{3\sqrt{2}}{2\pi} |x-z| \max\left[|x-y|^{-2}, |z-y|^{-2}\right].
$$

Thus, since, for $y \in \Gamma \setminus \Gamma_2$, $|z-y| \geqslant |x-y| - |x-z| \geqslant |x-y| - 1 \geqslant \frac{1}{2}|x-y|$, it holds that

$$
\left|\frac{\partial \Phi(x,y)}{\partial \mathbf{n}(y)} - \frac{\partial \Phi(x,y)}{\partial \mathbf{n}(y)}\right| \leqslant \frac{6\sqrt{2}}{\pi} |x-z| \, |x-y|^{-2}. \qquad (15)
$$

Hence,

$$
\begin{aligned}
|I_3(x,z)| &\leqslant \frac{12\sqrt{2}}{\pi} C_{\mathrm{int}} \|\psi\|_\infty |x-z| \int_2^\infty s^{-2} \mathrm{d}s \\
&\leqslant \frac{6\sqrt{2}}{\pi} C_{\mathrm{int}} \|\psi\|_{BC^{0,\beta}(\Gamma)} |x-z|.
\end{aligned}
$$

Now turning to $I_4(x,z)$, we can apply (15) once more to get, since $|x-y| \geqslant 1$ and $|z-y| \geqslant 1$ on $\Gamma_\cup$,

$$
\begin{aligned}
|I_4(x,z)| &\leqslant \frac{3\sqrt{2}}{2\pi} \|\psi\|_\infty |x-z| \int_{\Gamma_\cup} \mathrm{d}\mathbf{s}(y) \\
&\leqslant \frac{3\sqrt{2}}{\pi}(3 + f_+ - f_-) \|\psi\|_{BC^{0,\beta}(\Gamma)} |x-z|.
\end{aligned}
$$

Finally, we examine the reflected portion. By applying (15) and noting that $I(x^r, z^r)$ is never singular we can see that

$$\begin{aligned}
|I(x^r, z^r)| &\leqslant \|\psi\|_\infty \int_\Gamma \left| \frac{\partial \Phi(x^r, y)}{\partial \mathbf{n}(y)} - \frac{\partial \Phi(z^r, y)}{\partial \mathbf{n}(y)} \right| \mathrm{d}\mathbf{s}(y) \\
&\leqslant \frac{3\sqrt{2}}{2\pi} \|\psi\|_\infty |x - z| \int_\Gamma \max \left[|x - y^r|^{-2}, |z^r - y|^{-2} \right] \mathrm{d}\mathbf{s}(y) \\
&\leqslant \frac{3\sqrt{2}}{\pi} C_{\mathrm{int}} \|\psi\|_\infty |x - z| \int_0^\infty \frac{1}{\delta_-^2 + s^2} \mathrm{d}s \\
&\leqslant \frac{3\sqrt{2} C_{\mathrm{int}}}{\delta_-} \|\psi\|_{BC^{0,\beta}(\Gamma)} |x - z| .
\end{aligned}$$

Therefore the required bound (10) holds. □

In the next theorem, and in the rest of the paper, for $x \in \Gamma$, let $\mathbf{t}(x)$ denote the unit tangent vector, with the horizontal component of $\mathbf{t}(x)$ chosen to be positive (explicitly, $\mathbf{t}(x) = (1, f'(x_1))/\sqrt{1 + f'(x_1)^2}$). For $\phi \in C^1(\Gamma)$, $x \in \Gamma$, we define

$$\frac{\partial \phi}{\partial s}(x) := \lim_{h \to 0} \frac{\phi((x_1 + h, f(x_1 + h))) - \phi(x)}{l},$$

where

$$l := \int_{x_1}^{x_1 + h} \sqrt{1 + f'(s)^2} ds.$$

Let

$$d(x, y) := -\frac{1}{2\pi} \left[\frac{\mathbf{t}(x).\mathbf{n}(y)}{|x - y|^2} - 2 \frac{(x - y).\mathbf{n}(y)(x - y).\mathbf{t}(x)}{|x - y|^4} \right],$$

for $x, y \in \Gamma$.

Theorem 2.8
If $\beta \in (0, 1)$ and $\alpha + \beta > 1$ then $K : BC^{0,\beta}(\Gamma) \to BC^{1,\alpha+\beta-1}(\Gamma)$ and is bounded. Precisely,

$$\frac{\partial K\psi(x)}{\partial s} = \int_\Gamma [d(x, y) - d(x^r, y)] [\psi(y) - \psi(x)] \mathrm{d}\mathbf{s}(y), \qquad x \in \Gamma,$$

and for some constant $C > 0$, dependent only on $\alpha, \beta, f_\pm, H, C_\alpha$ and $C_{f'}$,

$$\left| \frac{\partial K\psi(x)}{\partial s} - \frac{\partial K\psi(z)}{\partial s} \right| \leqslant C \|\psi\|_{BC^{0,\beta}(\Gamma)} |x - z|^{\alpha+\beta-1},$$

for $x, z \in \Gamma$.

Proof
Let $x = (x_1, f(x_1)) \in \Gamma$. For $h \in \mathbb{R}$, let $x_h = (x_1 + h, f(x_1 + h))$. Then $x_h \in \Gamma$ is distance l along Γ from x and we need to examine $\frac{1}{l}[K\psi(x_h) - K\psi(x)]$. Choose $m \in \mathbb{Z}$ so that $m \leqslant x_1 < m + 1$ and, for $t > 1$, set

$$\begin{aligned}
\Gamma'_t &:= \{y \in \Gamma : m - t - 1 < y_1 < m + t + 1\}, \\
\Gamma'_{\cup,t} &:= \left\{y \in \overline{\Omega} : y_1 = m \pm (t + 1), y_2 \geqslant f_- - 1\right\} \\
&\qquad \cup \{y : |(m, f_- - 1) - y| = t + 1, y_2 \leqslant f_- - 1\},
\end{aligned}$$

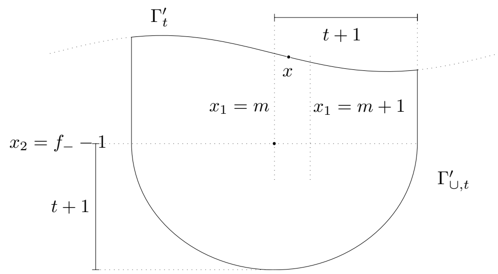


Figure 3: The point x, the surfaces Γ'_t and $\Gamma'_{\cup,t}$.

see Figure 3.

For $t > 1$ let K_t denote the double layer potential operator over the surface Γ'_t, given by

$$K_t\psi(x) := \int_{\Gamma'_t} \frac{\partial\Phi(x,y)}{\partial\mathbf{n}(y)}\psi(y)\mathrm{d}\mathbf{s}(y).$$

Clearly, $\left|\Gamma'_{\cup,t}\right| \leqslant \pi t + \delta_+$, and if $t \geqslant \frac{\delta_+}{4-\pi}$ then $\left|\Gamma'_{\cup,t}\right| \leqslant 4t$. The domain enclosed by $\Gamma'_t \cup \Gamma'_{\cup,t}$ is a closed Lipschitz domain and applying (11) we have,

$$\begin{aligned}
\frac{1}{l}&\left[K_t\psi(x_h) - K_t\psi(x)\right] \\
&= \frac{1}{l}\int_{\Gamma'_t}\left[\frac{\partial\Phi(x_h,y)}{\partial\mathbf{n}(y)} - \frac{\partial\Phi(x,y)}{\partial\mathbf{n}(y)}\right]\psi(y)\mathrm{d}\mathbf{s}(y) \\
&= \frac{1}{l}\int_{\Gamma'_t}\left[\frac{\partial\Phi(x_h,y)}{\partial\mathbf{n}(y)} - \frac{\partial\Phi(x,y)}{\partial\mathbf{n}(y)}\right]\left[\psi(y)-\psi(x)\right]\mathrm{d}\mathbf{s}(y) \\
&\qquad - \frac{1}{l}\psi(x)\int_{\Gamma'_{\cup,t}}\left[\frac{\partial\Phi(x_h,y)}{\partial\mathbf{n}(y)} - \frac{\partial\Phi(x,y)}{\partial\mathbf{n}(y)}\right]\mathrm{d}\mathbf{s}(y) \\
&= \int_{\Gamma'_t} d_h(x,y)\left[\psi(y)-\psi(x)\right]\mathrm{d}\mathbf{s}(y) \\
&\qquad - \psi(x)\int_{\Gamma'_{\cup,t}} d_h(x,y)\mathrm{d}\mathbf{s}(y) \\
&= I_a(x) + I_b(x) - \psi(x)I_c(x)
\end{aligned}$$

where

$$d_h(x,y) := \frac{1}{l}\left[\frac{\partial\Phi(x_h,y)}{\partial\mathbf{n}(y)} - \frac{\partial\Phi(x,y)}{\partial\mathbf{n}(y)}\right],$$

and

$$\begin{aligned}
I_a(x) &:= \int_{\Gamma_{2h}} d_h(x,y)\left[\psi(y)-\psi(x)\right]\mathrm{d}\mathbf{s}(y), \\
I_b(x) &:= \int_{\Gamma'_t\setminus\Gamma_{2h}} d_h(x,y)\left[\psi(y)-\psi(x)\right]\mathrm{d}\mathbf{s}(y), \\
I_c(x) &:= \int_{\Gamma'_{\cup,t}} d_h(x,y)\mathrm{d}\mathbf{s}(y).
\end{aligned}$$

Now the integral $I_1(x,z)$ in the proof of Theorem 2.7 is identical to $\frac{1}{l}I_a(x)$, if we set $r = h$. We again use the notation $\Gamma_{2h} = \{y \in \Gamma : |x_1 - y_1| < 2h\}$. So substituting $z = x_h$ in (12) and as $\alpha + \beta > 1$ we can see that $|I_a(x)| \leqslant \frac{1}{\pi}2^{1+\beta}(2^\alpha + 3^\alpha){C_{\rm int}}^{\alpha+2\beta}C_\alpha \|\psi\|_{BC^{0,\beta}(\Gamma)} h^{\alpha+\beta-1} \to 0$ as $h \to 0$.

We now define

$$\widetilde{d}_h(x,y) := \begin{cases} 0, & y \in \Gamma_{2h}, \\ d_h(x,y), & y \in \Gamma_t' \setminus \Gamma_{2h}, \end{cases}$$

so that

$$I_b(x) = \int_{\Gamma_t'} \widetilde{d}_h(x,y)\,[\psi(y) - \psi(x)]\,\mathrm{d}\mathbf{s}(y).$$

Taking $z = x_h$, $t > 1$ and using (13) we see that $\left|\widetilde{d}_h(x,y)\right| \leqslant C_1 C_\alpha |x_1 - y_1|^{\alpha-2}$ for $y \in \Gamma_1 \setminus \Gamma_{2h}$ and with C_1 defined as in the proof of Theorem 2.7. Similarly, using (15), $\left|\widetilde{d}_h(x,y)\right| \leqslant \frac{6\sqrt{2}}{\pi}|x_1 - y_1|^{-2}$ for $y \in \Gamma_t' \setminus \Gamma_1$. Thus, for $0 < h \leqslant 1$,

$$\left|\widetilde{d}_h(x,y)\right| \leqslant D(x,y) \;:=\; \begin{cases} C_1 C_\alpha |x_1 - y_1|^{\alpha-2}, & y \in \Gamma_1, \\ \frac{6\sqrt{2}}{\pi}|x_1 - y_1|^{-2}, & y \in \Gamma_t' \setminus \Gamma_1. \end{cases}$$

Now $D(x,\cdot)\,[\psi(\cdot) - \psi(x)] \in L^1(\Gamma_t')$ and, for $x \neq y$, $d_h(x,y) \to d(x,y)$ as $h \to 0$. Hence by the dominated convergence theorem we have, as $h \to 0$,

$$I_b(x) \to \int_{\Gamma_t'} d(x,y)\,[\psi(y) - \psi(x)]\,\mathrm{d}\mathbf{s}(y).$$

It is clear that

$$I_c(s) \to \int_{\Gamma_{\cup,t}'} d(x,y)\mathrm{d}\mathbf{s}(y)$$

as $h \to 0$ so that

$$\begin{aligned} \frac{\partial K_t\psi(x)}{\partial s} &:= \lim_{h\to 0}\frac{1}{l}\,[K_t(x_h) - K_t(x)] \\ &= \int_{\Gamma_t'} d(x,y)\,[\psi(y) - \psi(x)]\,\mathrm{d}\mathbf{s}(y) - \psi(x)\int_{\Gamma_{\cup,t}'} d(x,y)\mathrm{d}\mathbf{s}(y). \end{aligned} \tag{16}$$

Now for $x, y \in \Gamma$ and $x \neq y$, we have the bound,

$$|d(x,y)| \leqslant \frac{3}{2\pi}|x - y|^{-2}. \tag{17}$$

Hence by (15) with $z = x_h$ and letting $t \geqslant \frac{\delta_+}{4-\pi}$, so that $\left|\int_{\Gamma_{\cup,t}'}\mathrm{d}\mathbf{s}(y)\right| \leqslant 4t$, we have

$$\begin{aligned} \left|\int_{\Gamma_{\cup,t}'} d(x,y)\mathrm{d}\mathbf{s}(y)\right| &\leqslant \frac{6\sqrt{2}}{\pi}\int_{\Gamma_{\cup,t}'}|x - y|^{-2}\,\mathrm{d}\mathbf{s}(y) \\ &\leqslant \frac{6\sqrt{2}}{\pi}t^{-2}\int_{\Gamma_{\cup,t}'}\mathrm{d}\mathbf{s}(y) \leqslant \frac{24\sqrt{2}}{\pi}t^{-1}. \end{aligned} \tag{18}$$

Arguing similarly to the derivation of (16) and (18), we have

$$\begin{aligned}\int_{\Gamma'_t} d(x^r,y)\psi(y)\mathrm{d}\mathbf{s}(y) &= \int_{\Gamma'_t} d(x^r,y)\left(\psi(y)-\psi(x)\right)\mathrm{d}\mathbf{s}(y) \\ &\quad - \psi(x)\int_{\Gamma'_{\cup,t}} d(x^r,y)\mathrm{d}\mathbf{s}(y)\end{aligned}$$

and

$$\left|\int_{\Gamma'_{\cup,t}} d(x,y)\mathrm{d}\mathbf{s}(y)\right| \leqslant \frac{24\sqrt{2}}{\pi}t^{-1}.$$

Thus, and by (16) and (18),

$$\begin{aligned}\frac{\partial K\psi(x)}{\partial s} &= \int_{\Gamma\setminus\Gamma'_t}\left(d(x,y)-d(x^r,y)\right)\psi(y)\mathrm{d}\mathbf{s}(y) \\ &\quad + \int_{\Gamma'_t}\left(d(x,y)-d(x^r,y)\right)\left(\psi(y)-\psi(x)\right)\mathrm{d}\mathbf{s}(y) \\ &\quad - \psi(x)\int_{\Gamma'_{\cup,t}}\left(d(x,y)-d(x^r,y)\right)\mathrm{d}\mathbf{s}(y).\end{aligned}$$

Letting $t\to\infty$ we see that

$$\frac{\partial K\psi(x)}{\partial s} = \int_\Gamma \left[d(x,y)-d(x^r,y)\right]\left[\psi(y)-\psi(x)\right]\mathrm{d}\mathbf{s}(y),$$

for $x\in\Gamma$, as required.

We now show that the surface derivative is Hölder continuous. $(K\psi)'(x) := \partial K\psi(x)/\partial s$ will be a useful shorthand to employ for the next section of the proof.

For $x,y\in\Gamma$ we show a further inequality for $d(x,y)$. As $\mathbf{t}(x).\mathbf{n}(y) = \mathbf{t}(x).(\mathbf{n}(y)-\mathbf{n}(x))$ and by applying Lemma 1.1 we have

$$|d(x,y)| \leqslant \frac{1}{2\pi}(2+\sqrt{5})C_\alpha\left|x_1-y_1\right|^{\alpha-2}. \tag{19}$$

Using (17), (19) and as $\psi\in BC^{0,\beta}(\Gamma)$ we have

$$\begin{aligned}|(K\psi)'(x)| &\leqslant \frac{1}{\pi}(2+\sqrt{5})C_\alpha {C_{\mathrm{int}}}^{1+\beta}\left\|\psi\right\|_{BC^{0,\beta}(\Gamma)}\int_{-1}^{1}|s|^{\alpha+\beta-2}\,\mathrm{d}s \\ &\quad + \frac{3}{\pi}C_{\mathrm{int}}\left\|\psi\right\|_{BC^{0,\beta}(\Gamma)}\int_1^\infty |s|^{-2}\,\mathrm{d}s \\ &\leqslant \frac{1}{\pi}\left((2+\sqrt{5})C_\alpha {C_{\mathrm{int}}}^{\beta}+3\right)C_{\mathrm{int}}\left\|\psi\right\|_{BC^{0,\beta}(\Gamma)}.\end{aligned}$$

Let $C_2 := \frac{1}{\pi}\left((2+\sqrt{5})C_\alpha {C_{\mathrm{int}}}^{\beta}+3\right)C_{\mathrm{int}}$ and therefore $(K\psi)'(x)$ exists as an improper integral.

We now consider $|(K\psi)'(x)-(K\psi)'(z)|$. For $|x_1-z_1|\geqslant 1$, we can utilise the above bound to obtain

$$|(K\psi)'(x)-(K\psi)'(z)| \leqslant 2C_2\left\|\psi\right\|_{BC^{0,\beta}(\Gamma)} \leqslant 2C_2\left\|\psi\right\|_{BC^{0,\beta}(\Gamma)}|x-z|^{\alpha+\beta-1}.$$

Similarly to Theorem 2.7, we split the integral into parts for $|x_1 - z_1| \leqslant 1$, namely

$$(K\psi)'(x) - (K\psi)'(z) = 2\left[I_1(x,z) + I_2(x,z) + I_3(x,z) + I_4(x,z)\right]$$

where

$$\begin{aligned}
I_1(x,z) &:= \int_{\Gamma_{2r}} d(x,y)\left[\psi(y) - \psi(x)\right] - d(z,y)\left[\psi(y) - \psi(z)\right] \mathrm{d}\mathbf{s}(y), \\
I_2(x,z) &:= \int_{\Gamma_2 \setminus \Gamma_{2r}} \left[d(x,y) - d(z,y)\right]\left[\psi(y) - \psi(x)\right] \mathrm{d}\mathbf{s}(y), \\
I_3(x,z) &:= \int_{\Gamma \setminus \Gamma_2} \left[d(x,y) - d(z,y)\right]\left[\psi(y) - \psi(x)\right] \mathrm{d}\mathbf{s}(y), \\
I_4(x,z) &:= -\int_{\Gamma \setminus \Gamma_{2r}} d(z,y)\left[\psi(z) - \psi(x)\right] \mathrm{d}\mathbf{s}(y),
\end{aligned}$$

with $r = |x_1 - y_1|$. We bound I_1 in the same manner as in Theorem 2.7. By (19) and as $|x_1 - y_1| < 2r$ implies $|z_1 - y_1| < 3r$,

$$\begin{aligned}
|I_1(x,z)| &\leqslant \frac{1}{2\pi} {C_{\mathrm{int}}}^{1+\beta} C_\alpha \|\psi\|_{BC^{0,\beta}(\Gamma)} \left[\int_{-2r}^{2r} |s|^{\alpha+\beta-2} + \int_{-3r}^{3r} |s|^{\alpha+\beta-2}\, \mathrm{d}s\right] \\
&\leqslant \frac{1}{\pi}(2^{\alpha+\beta-1} + 3^{\alpha+\beta-1}) {C_{\mathrm{int}}}^{1+\beta} C_\alpha \|\psi\|_{BC^{0,\beta}(\Gamma)} |x - z|^{\alpha+\beta-1}.
\end{aligned}$$

Before we can proceed to the next two integrals, we must prove a final inequality for $d(x,y)$. We again note that $|x_1 - y_1| \leqslant 2|\xi_1 - y_1|$, if $2r \leqslant |x_1 - y_1|$ and $\xi_1 \in [x_1, z_1]$, $\xi \in \Gamma$. So by the mean value theorem we have,

$$\left|\frac{1}{|x-y|^4} - \frac{1}{|z-y|^4}\right| \leqslant 32\frac{|x-z|}{|x_1 - y_1|^5}.$$

Further, from Lemma 1.1, we can deduce that $|\mathbf{t}(x) - \mathbf{t}(y)| \leqslant \sqrt{5} C_\alpha |x_1 - y_1|^\alpha$ and $|(x-y).\mathbf{t}(y)| \leqslant C_\alpha |x_1 - y_1|^{1+\alpha}$. Now by applying the above bounds, $|z_1 - y_1| \leqslant \frac{3}{2}|x_1 - y_1|$, $|x_1 - z_1| \leqslant \frac{1}{2}|x_1 - y_1|$, $|z - y| \leqslant C_{\mathrm{int}} |z_1 - y_1|$ and $|x_1 - z_1| \leqslant |x - z|$, so that

$$\begin{aligned}
&\left|\frac{\mathbf{n}(y).(x-y)\mathbf{t}(x).(x-y)}{|x-y|^4} - \frac{\mathbf{n}(y).(z-y)\mathbf{t}(z).(z-y)}{|z-y|^4}\right| \\
&\qquad = \Bigg|\left(\frac{1}{|x-y|^4} - \frac{1}{|z-y|^4}\right)\mathbf{n}(y).(z-y)\mathbf{t}(z).(z-y) \\
&\qquad\qquad + \frac{1}{|x-y|^4}\Big(\mathbf{n}(y).(x-y)(\mathbf{t}(x) - \mathbf{t}(z)).(x-y) \\
&\qquad\qquad\qquad + \mathbf{n}(y).(x-z)\mathbf{t}(z).(z-y) \\
&\qquad\qquad\qquad + \mathbf{n}(y).(x-y)\mathbf{t}(z).(x-z)\Big)\Bigg| \\
&\qquad \leqslant C_3 C_\alpha |x-z|^\alpha |x_1 - y_1|^{-2}
\end{aligned}$$

and

$$\left|\frac{\mathbf{t}(x).\mathbf{n}(y)}{|x-y|^2}-\frac{\mathbf{t}(z).\mathbf{n}(y)}{|z-y|^2}\right|$$

$$\begin{aligned}
&\leqslant \left|\frac{1}{|x-y|^2}-\frac{1}{|z-y|^2}\right||(\mathbf{t}(z)-\mathbf{t}(y)).\mathbf{n}(y)|+\frac{|\mathbf{n}(y).(\mathbf{t}(x)-\mathbf{t}(z))|}{|x-y|^2}\\
&\leqslant C_4C_\alpha\,|x-z|^\alpha\,|x_1-y_1|^{-2}.
\end{aligned}$$

where $C_3 = 2^2 3^{2+\alpha}{C_{\mathrm{int}}}^{2-\alpha}+\sqrt{5}{C_{\mathrm{int}}}^{\alpha}+2^{\alpha-1}(3^{1+\alpha}+1)$ and $C_4 = 2^1 3^{1+\alpha}{C_{\mathrm{int}}}^{1-\alpha}$ $+\sqrt{5}{C_{\mathrm{int}}}^{\alpha}$. Hence

$$|d(x,y)-d(z,y)|\leqslant\frac{1}{2\pi}(2C_3+C_4)C_\alpha\,|x-z|^\alpha\,|x_1-y_1|^{-2}.$$

Thus turning to I_2 we have

$$\begin{aligned}
|I_2(x,z)| &\leqslant \frac{1}{\pi}(2C_3+C_4)C_\alpha C_{\mathrm{int}}\,\|\psi\|_{BC^{0,\beta}(\Gamma)}\,|x-z|^\alpha\int_{2r}^{2}|s|^{\beta-2}\,\mathrm{d}s\\
&\leqslant \frac{1}{\pi}2^{\beta-1}(2C_3+3)C_\alpha C_{\mathrm{int}}\,\|\psi\|_{BC^{0,\beta}(\Gamma)}\left[|x-z|^{\alpha+\beta-1}+|x-z|^\alpha\right].
\end{aligned}$$

For I_3 we utilise (17) and the mean value theorem again to obtain

$$\begin{aligned}
|I_3(x,z)| &\leqslant \frac{3}{\pi}C_{\mathrm{int}}\,\|\psi\|_{BC^{0,\beta}(\Gamma)}\,|x-z|\int_{2}^{\infty}|s|^{\beta-2}\,\mathrm{d}s\\
&\leqslant \frac{3}{\pi}(2^{\beta-1})C_{\mathrm{int}}\,\|\psi\|_{BC^{0,\beta}(\Gamma)}\,|x-z|.
\end{aligned}$$

To bound $I_4(x,z)$, by (19) we obtain

$$\begin{aligned}
|I_4(x,z)| &\leqslant \frac{1}{\pi}(2+\sqrt{5})C_{\mathrm{int}}\,\|\psi\|_{BC^{0,\beta}(\Gamma)}\,|x-z|^\beta\int_{2r}^{\infty}|s|^{\alpha-2}\,\mathrm{d}s\\
&\leqslant \frac{1}{\pi}2^{\alpha-1}(2+\sqrt{5})C_{\mathrm{int}}\,\|\psi\|_{BC^{0,\beta}(\Gamma)}\,|x-z|^{\alpha+\beta-1}.
\end{aligned}$$

Thus we conclude that $K\psi\in BC^{0,\alpha+\beta-1}(\Gamma)$. □

Remark 2.9
Let $N\in\mathbb{N}$ be such that $N\alpha\geqslant 1$. Then we can apply Theorem 2.7 repeatedly to see that $K^N: BC(\Gamma)\to BC^{0,\beta}(\Gamma)$, for all $\beta\in(0,1)$. Finally, we now apply Theorem 2.8 to this result, to see that $K^{N+1}: BC(\Gamma)\to BC^{1,\beta}(\Gamma)$, for every $\beta\in(0,\alpha)$.

Theorem 2.10
If $\mu\in BC^{1,\beta}(\Gamma)$ and $\beta\in(0,1)$ then, for the double-layer potential ϕ, defined by (2), it holds that $\phi\in BC^{1,\beta}(\overline{\Omega})$ and $\phi\in BC^{1,\beta}(\Omega_H\setminus\overline{\Omega})$.

Proof
From Theorem 2.3 and Theorem 2.6 we see that $\phi\in BC(\overline{\Omega})$ and $\phi\in BC$ $(\Omega_H\setminus\Omega)$. It remains to show that $\nabla\phi\in BC^{0,\beta}(\overline{\Omega})$ and $\nabla\phi\in BC^{0,\beta}(\Omega_H\setminus\overline{\Omega})$. Let $x^*:=(x_1^*,x_2^*)\in\Gamma$ and define $\chi\in C^\infty(\mathbb{R})$ such that, for $y_1\in\mathbb{R}$,

$$\chi(y_1)=\begin{cases}1, & |y_1|\leqslant\frac{1}{2},\\ 0, & |y_1|>1.\end{cases}$$

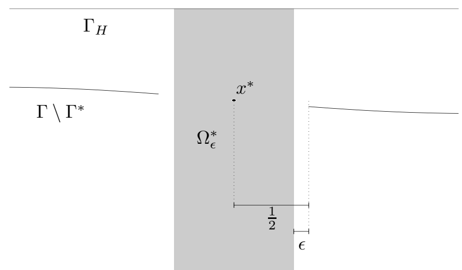


Figure 4: The point x^*, the surface $\Gamma \setminus \Gamma^*$ and the domain Ω_ϵ^*.

Let $\Gamma_\chi := \{y \in \Gamma : |y_1 - x_1^*| \leqslant 1\}$ and $\Gamma^* := \left\{y \in \Gamma : |y_1 - x_1^*| \leqslant \frac{1}{2}\right\} \subset \Gamma_\chi$ and, fixing $0 < \epsilon < \frac{1}{2}$, $\Omega_\epsilon^* := \left\{y \in \Omega_H : |y_1 - x_1^*| < \frac{1}{2} - \epsilon\right\}$, see Figure 4.

Next we split ϕ into three parts,

$$\phi(x) = I_1(x) + I_2(x) + I_3(x), \qquad x \in \Omega_H,$$

where

$$\begin{aligned} I_1(x) &:= \int_\Gamma \frac{\partial \Phi_H(x,y)}{\partial \mathbf{n}(y)} \mu(y)(1 - \chi(x_1^* - y_1)) \mathrm{d}\mathbf{s}(y), \\ I_2(x) &:= \int_{\Gamma_\chi} \frac{\partial \Phi(x,y)}{\partial \mathbf{n}(y)} \mu(y)\chi(x_1^* - y_1) \mathrm{d}\mathbf{s}(y), \\ I_3(x) &:= \int_{\Gamma_\chi} \frac{\partial \Phi(x^r,y)}{\partial \mathbf{n}(y)} \mu(y)\chi(x_1^* - y_1) \mathrm{d}\mathbf{s}(y). \end{aligned}$$

We now examine I_1, I_2 and I_3 separately. We take $C > 0$ to be a constant independent of x^*, but not necessarily equal at each use. By Lemma 2.2, $I_1(x)$ is harmonic in Ω_0^* and, by Theorem 2.6, I_1 is bounded, for $x \in \Omega_H$, by

$$|I_1(x)| \leqslant C_\Gamma \|1 - \chi\|_\infty \|\mu\|_\infty = C \|\mu\|_\infty .$$

Furthermore, by the elliptic regularity results [GT, Theorem 3.9, Lemma 4.1] and as Ω_ϵ^* is at least distance ϵ away from $\Gamma \setminus \Gamma^*$, then $\nabla I_1 \in BC^{0,1}(\Omega_\epsilon^*)$ and

$$\|\nabla I_1\|_{BC^{0,1}(\Omega_\epsilon^*)} \leqslant C \frac{1}{\epsilon} \|\mu\|_\infty .$$

Now, by [G1, Chapter 2, §19, Theorem 3] and as $\mu \in BC^{1,\beta}(\Gamma)$ then $\nabla I_2 \in BC^{0,\beta}(\overline{\Omega})$ and

$$\|\nabla I_2\|_{BC^{0,\beta}(\overline{\Omega})} \leqslant C \|\chi\|_{BC^{1,\beta}(\Gamma)} \|\mu\|_{BC^{1,\beta}(\Gamma)} .$$

Similarly, $\nabla I_2 \in BC^{0,\beta}(\Omega_H \setminus \Omega)$ and $\|\nabla I_2\|_{BC^{0,\beta}(\Omega_H \setminus \Omega)} \leqslant C \|\mu\|_{BC^{1,\beta}(\Gamma)}$. It is clear that $I_3 \in BC^2(\Omega_H)$, *i.e.* I_3 is bounded and continuous with bounded and continuous first and second derivatives, and we have

$$\|\nabla I_3\|_{BC^{0,1}(\Omega_H)} \leqslant \|I_3\|_{BC^2(\Omega_H)} \leqslant C.$$

Thus, there exists $C > 0$ such that $\|\nabla \phi\|_{BC^{0,\beta}(\overline{\Omega \cap \Omega_\epsilon^*})} \leqslant C$, for all $x^* \in \Gamma$. Hence $\nabla \phi \in BC^{0,\beta}(\overline{\Omega})$ and, similarly, $\nabla \phi \in BC^{0,\beta}(\Omega_H \setminus \Omega)$. □

3 The Boundary Integral Equation and Well-posedness

In this section we begin by reformulating the boundary value problem as a boundary integral equation. The results of the previous section, in particular Lemma 2.2, Corollary 2.4 and Theorem 2.6, imply the following theorem.

Theorem 3.1
Suppose $\mu \in BC(\Gamma)$ and ϕ is defined by (2). Then ϕ satisfies the Dirichlet boundary value problem (1) if and only if

$$\phi_0(x) = \int_\Gamma \frac{\partial \Phi_H(x,y)}{\partial \mathbf{n}(y)} \mu(y) \mathrm{d}\mathbf{s}(y) - \frac{1}{2}\mu(x), \qquad x \in \Gamma. \tag{20}$$

We can write (20) in operator notation as

$$(I - K)\mu = -2\phi_0,$$

where I is the identity operator and K is defined by (9).

We next show that the boundary value problem and boundary integral equation are well-posed, this notion dating back to Hadamard [H1] who wrote that a problem is well-posed if a solution exists, is unique and depends continuously on the data.

To prove uniqueness for the boundary value problem, we use the following Phragmén-Lindelöf-type result, an extension of the maximum-minimum principle from bounded to unbounded domains (cf. [F1, §2.5]).

Theorem 3.2
If $u \in BC(\overline{\Omega}) \cap C^2(\Omega)$ is harmonic in Ω then

$$\sup_{x\in\Omega} u(x) \leqslant \sup_{x\in\Gamma} u(x).$$

Proof
We assume, without loss of generality, that $\sup_{x\in\Gamma} u(x) = 0$ and suppose that there exists $x_+ \in \Omega$ such that $u(x_+) > 0$. Choose $x_0 \in \mathbb{R}^2 \setminus \overline{\Omega}$ with $\mathrm{dist}(x_0, \Omega) > 1$ so that $\ln|x - x_0| > 0$ for all $x \in \overline{\Omega}$. Let $a > 1$ be sufficiently large that $|x_+ - x_0| < a$ and define $G_a := \Omega \cap \{x : |x - x_0| < a\}$ (see Figure 5).

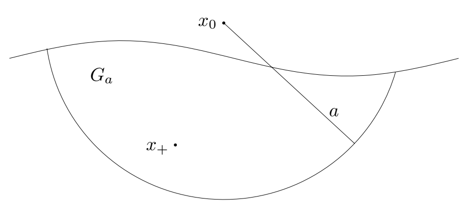


Figure 5: The domain G_a and points x_0, x_+.

Defining $v \in C(\overline{\Omega})$ by

$$v(x) := u(x) - \frac{1}{2}u(x_+)\frac{\ln|x - x_0|}{\ln|x_+ - x_0|},$$

clearly v is harmonic in Ω. Further $v \leqslant 0$ on Γ and, if a is chosen large enough, $v \leqslant 0$ on the boundary of G_a. Thus it follows from the maximum principle for finite domains [K1, Corollary 6.9] that $v(x) \leqslant 0$ for all $x \in G_a$. However $v(x_+) = \frac{1}{2}u(x_+) > 0$ which is a contradiction since $x_+ \in G_a$. $\square$

The above result has the following corollary.

Corollary 3.3
The Dirichlet boundary value problem (1) has at most one solution.

We next prove that the integral operator $(I - K)$ is injective, where K is given by (9). To do this we initially study a mixed homogeneous boundary value problem in the infinite strip domain between Γ_H and Γ.

Theorem 3.4
Suppose that

$$\phi(x) = \int_\Gamma \frac{\partial \Phi_H(x,y)}{\partial \mathbf{n}(y)} \mu(y) \mathrm{d}\mathbf{s}(y), \qquad x \in \Omega_H \setminus \overline{\Omega},$$

where $\mu \in BC^{1,\beta}(\Gamma)$, for some $\beta \in (0,1)$, and that $\frac{\partial \phi}{\partial \mathbf{n}} = 0$ on Γ in the sense that

$$\lim_{h\to 0, h>0} \mathbf{n}(x). \frac{\nabla \phi(x + h\mathbf{n}(x))}{h} = 0, \qquad x \in \Gamma.$$

Then $\phi = 0$ in $\Omega_H \setminus \overline{\Omega}$.

Proof
We define $S := \Omega_H \setminus \overline{\Omega}$, $S_A := \{x \in S : |x_1| < A\}$ and $E_A := \{x \in S : |x_1| = A\}$. Since $\mu \in BC^{1,\beta}(\Gamma)$, it follows from Lemma 2.2 and Theorem 2.10 that $\phi \in BC^{1,\beta}(\overline{S}) \cap C^2(S)$ and that ϕ is harmonic in S. Noting also that $\phi = 0$ on Γ_H and $\frac{\partial \phi}{\partial \mathbf{n}} = 0$ on Γ, applying Green's first theorem we see that

$$\int_{S_A} (\nabla \phi)^2 \mathrm{d}x = \int_{E_A} \phi \frac{\partial \phi}{\partial \mathbf{n}} \mathrm{d}s = O(1)$$

as $A \to \infty$, so that $\nabla \phi \in L^2(S)$. Also, since

$$\int_S (\nabla \phi)^2 \mathrm{d}x \geqslant \int_S \left| \frac{\partial \phi(x)}{\partial x_1} \right|^2 \mathrm{d}x = \int_{-\infty}^{\infty} \left\{ \int_{f(x_1)}^H \left| \frac{\partial \phi(x)}{\partial x_1} \right|^2 \mathrm{d}x_2 \right\} \mathrm{d}x_1$$

there exists a sequence $A_n \to \infty$ such that

$$\int_{E_{A_n}} \left| \frac{\partial \phi}{\partial \mathbf{n}} \right|^2 \mathrm{d}s = \int_{f(A_n)}^H \left| \frac{\partial \phi(x)}{\partial x_1} \right|^2 \mathrm{d}x_2 + \int_{f(-A_n)}^H \left| \frac{\partial \phi(x)}{\partial x_1} \right|^2 \mathrm{d}x_2 \to 0,$$

as $n \to \infty$. Using the same sequence $\{A_n\}$ we have, by the Cauchy-Schwarz inequality,

$$\begin{aligned} \int_S (\nabla \phi)^2 \mathrm{d}x &= \lim_{n\to\infty} \int_{E_{A_n}} \phi \frac{\partial \phi}{\partial \mathbf{n}} \mathrm{d}s \\ &\leqslant \lim_{n\to\infty} \sqrt{\int_{E_{A_n}} |\phi|^2 \, \mathrm{d}s} \sqrt{\int_{E_{A_n}} \left| \frac{\partial \phi}{\partial \mathbf{n}} \right|^2 \mathrm{d}s} = 0. \end{aligned}$$

Hence $\nabla\phi = 0$ in S and, since $\phi \in C^1(\overline{S})$ and $\phi = 0$ on Γ_H, $\phi = 0$ in S as required. $\square$

Now combining the above result with the properties of the double layer potential and the mapping properties of the integral operator K, we prove the injectivity of $(I - K)$.

Theorem 3.5
$(I - K)$ is injective on $BC(\Gamma)$.

Proof
Let $\mu \in BC(\Gamma)$ satisfy $(I - K)\mu = 0$, that is $\mu = K\mu$. By theorems 2.7 and 2.8, $\mu \in BC^{1,\beta}(\Gamma)$ for all $\beta \in (0, \alpha)$.

By Theorem 3.1, ϕ, defined by (2), satisfies the boundary value problem (1) with $\phi_0 = 0$. Using Corollary 3.3 it follows that $\phi = 0$ in Ω so that $\nabla\phi = 0$ in Ω. Hence, by Theorem 2.5, $\frac{\partial\phi}{\partial\mathbf{n}} = 0$ on Γ in the sense of Theorem 3.4. Thus, by Theorem 3.4, $\phi = 0$ in $\Omega_H \setminus \overline{\Omega}$ and so, in the $\phi_\pm$ notation used in Theorem 2.3, $\phi_+ = 0$ on Γ. Finally, by Corollary 2.4, $\mu = \phi_+ - \phi_- = 0$. Therefore $(I - K)\mu = 0 \Rightarrow \mu = 0$ and hence $(I - K)$ is injective. $\square$

To prove the surjectivity of the integral operator $(I - K)$ we use a result from Arens et al. [A2], which applies to integral equations on the real line of the form

$$\lambda\psi(s) - \int_{-\infty}^{\infty} \kappa(s - t)z(s, t)\psi(t)\mathrm{d}t = \phi(s), \qquad s \in \mathbb{R}, \tag{21}$$

where $\lambda \in \mathbb{C}$, the functions κ, z and ϕ are assumed known and ψ is the solution to be determined, in the case that $\kappa \in L^1(\mathbb{R})$ and $z \in BC(\widetilde{\mathbb{R}^2})$ where $\widetilde{\mathbb{R}^2} := \left\{(s, t) \in \mathbb{R}^2 : s \neq t\right\}$.

In general, the integral operator in (21) is non-compact (for example, this is certainly the case if $\kappa \neq 0$ and $z \equiv 1$). In [A1, A2] the invertibility of individual integral operators of the form (21) is deduced by embedding them in a larger family of integral operators, where this family is chosen so that it has certain compactness and translation invariance properties. Let $B := B(f_+, f_-, C_{f'}, C_\alpha)$ denote the set of $f \in BC^{1,\alpha}(\mathbb{R})$ such that $f_- \leqslant f(x_1) \leqslant f_+$, $x_1 \in \mathbb{R}$, and such that (3) holds. We will apply [A2, Theorem 3.18] to a family of operators generated by the set B.

We note that the integral equation (20) is equivalent to the integral equation on the real line

$$\widetilde{\mu}(s) - 2\int_{-\infty}^{\infty} \frac{\partial\Phi_H(x, y)}{\partial\mathbf{n}(y)}\sqrt{1 + f'(t)^2}\,\widetilde{\mu}(t)\mathrm{d}t = -2\widetilde{\phi_0}(s), \qquad s \in \mathbb{R}. \tag{22}$$

In this equation x and y are the points $x = (s, f(s))$, $y = (t, f(t))$ and $\widetilde{\mu}, \widetilde{\phi_0} \in BC(\mathbb{R})$ are defined by $\widetilde{\mu}(s) := \mu((s, f(s)))$ and $\widetilde{\phi_0}(s) := \phi_0((s, f(s)))$, $s \in \mathbb{R}$. So we define the kernel k_f, dependent on $f \in B$, by

$$k_f(s, t) := 2\frac{\partial\Phi_H(x, y)}{\partial\mathbf{n}(y)}\bigg|_{x=(s,f(s)),y=(t,f(t))}\sqrt{1 + f'(t)^2}, \qquad s, t \in \mathbb{R}, s \neq t.$$

Define $\kappa \in L^1(\mathbb{R})$ by

$$\kappa(s) := \begin{cases} |s|^{\alpha-1}, & 0 < |s| \leqslant 1, \\ |s|^{-2}, & 1 < |s|, \end{cases}$$

and let

$$z_f(s,t) := \frac{k_f(s,t)}{\kappa(s-t)}, \qquad s,t \in \mathbb{R}, s \neq t,$$

so $k_f(s,t) = \kappa(s-t)z_f(s,t)$ and the integral equation (22) takes the form (21) with $\lambda = 1$. Further, by Lemma 2.1 and Theorem 2.6, it holds that, for some constant $C > 0$,

$$|k_f(s,t)| \leqslant C\kappa(s-t), \tag{23}$$

for all $s,t \in \mathbb{R}$ with $s \neq t$ and all $f \in B$. Thus $z_f \in BC(\widetilde{\mathbb{R}^2})$ for $f \in B$.

Theorem 3.6
The operator $(I-K)$ is invertible on $BC(\Gamma)$. Further, for some constant C_I, dependent only on $f_\pm$, $C_{f'}$, C_α and H, we have that

$$\left\|(I-K)^{-1}\right\|_{BC(\Gamma)} \leqslant C_I.$$

Proof
To prove this theorem we show that all five conditions of [A2, Theorem 3.18] are satisfied by the family of operators generated by B, namely the set of integral operators with kernel $k_f = \kappa(s-t)z_f(s,t)$, for some $f \in B$. Let $V := \{z_f : f \in B\}$.

By (23) condition 1 holds with $b = 2$. Using [A2, Lemma 4.4] it follows that V is $\tilde{s}$-sequentially compact (as defined in [A2]), so condition 2 is satisfied. It is clear, from the definition of B, that V is closed under the action of the translation operator $T_a : BC(\widetilde{\mathbb{R}^2}) \to BC(\widetilde{\mathbb{R}^2})$, defined by $T_a z = z(\cdot - a, \cdot - a)$, $a \in \mathbb{R}$, $z \in BC(\widetilde{\mathbb{R}^2})$, so condition 3 is satisfied. To show that condition 4 is satisfied we note that the definition of B is the same as in [A2] and apply Corollary 4.5 of [A2]. Theorem 3.5 ensures that condition 5, injectivity of $I - K_f$ on $BC(\Gamma)$, for $f \in B$, is satisfied.

Thus [A2, Theorem 3.18] applies and the result follows. $\square$

Remark 3.7
Let K^* denote the adjoint operator of K, defined by (20) with $\frac{\partial \Phi_H(x,y)}{\partial \mathbf{n}(y)}$ replaced by $\frac{\partial \Phi_H(x,y)}{\partial \mathbf{n}(x)}$. [A2, Theorem 3.18] implies that $I - K^*$ is also invertible as an operator on $BC(\Gamma)$ and that $I-K$ and $I-K^*$ are invertible operators on $L^p(\Gamma)$, $1 \leqslant p \leqslant \infty$. Further, the constant C_I can be chosen, dependent only on $f_\pm, C_{f'}, C_\alpha$ and H, so that

$$\max_{1\leqslant p \leqslant \infty} \left\|(I-K)^{-1}\right\|_{L^p(\Gamma)} = \max_{1\leqslant p \leqslant \infty} \left\|(I-K^*)^{-1}\right\|_{L^p(\Gamma)} \leqslant C_I.$$

We can combine the previous results into a final theorem.

Theorem 3.8
The Dirichlet boundary value problem defined in §1 has exactly one solution. This solution satisfies the bound

$$|\phi(x)| \leqslant \sup_{y\in\Gamma} |\phi_0(y)|, \qquad x \in \overline{\Omega}, \tag{24}$$

that is, the problem is well-posed.

Proof
Theorem 3.6 implies that the integral equation (20), equivalent to (22), has exactly one solution μ. Therefore, by Theorem 3.1, there exists a solution to the boundary value problem and Corollary 3.3 tells us that this is the unique solution. Finally, applying Theorem 3.2 we obtain (24). $\square$

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