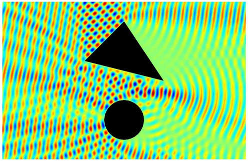
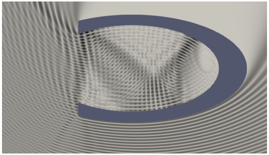
Welcome to ...

At the Interface Between Semiclassical Analysis (SAC) and Numerical Analysis (NA) of Wave Scattering Problems



Simon Chandler-Wilde[1], Monique Dauge[2], Euan Spence[3] & Jared Wunsch[4] (organisers)

[1]University of Reading
[2]Université de Rennes
[3]University of Bath
[4]Northwestern University



Mathematisches Forschungsinstitut Oberwolfach, September 2022

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- Housekeeping: lunch, tables/napkins, MFO abstract book, MFO Friends

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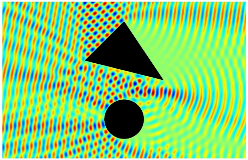
Our aim today: to give some overview of the SCA/NA interface (and existing successes) from the NA side, then the SCA side this pm.

NA at the SCA/NA interface: issues and case studies



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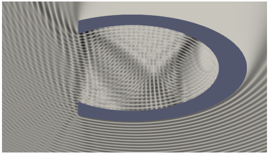

At the Interface Between Semiclassical Analysis (SAC) and Numerical Analysis (NA) of Wave Scattering Problems, MFO, September 2022

At the Interface between Semiclassical Analysis and Numerical Analysis of Wave Scattering Problems: Schedule of Talks

Monday

0900-0950 Chandler-Wilde (First "introductory" numerical-analysis talk)

0950-1000 Questions/discussion.

1000-1050 Moiola (Second "introductory" numerical-analysis talk)

1050-1100 Questions/discussion.

1100-1130 Coffee

1130-1220 Gander (Third "introductory" numerical-analysis talk)

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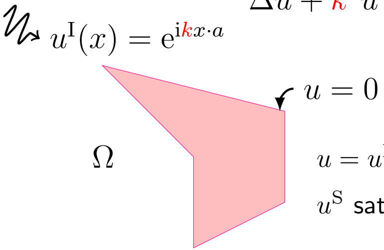
- The NA side of the NA/SCA interface - bringing SCA colleagues up to speed with our (NA) methods and our issues (that likely need SCA input)
- Opportunities at the NA/SCA interface (case studies) - why we are here!

Model problem (to get across main ideas)

$$\Delta u + k^2 u = 0 \qquad k \gg 1$$

$u^{\mathrm{I}}(x) = \mathrm{e}^{\mathrm{i}kx\cdot a}$



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u^{S} satisfies Sommerfeld radiation condition

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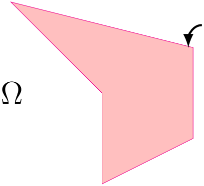

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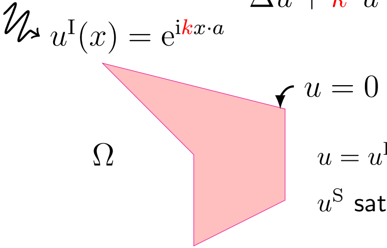

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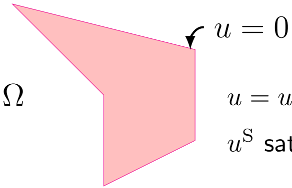

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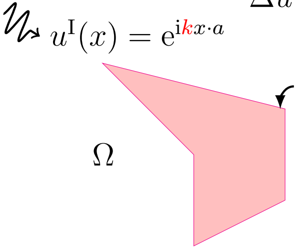

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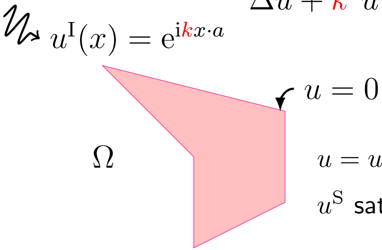

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Geometry: SCA most comfortable with C^∞ (but large body of work on diffraction by corners/edges, including **Baskin** & **Wunsch** 2013)
NA likes C^∞ too, but also keen on piecewise smooth/analytic + rougher boundaries

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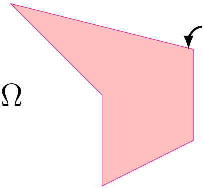

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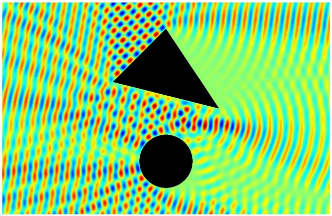
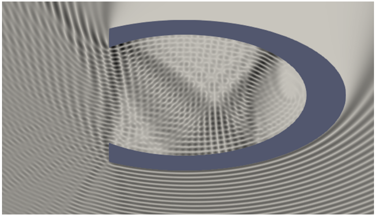
Example solutions:





The **Galerkin Method** (the focus of this talk)

This starts from a **variational formulation** of the above problem as:

find $v \in \mathcal{H}$ (some Hilbert space) such that

$$a(v, w) = F(w) \quad \forall w \in \mathcal{H},$$

where $a(\cdot, \cdot)$ and $F(\cdot)$ are some specified sesquilinear form and anti-linear functional on $\mathcal{H}$.

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(See next talk (**Moiola**) for cases where $\mathcal{H}_N \subset \mathcal{H}$ **violated**.)

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Goal: SCA to help with all three choices!

Variational formulation in domain: starting point of FEM

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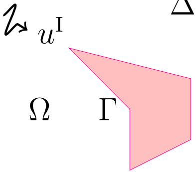

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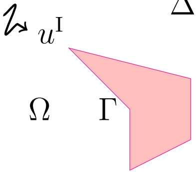

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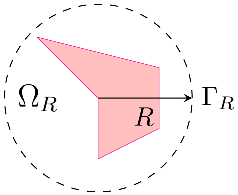

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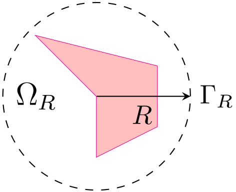

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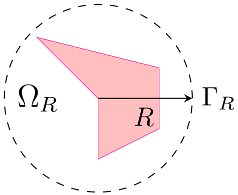
F(w) &:= \int_{\Gamma_R} \left(\partial_n u^{\mathrm{I}} - \mathrm{DtN}_k(\gamma u^{\mathrm{I}})\right) \gamma \bar{w} \, \mathrm{d}s, \quad v, w \in \mathcal{H},
\end{aligned}$$

Variational formulation in domain: starting point of FEM



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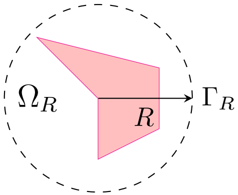

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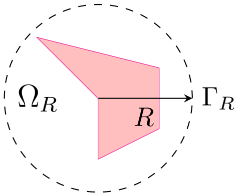

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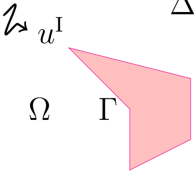
ii) **PML** $=$ complex scaling in SCA $+$ truncating domain with $u = 0$ b.c.

Variational form. on boundary: starting point of BEM

$$\Delta u + k^2 u = 0 \qquad k \gg 1$$



$u = 0$

$u = u^{\mathrm{I}} + u^{\mathrm{S}}$

u^{S} satisfies SRC

Green's representation theorem:

$$u^{\mathrm{S}}(x) = \int_\Gamma \left(\Phi(x,y)\partial_n u^{\mathrm{S}}(y) - \frac{\partial \Phi(x,y)}{\partial n(y)} u^{\mathrm{S}}(x) \right) \mathrm{d}s(y), \quad x \in \Omega,$$

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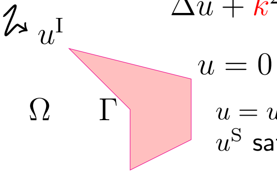

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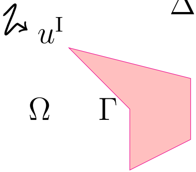

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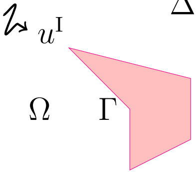

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$$Av = f \Rightarrow \int_\Gamma Av\bar{w}\,\mathrm{d}s = \int_\Gamma f\bar{w}\,\mathrm{d}s, \quad w \in \mathcal{H}, \quad \text{i.e.}$$

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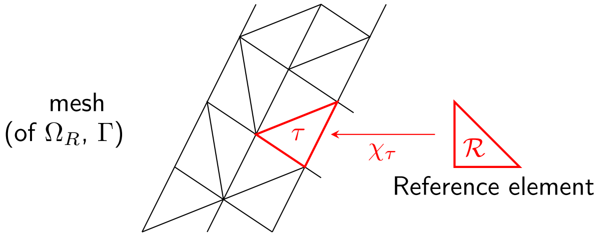
Lots of choices/options, design of new BIEs an active area which can be/is informed by SCA, e.g., **Epstein** talk & work of **Darbas**.

Piecewise polynomial spaces $\mathcal{H}_N$ for FEM/BEM



On our domain we construct a **mesh**, a collection of **elements** τ, each the image of a fixed **reference element** under a diffeomorphism $\chi_\tau : \mathcal{R} \to \tau$.

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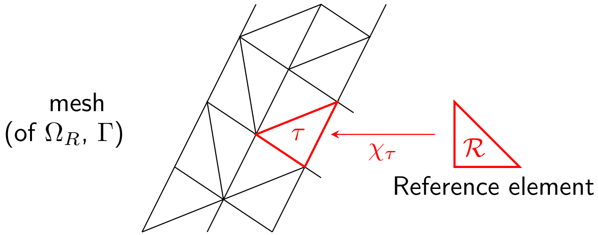

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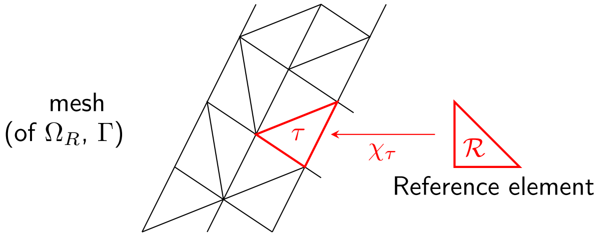

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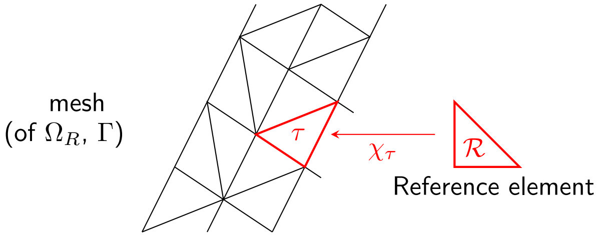

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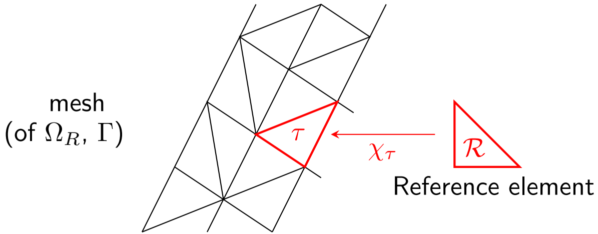

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- $h := \max \mathrm{diam}(\tau) \to 0$ (h-FEM/BEM); or
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as function of Ω, k, h, and p – NA and SCA input valuable and complementary.

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as function of Ω, k, h, and p – NA and SCA input valuable and complementary. By Whittaker-Shannon-Nyquist criterion we expect

$$\dim(\mathcal{H}_N) \sim k^{\text{dimension of domain}}$$

necessary and sufficient to maintain accuracy as $k \to \infty$.

Comparison of FEM/BEM

FEM	BEM
Computational domain (Ω_R) is d-dimensional	Computational domain ($\Gamma = \partial\Omega$) is $d-1$-dimensional
$\dim(\mathcal{H}_N) \sim k^d$ needed for small best approx. error	$\dim(\mathcal{H}_N) \sim k^{d-1}$ needed for small best approx. error
Sparse linear system (see **Gander** talk)	Dense linear system (see, e.g., **Chaillat-Loseille** talk)
Works for variable coefficients, e.g., $\nabla \cdot (A(x)\nabla u) + k^2 n(x) u = 0$	Needs constant coefficients

Numerical analysis of the Galerkin method

Goal: prove **quasi-optimality**, that the **Galerkin solution** $v_N \in \mathcal{H}_N$ satisfies

$$\underbrace{\|v - v_N\|_{\mathcal{H}}}_{\text{Error in } v_N} \le C_{\mathrm{qo}} \underbrace{\min_{w_N \in \mathcal{H}_N} \|v - w_N\|_{\mathcal{H}}}_{\text{Best approximation error}} \qquad (*)$$

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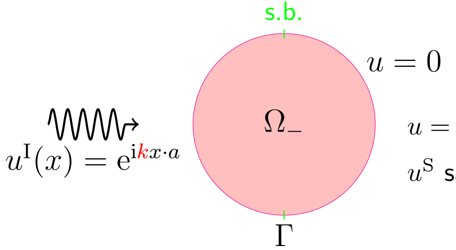
Compact perturbation arguments show $\exists N_0$ s.t. $(*)$ holds for $N \ge N_0$ but need k-explicit (e.g., SCA) info about Helmholtz solution to quantify how N_0 and $C_{\rm qo}$ depend on k (and Ω, h, p).

SCA applied in NA: Ex. 1. Hybrid NA-asymptotic methods

$$\Delta u + k^2 u = 0 \qquad\qquad k \gg 1$$



$u = u^{\mathrm{I}} + u^{\mathrm{S}}$

u^{S} satisfies Sommerfeld rad. cond.

Melrose & Taylor 1985, through SCA, study $k \to \infty$ asymptotics of

$$\eta^{\mathrm{slow}}(x) = \frac{\partial_n u(x)}{\mathrm{e}^{\mathrm{i}kx\cdot a}}, \quad x \in \Gamma,$$

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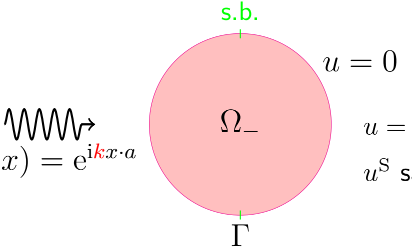

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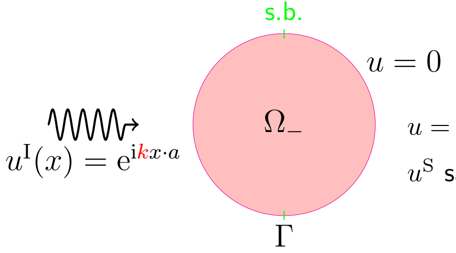

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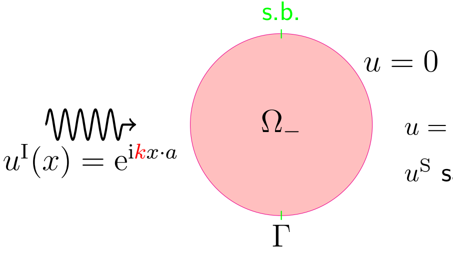

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Open problems: extension of theory/algorithms to 3D, non-convex scatterers

SCA applied in NA: Ex. 2. "Pollution" in FEM/BEM

Q. In **quasi-optimality** estimate

$$\underbrace{\|v - v_N\|_{\mathcal{H}}}_{\text{Error in } v_N} \leq C_{\rm qo} \underbrace{\min_{w_N \in \mathcal{H}_N} \|v - w_N\|_{\mathcal{H}},}_{\text{Best approximation error}} \quad N \geq N_0, (*)$$

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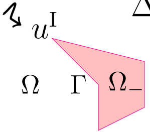
Open problems: Elasticity, Maxwell (though see **Melenk**, **Sauter** 2022)? For BEM case, other b.c.'s, Ω trapping?

SCA applied in NA: Ex. 3. k-dependence of BIOs

$$\Delta u + k^2 u = 0 \qquad\qquad k \gg 1$$



$u = 0$

Try as ansatz

$$u^{\mathrm{S}}(x) = \int_\Gamma \left(\frac{\partial \Phi(x,y)}{\partial n(y)} - \mathrm{i}\eta \Phi(x,y) \right) \varphi(y)\, \mathrm{d}s(y), \quad x \in \Omega. \quad (*)$$

Imposing b.c. $u^{\mathrm{S}} = -u^{\mathrm{I}}$ on Γ gives a BIE $A\varphi = f := -\gamma u^{\mathrm{I}}$.

Q. How do $\|A\|$, $\|A^{-1}\|$, $\operatorname{cond} A := \|A\|\,\|A^{-1}\|$ depend on k, η, Ω?

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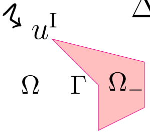

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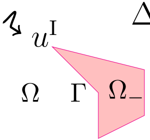

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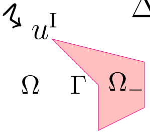

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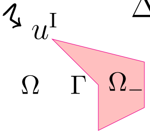

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Open problems: rigorous extensions to other b.c.'s, elastic waves, where η is an (SCA-inspired) operator – see, e.g., **Darbas**, **Chaillat**, Le Louër 2021

Conclusions

Coming from the NA side it is clear that:

- There are many problems in NA of wave problems that SCA can help with
- There are successes at the SCA/NA interface already, but many open problems