Chapter 1

A Brakhage-Werner-Type Integral Equation Formulation of a Rough Surface Scattering Problem

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Abstract. *We consider the problem of scattering of time-harmonic acoustic waves by an unbounded sound-soft rough surface. Although integral equation methods are widely used for the numerical solution of such problems, to date there is no formulation which is known to be uniquely solvable in the 3D case. We consider a novel Brakhage-Werner type integral equation formulation of this problem, based on an ansatz as a combined single- and double-layer potential, but replacing the usual fundamental solution of the Helmholtz equation with an appropriate half-space Green's function. In the case when the surface Γ is sufficiently smooth (Lyapunov), we sketch how it can be shown that the integral operators are bounded as operators on $L^2(\Gamma)$ and, moreover, how it can be shown that the integral equation is uniquely solvable in the space $L^2(\Gamma)$. The proof of this latter result uses novel, direct arguments, leading to explicit bounds on the inverse in terms of the wave number, the parameter coupling the single- and double-layer potentials, and the maximum surface slope. These bounds show that the norm of the inverse operator is bounded uniformly in the wave number if the coupling parameter is chosen proportional to the wave number.*
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This paper is concerned with boundary integral equation methods for what are termed *rough surface scattering problems* in the engineering literature. We use the phrase *rough surface* to denote a surface which is a perturbation of an infinite plane surface such that the whole surface lies within a finite distance of the original plane. In particular, we are concerned with the case where the scattering surface Γ is the graph of some bounded continuous function $f : \mathbb{R}^2 \to \mathbb{R}$, i.e.

$$\Gamma := \big\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = f(x_1, x_2)\big\}. \tag{1.1}$$

We will consider a typical problem of this type, namely acoustic scattering by a rough, sound soft surface, the acoustic medium of propagation occupying the perturbed half-space

$$D := \{x = (x_1, x_2, x_3) : x_3 > f(x_1, x_2)\} \tag{1.2}$$

above the scattering surface Γ. We will suppose throughout that f is a moderately smooth function, i.e. is continuously differentiable with Hölder continuous first derivative (Γ is Lyapunov).

Rough surface scattering problems arise frequently in applications. They model acoustic and electromagnetic wave propagation over outdoor ground and sea surfaces and, at a very different scale, problems of optical scattering by surfaces in nano-technology. The extensive literature on the mathematical and computational modelling of these problems has been reviwed recently by Saillard & Sentenac [15], Warnick & Chew [16], and DeSanto [10], these reviews making it clear that boundary integral equation methods are very popular, with many effective, specialised numerical algorithms developed.

Despite this interest in the application of the BIE method, the associated mathematical and numerical analysis to support these practical computations is largely absent, in the more important 3D case at least. For example, to date there is no integral equation formulation that is known to be uniquely solvable for a general 3D rough surface scattering problem.

For the 2D rough surface scattering problem much progress has been made in the last ten years in terms of deriving well-posed boundary integral equations for a variety of acoustic, electromagnetic, and elastic wave problems (see e.g. [4, 17, 2]). For general rough surface scattering problems the usual integral equation formulations for scattering by bounded surfaces are unattractive from a theoretical point of view since the standard boundary integral operators (e.g. the standard single- and double-layer potential operators) are not bounded operators on any of the usual function spaces when the scattering surface is unbounded. This has important practical consequences, in particular implying large condition numbers when the standard integral equations are discretised on large sections of rough surfaces.

In the 2D case alternative integral equations, with bounded integral operators, have been obtained by replacing the standard fundamental solution by the Dirichlet or impedance Green's function for a half-plane that contains the domain D of propagation (see e.g. [4, 17]). This modification leads to kernels of boundary integral operators that are weakly singular in their asymptotic behaviour at infinity so that the integral operators are bounded on $BC(\Gamma)$, the space of bounded continuous functions on Γ, and on the other usual function spaces. In the case of a 2D sound-soft rough surface, [17] follows the approach that was proposed much earlier for the sound-soft bounded obstacle, for example by Brakhage and Werner [3]. This approach, to seek the solution to the exterior Dirichlet problem as a linear combination of double and single-layer potentials, we will term the Brakhage-Werner method. It is used in [17], with the twist that the standard fundamental solution is replaced by the Dirichlet Green's function for a half-plane.

In this paper we will discuss the analogous modification in the 3D case, summarising recent work in [5, 6]. Following [17], we derive a Brakhage-Werner-type integral equation, replacing the standard fundamental solution with the Dirichlet Green's function for a half-space that contains D. The complication in the 3D case is that this modification, while it improves the kernels of the integral operators significantly in terms of their behaviour at infinity, leads to kernels of the integral operators that are strongly singular rather than weakly singular as in the 2D case, even when the boundary is smooth. As a consequence, the boundary integral operators are no longer well-defined as operators on $BC(\Gamma)$. We are, by careful calculations, including computations of the Fourier transform of parts of the kernel, able to show the boundedness of the operators on $L^2(\Gamma)$.

To establish existence of solution and well-posedness in the 2D case generalisations of part of the Riesz theory of compact operators have been developed (e.g. [8]) which require only local compactness rather than compactness. These results enable existence of solution in $BC(\Gamma)$ to be deduced from uniqueness of solution. In fact, injectivity of the second kind BIE in $BC(\Gamma)$ implies well-posedness in $BC(\Gamma)$ and in the space $L^p(\Gamma)$, $1 \leq p \leq \infty$ (see e.g. Preston et al. [14] in this proceedings). But this theory is not relevant for 3D rough surface scattering problems as the boundary integral operators are not even well-defined as operators on $BC(\Gamma)$.

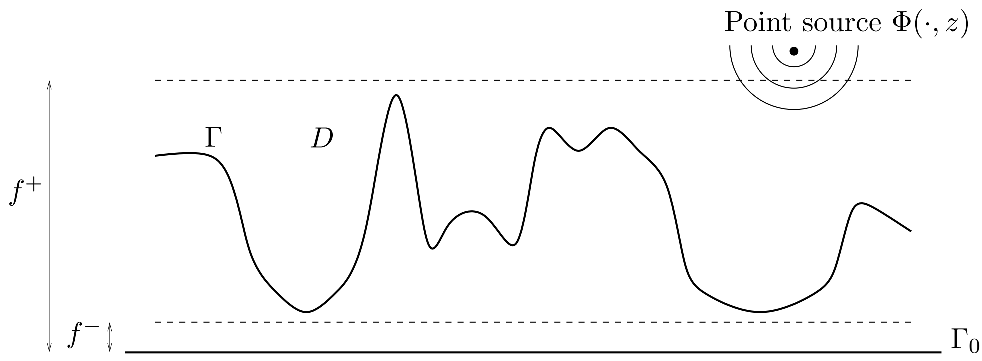

Figure 1.1: Geometrical setting of the scattering problem

In the next section we give a rigorous formulation of the scattering problem, and derive the Brakhage-Werner type integral equation. This integral equation, in operator form, is

$$(I + K - i\eta S)\varphi = 2g, \tag{1.3}$$

where I is the identity operator, S and K are single- and double-layer potential operators, defined by (1.12) and (1.13) below, g is the Dirichlet data for the scattered field on Γ, and $\eta > 0$ is the coupling parameter. We sketch how it can be established that S and K are bounded operators on $L^2(\Gamma)$; for further details see [5].

Next we indicate how one can show that

$$A := I + K - i\eta S \tag{1.4}$$

is invertible as an operator on $L^2(\Gamma)$, obtaining the explicit bound that

$$||A^{-1}||_{L^2(\Gamma)\to L^2(\Gamma)} < B(L, \kappa/\eta) := 2 + 2L + 4L^2 + \frac{\kappa}{\eta}\left(2 + 5L + 3L^{3/2}\right), \tag{1.5}$$

where $L := \sup_{\mathbf{x}\in\mathbb{R}^2} |\nabla f(\mathbf{x})|$ is the maximum surface slope. In fact, the tools needed to obtain this explicit bound are rather elementary, namely standard jump relations for layer-potentials (e.g. [9]) combined with carefully chosen applications of the divergence theorem. Our techniques of argument are inspired by the somewhat similar methods used to prove invertibility for second kind boundary integral equations for potential problems in Lipschitz domains (e.g. [13]), and by arguments used to obtain a priori bounds for variational formulations of rough surface scattering problems [11, 7]. In particular, a similarly explicit lower bound for the inf-sup (LBB) constant of a variational formulation of this rough surface scattering problem is shown in [7].

The bound (1.5) is attractive in its explicitness. One consequence of (1.5) is that if η is chosen proportional to the wave number κ, as recommended for larger frequencies in the bounded obstacle case in [12, 1], then the inverse operator is bounded by an explicit function of the Lipschitz constant of f and, in particular, is independent of the wave number, for $\kappa > 0$.

Notation. Throughout the paper x and y denote points in $\mathbb{R}^3$ with components $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The image of $y \in \mathbb{R}^3$ in the plane $\Gamma_0 := \{x \in \mathbb{R}^3 : x_3 = 0\}$ is denoted by $y' := (y_1, y_2, -y_3)$. By $\mathbf{x}$ we will denote $(x_1, x_2) \in \mathbb{R}^2$, so that $x = (\mathbf{x}, x_3)$. Similarly $\mathbf{y}$ denotes (y_1, y_2). $BC(\Gamma)$ will denote the set of bounded continuous real- or complex-valued functions on Γ, a Banach space with the norm $||\cdot||_{BC(\Gamma)}$ defined by $||F||_{BC(\Gamma)} = \sup_{x\in\Gamma} |F(x)|$. For $0 < \alpha \leq 1$ let $BC^{1,\alpha}(\mathbb{R}^2)$ denote the set of those bounded continuously differentiable functions $F : \mathbb{R}^2 \to \mathbb{R}$ that have the property that ∇F is bounded and uniformly Hölder continuous with index α. It is convenient also to have a shorthand for the intersection of the sets $L^2(\Gamma)$ and $BC(\Gamma)$, so we define

$$X := L^2(\Gamma) \cap BC(\Gamma).$$

Since $L^2(\Gamma)$ and $BC(\Gamma)$ are Banach spaces equipped with their respective norms, so also is X, equipped with the norm $||\cdot||_X$ defined by $||F||_X := \max(||F||_{L^2(\Gamma)}, ||F||_{BC(\Gamma)})$.

1.1 The rough surface scattering problem

Time-harmonic ($e^{-i\omega t}$ time dependence) acoustic waves are modelled by the Helmholtz equation

$$\triangle u + \kappa^2 u \;=\; 0. \tag{1.6}$$

In this equation $\kappa = \omega/c > 0$, where c is the speed of sound, is the *wave number*. We consider acoustic wave motion in the *domain of propagation* D defined by (1.2), throughout assuming that $f \in BC^{1,\alpha}(\mathbb{R}^2)$, for some $\alpha \in (0,1]$, and that f is a strictly positive function, so that there exist constants $f^+ > f^- > 0$ with $f^- \leq f(\mathbf{x}) \leq f^+$, $\mathbf{x} \in \mathbb{R}^2$. We denote the boundary of D by Γ, so that Γ is given by (1.1). We use the notation Γ_h, for $h \in \mathbb{R}$, to denote the plane

$$\Gamma_h := \{x = (x_1, x_2, x_3) : x_3 = h\}.$$

By U_h we denote the half-space above Γ_h and by S_h the part of D below Γ_h, so that

$$U_h := \{x : x_3 > h\}, \quad S_h := D \setminus \bar{U}_h.$$

We consider the scattering of a wave u^i incident on the surface Γ. We assume that the total field $u := u^i + u^s$, which is the sum of the incident field and the scattered field u^s, satisfies on Γ the Dirichlet boundary condition

$$u(x) \;=\; 0, \quad x \in \Gamma. \tag{1.7}$$

We require that the scattered field is bounded in D. We also require that u satisfies the following *limiting absorption principle*: denoting u temporarily by $u^{(\kappa)}$ to indicate its dependence on κ, we suppose that for all sufficiently small $\epsilon > 0$ a solution $u^{(\kappa+i\epsilon)}$ exists which satisfies (1.6) and (1.7) (with κ replaced by $\kappa + i\epsilon$) and that, for all $x \in D$,

$$u^{(\kappa+i\epsilon)}(x) \to u^{(\kappa)}(x), \quad \epsilon \to 0. \tag{1.8}$$

The limiting absorption principle plays the role of a radiation condition, singling out the correct physical solution.

Let

$$\Phi(x,y) \;:=\; \frac{1}{4\pi}\frac{e^{i\kappa|x-y|}}{|x-y|}, \quad x, y \in \mathbb{R}^3,\; x \neq y, \tag{1.9}$$

denote the standard fundamental solution of the Helmholtz equation. In order to get kernels of our boundary integral operators which have faster decay at infinity we will, following [5], replace $\Phi(x,y)$ by an appropriate half-space Green's function for the Helmholtz equation. Specifically, we will work with the function

$$G(x,y) \;:=\; \Phi(x,y) - \Phi(x,y'), \tag{1.10}$$

with $y' = (y_1, y_2, -y_3)$, which is the Dirichlet Green's function for the half space $\{x : x_3 > 0\}$. The faster decay of $G(x,y)$ compared to $\Phi(x,y)$, as $|x|, |y| \to \infty$ with $x, y \in \Gamma$, is captured in the bound [5, equation (3.8)] that, for some constant $C > 0$,

$$|G(x,y)| \leq \frac{C(1+x_3)(1+y_3)}{|x-y|^2}, \tag{1.11}$$

for all $x, y \in \mathbb{R}^3$ with $x \neq y$ and $x_3, y_3 \geq 0$.

Thus we will use layer potentials with $\Phi(x,y)$ replaced by $G(x,y)$, so that we define the *single-layer potential operator* by

$$(S\varphi)(x) \;:=\; 2\int_\Gamma G(x,y)\varphi(y)\,ds(y), \quad x \in \Gamma, \tag{1.12}$$

and the *double-layer potential operator* by

$$(K\varphi)(x) \;:=\; 2\int_\Gamma \frac{\partial G(x,y)}{\partial \nu(y)}\varphi(y)\,ds(y), \quad x \in \Gamma, \tag{1.13}$$

where the normal $\nu(y)$ is directed into D.

We will concentrate on the case when the incident field is that due to a point source located at some point $z \in D$, i.e. $u^i = \Phi(\cdot, z)$. Thus the following is the problem we consider:

PROBLEM 1 (**Point source rough surface scattering problem**). *Let $u^i = \Phi(\cdot, z)$ be the incident field due to a point source at $z \in D$. Then we seek a scattered field $u^s \in C^2(D) \cap BC(\bar{D})$ such that u^s is a solution to the Helmholtz equation (1.6) in D, the total field satisfies the sound-soft boundary condition (1.7), and the limiting absorption principle (1.8) holds.*

To convert this scattering problem to a boundary value problem we seek the scattered field as the sum of the solution in the case when Γ is the flat plane Γ_0, namely $-\Phi(\cdot, z')$, where z' is the image of z in Γ_0, plus some unknown remainder v. Since u vanishes on Γ we have that

$$v(x) = -\{\Phi(x,z) - \Phi(x,z')\} = -G(x,z) =: g(x), \quad x \in \Gamma. \tag{1.14}$$

Clearly $g \in BC(\Gamma)$ and it follows from (1.11) that $g \in L^2(\Gamma)$, so that $g \in X = L^2(\Gamma) \cap BC(\Gamma)$. Further, by the dominated convergence theorem we see that $||g_\epsilon - g||_{L^2(\Gamma)} \to 0$ as $\epsilon \to 0^+$, where g_ϵ is $-G(\cdot, z)$ with κ replaced by $\kappa + i\epsilon$. Thus u^s satisfies the above scattering problem if and only if v satisfies the following Dirichlet problem, with g given by (1.14) and g_ϵ defined as $-G(\cdot, z)$ with κ replaced by $\kappa + i\epsilon$.

PROBLEM 2 (**BVP**). *Given $g \in X$ and $g_\epsilon \in X$, for $\epsilon > 0$, with $||g_\epsilon - g||_{L^2(\Gamma)} \to 0$ as $\epsilon \to 0$, find $v \in C^2(D) \cap BC(\bar{D})$ which satisfies the Helmholtz equation (1.6) in D, the Dirichlet boundary condition $v = g$ on Γ, and the following limiting absorption principle: that, for all sufficiently small $\epsilon > 0$, there exists $v_\epsilon \in C^2(D) \cap BC(\bar{D})$ satisfying $v_\epsilon = g_\epsilon$ on Γ, and (1.6), with κ replaced by $\kappa + i\epsilon$, such that, for all $x \in D$, $v_\epsilon(x) \to v(x)$ as $\epsilon \to 0$.*

In this paper we will, following [5], look for a solution to this boundary value problem as the *combined single- and double-layer potential*

$$v(x) \;:=\; u_2(x) - i\eta\, u_1(x), \quad x \in D, \tag{1.15}$$

with some parameter $\eta > 0$, where for a given function $\varphi \in X$ we define the *single-layer potential*

$$u_1(x) \;:=\; \int_\Gamma G(x,y)\varphi(y)\,ds(y), \quad x \in \mathbb{R}^3, \tag{1.16}$$

and the *double-layer potential*

$$u_2(x) \;:=\; \int_\Gamma \frac{\partial G(x,y)}{\partial \nu(y)}\varphi(y)\,ds(y), \quad x \in \mathbb{R}^3. \tag{1.17}$$

Seeking the solution in this form it is shown in [5], as a consequence of jump relations for the layer-potentials, that, for $g \in X$, the boundary condition $v = g$ on Γ is satisfied if and only if the boundary integral equation

$$A\varphi = 2g \tag{1.18}$$

holds on Γ, where A is the operator defined by (1.4).

A main result of [5], crucial to the arguments that we will make in the next section, is:

THEOREM 1. *The single- and double-layer potential operators S and K, defined by (1.12) and (1.13), are bounded operators on $L^2(\Gamma)$ and on X.*

The technique of argument used is to show this result is as follows. Where $a(x,y)$ denotes the kernel of S or K, via Taylor expansions with respect to x_3 and y_3 we can show that, for some (small) integer N,

$$a(x,y) = \tilde{a}(x,y) + \sum_{m=1}^{N} b_m(\mathbf{x})\ell_m(\mathbf{x}-\mathbf{y})c_m(\mathbf{y}),$$

where $b_m, c_m \in BC(\mathbb{R}^2)$, $\ell_m \in L^2(\mathbb{R}^2) \cap C(\mathbb{R}^2)$ and the Fourier transform of ℓ_m is bounded (this is established via explicit computations). The remaining part of $a(x,y)$ after the finite sum is subtracted, namely $\tilde{a}(x,y)$, is relatively well-behaved, precisely $\tilde{a}(x,y)$ is continuous for $x \neq y$ and

$$|\tilde{a}(x,y)| \leq \ell(\mathbf{x}-\mathbf{y}),$$

for some $\ell \in L^1(\mathbb{R}^2)$. These properties guarantee the boundedness of the integral operator with kernel a on the space $L^2(\Gamma)$.

In [5] the following result is also shown, establishing that invertibility of A on X and existence of solution to the boundary value problem and scattering problem follow once we show that A is invertible on $L^2(\Gamma)$.

THEOREM 2. *If A is invertible as an operator on $L^2(\Gamma)$, then A is invertible as an operator on X. Moreover, if A is invertible, then the boundary value problem has exactly one solution v, defined by (1.15)-(1.17) with $\varphi \in X$ given by $\varphi = 2A^{-1}g$. Further, for some constant $c > 0$, independent of g,*

$$|v(x)| \leq c\, ||g||_X, \quad x \in \bar{D}.$$

1.2 Invertibility of A

In this section of the paper we indicate how one establishes that A is invertible as an operator on $L^2(\Gamma)$, with the explicit bound (1.5) on A^{-1}.

We work with the operators S and K and with their adjoints. The adjoint of K is the operator K' defined by

$$(K'\varphi)(x) \;:=\; 2\int_\Gamma \frac{\partial G(x,y)}{\partial \nu(x)}\varphi(y)\, ds(y), \quad x \in \Gamma. \tag{1.19}$$

Arguing in the same way as for K we see that K' is a bounded operator on $L^2(\Gamma)$. The adjoint of A is $A' = I + K' - i\eta S$. From standard properties of adjoint operators on Hilbert spaces we have that A and A' have the same norm, that A is invertible if and only if A' is invertible, and that if they are both invertible then

$$||A^{-1}||_{L^2(\Gamma)\to L^2(\Gamma)} = ||A'^{-1}||_{L^2(\Gamma)\to L^2(\Gamma)}. \tag{1.20}$$

Thus, we can proceed by bounding A'^{-1}. The following is the main step in establishing that A'^{-1} is bounded.

LEMMA 3. *Suppose that, in addition to our assumptions throughout on f, it holds that $f \in C^\infty(\Gamma)$. Then, for all $\varphi \in L^2(\Gamma)$ there holds*

$$||A'\varphi||_{L^2(\Gamma)} \geq B^{-1}||\varphi||_{L^2(\Gamma)}, \tag{1.21}$$

where $B = B(L, \kappa/\eta)$ is defined by (1.5).

This result is shown as follows. A simplifying observation is that, since A' is bounded on $L^2(\Gamma)$, it is enough to show (1.21) holds for all $\varphi \in Y$, where Y is a dense subset of $L^2(\Gamma)$, the set of those φ that are Hölder continuous and compactly supported. Then one studies the behaviour, for $\varphi \in Y$, of the single-layer potential u defined by

$$u(x) := \int_\Gamma G(x,y)\varphi(y)ds(y) = \int_{\tilde{\Gamma}} \Phi(x,y)\varphi(y)ds(y) - \int_{\tilde{\Gamma}'} \Phi(x,y)\varphi(y)ds(y), \quad x \in \mathbb{R}^3,$$

where $\tilde{\Gamma} \subset \Gamma$ denotes the bounded support of φ and $\tilde{\Gamma}' := \{y' : y \in \tilde{\Gamma}\}$ denotes the image of $\tilde{\Gamma}$ in Γ_0. From standard properties of the single-layer potential (e.g. [9]) we have that $u \in C(\mathbb{R}^3)$ and that u satisfies the Helmholtz equation in $\mathbb{R}^3 \setminus (\tilde{\Gamma} \cup \tilde{\Gamma}')$. Further, it follows from the bound (1.11) that

$$u(x) = O(|x|^{-2}), \quad \nabla u(x) = O(|x|^{-2}), \tag{1.22}$$

as $|x| \to \infty$ with $x \in \bar{U}_0$ and $x_3 = O(1)$. Moreover, where $M = \{x : 0 < x_3 < f(\mathbf{x})\}$ denotes the region between Γ and Γ_0, ∇u can be continuously extended from D to $\bar{D}$ and from M to $\bar{M}$, with limiting values on Γ given by

$$\nabla u_\pm(x) = \int_{\tilde{\Gamma}} \nabla_x G(x,y)\varphi(y)ds(y) \mp \frac{1}{2}\varphi(x)\nu(x), \quad x \in \Gamma, \tag{1.23}$$

where $\nu(x)$ is the unit normal vector at x, directed into D, and $\nabla u_\pm(x) := \lim_{\epsilon\to 0+} \nabla u(x \pm \epsilon\nu(x))$.

From (1.23), $\nabla_T u$, the tangential part of ∇u, is continuous across Γ. On the other hand, the normal derivative jumps across Γ, with

$$\frac{\partial u_\pm}{\partial \nu}(x) = \frac{1}{2}\left[(K'\varphi)(x) \mp \varphi(x)\right], \quad x \in \Gamma. \tag{1.24}$$

Since also $u = \frac{1}{2}S\varphi$ on Γ, defining

$$g := \frac{1}{2}A'\varphi = \frac{1}{2}(I + K' - i\eta S)\varphi,$$

we see that

$$\frac{\partial u_-}{\partial \nu}(x) - i\eta u(x) = g(x), \quad x \in \Gamma. \tag{1.25}$$

Further, from (1.24) we see that $\frac{\partial u_-}{\partial \nu} - \frac{\partial u_+}{\partial \nu} = \varphi$ on Γ. To complete the proof we have to show that

$$||\varphi||_{L^2(\Gamma)} \le 2B||g||_{L^2(\Gamma)}.$$

This can be achieved by bounding the normal derivatives of u on Γ via applications of the divergence theorem in M and D.

A first (and standard) application of the divergence theorem in M gives that, since $u = 0$ on Γ_0,

$$\operatorname{Im} \int_\Gamma \bar{u}\frac{\partial u_-}{\partial \nu} ds = 0. \tag{1.26}$$

Using (1.25) to replace $\partial u_-/\partial\nu$ in the above equation, and applying Cauchy-Schwarz, we see that

$$\eta||u||^2_{L^2(\Gamma)} = -\operatorname{Im}\int_\Gamma \bar{u} g ds \le ||u||^2_{L^2(\Gamma)}||g||^2_{L^2(\Gamma)},$$

so that

$$||u||_{L^2(\Gamma)} \le \eta^{-1}||g||_{L^2(\Gamma)}. \tag{1.27}$$

Alternatively, using (1.25) to replace u in (1.26) and using Cauchy-Schwarz, we see that

$$\left|\left|\frac{\partial u_-}{\partial \nu}\right|\right|_{L^2(\Gamma)} \leq ||g||_{L^2(\Gamma)}. \tag{1.28}$$

It remains to bound the L^2 norm of $\partial u_+/\partial \nu$ in terms of $||g||_{L^2(\Gamma)}$. We first make a second application of the divergence theorem in M. We have

$$0 = 2\mathrm{Re}\int_M (\Delta u + \kappa^2 u)\frac{\partial \bar{u}}{\partial x_3} dx = \int_M \nabla\cdot\left[e_3(\kappa^2|u|^2 - |\nabla u|^2) + 2\mathrm{Re}\left(\frac{\partial \bar{u}}{\partial x_3}\nabla u\right)\right] dx,$$

where e_3 is the unit vector in the x_3-direction. Applying the divergence theorem we get that

$$\begin{aligned}\int_{\Gamma_0}\left|\frac{\partial u}{\partial x_3}\right|^2 ds &= \int_\Gamma \left\{\nu_3(\kappa^2|u|^2 - |\nabla u_-|^2) + 2\mathrm{Re}\left(\frac{\partial \bar{u}_-}{\partial x_3}\frac{\partial u_-}{\partial \nu}\right)\right\} ds \\ &= \int_\Gamma\left\{\nu_3\left(\kappa^2|u|^2 + \left|\frac{\partial u_-}{\partial \nu}\right|^2 - |\nabla_T u|^2\right) + 2\mathrm{Re}\left(e_3\cdot\nabla_T\bar{u}\,\frac{\partial u_-}{\partial \nu}\right)\right\} ds, \end{aligned} \tag{1.29}$$

where $\nu_3 := e_3 \cdot \nu$ is the vertical component of ν. Since

$$\frac{1}{\tilde{L}} \leq \nu_3(x) \leq 1, \quad |e_3\cdot\nabla_T u(x)| \leq \frac{L}{\tilde{L}}|\nabla_T u(x)|, \quad x \in \Gamma, \tag{1.30}$$

where $\tilde{L} := (1+L^2)^{1/2}$, we deduce that

$$\frac{1}{\tilde{L}}\int_\Gamma |\nabla_T u|^2 ds \leq \int_\Gamma\left(\kappa^2|u|^2 + \left|\frac{\partial u_-}{\partial \nu}\right|^2\right) ds + \frac{2L}{\tilde{L}}\int_\Gamma |\nabla_T u|\left|\frac{\partial u_-}{\partial \nu}\right| ds. \tag{1.31}$$

Applying Cauchy-Schwartz we see that

$$\frac{1}{\tilde{L}}\int_\Gamma |\nabla_T u|^2 ds \leq \int_\Gamma\left(\kappa^2|u|^2 + \left|\frac{\partial u_-}{\partial \nu}\right|^2\right) ds + \frac{1}{2\tilde{L}}\int_\Gamma |\nabla_T u|^2\, ds + \frac{2L^2}{\tilde{L}}\int_\Gamma\left|\frac{\partial u_-}{\partial \nu}\right|^2 ds. \tag{1.32}$$

Using (1.27) and (1.28) it follows that

$$||\nabla_T u||_{L^2(\Gamma)} \leq \left(2\tilde{L} + \frac{2\tilde{L}\kappa^2}{\eta^2} + 4L^2\right)^{1/2} ||g||_{L^2(\Gamma)}. \tag{1.33}$$

To complete our argument we apply the divergence theorem in the region $S_H = D \setminus U_H$, for some $H > f^+$, in order to bound $\partial u_+/\partial \nu$ in terms of $||\nabla_T u||_{L^2(\Gamma)}$. Arguing exactly as we did to obtain (1.29) we find that

$$\begin{aligned}\int_{\Gamma_H}\left\{\kappa^2|u|^2 + \left|\frac{\partial u}{\partial x_3}\right|^2 - |\nabla_{\mathbf{x}} u|^2\right\} ds &= \int_\Gamma\left\{\nu_3\left(\kappa^2|u|^2 + \left|\frac{\partial u_+}{\partial \nu}\right|^2 - |\nabla_T u|^2\right)\right. \\ &\qquad \left. + 2\mathrm{Re}\left(e_3\cdot\nabla_T\bar{u}\,\frac{\partial u_+}{\partial \nu}\right)\right\} ds, \end{aligned} \tag{1.34}$$

where $\nabla_{\mathbf{x}}$ denotes the gradient operator on Γ_H. Lemma 2.2 in [7] implies that

$$\int_{\Gamma_H}\left\{\kappa^2|u|^2 + \left|\frac{\partial u}{\partial x_3}\right|^2 - |\nabla_{\mathbf{x}} u|^2\right\} ds \leq 2\kappa\,\mathrm{Im}\int_{\Gamma_H}\bar{u}\frac{\partial u}{\partial x_3}\, ds, \tag{1.35}$$

and a further application of the divergence theorem in S_H gives that

$$\mathrm{Im} \int_{\Gamma_H} \bar{u} \frac{\partial u}{\partial x_3}\, ds = \mathrm{Im} \int_{\Gamma} \bar{u} \frac{\partial u_+}{\partial \nu} ds. \tag{1.36}$$

Combining (1.34), (1.35), and (1.36), and noting (1.30), we see that

$$\begin{aligned} \frac{1}{\tilde{L}} \int_{\Gamma} \left| \frac{\partial u_+}{\partial \nu} \right|^2 ds &\leq \int_{\Gamma} \nu_3 \left| \frac{\partial u_+}{\partial \nu} \right|^2 ds \\ &\leq \int_{\Gamma} |\nabla_T u|^2 ds + \frac{2L}{\tilde{L}} \int_{\Gamma} |\nabla_T u| \left| \frac{\partial u_+}{\partial \nu} \right| ds + 2\kappa \int_{\Gamma} |u| \left| \frac{\partial u_+}{\partial \nu} \right| ds. \end{aligned}$$

Applying Cauchy-Schwartz it follows that

$$\frac{1}{3\tilde{L}} \int_{\Gamma} \left| \frac{\partial u_+}{\partial \nu} \right|^2 ds \leq \left(1 + \frac{3L^2}{\tilde{L}}\right) \int_{\Gamma} |\nabla_T u|^2 ds + 3\kappa^2 \tilde{L} \int_{\Gamma} |u|^2\, ds. \tag{1.37}$$

Bounding the right hand side using (1.27) and (1.33), we find that

$$\left|\left| \frac{\partial u_+}{\partial \nu} \right|\right|_{L^2(\Gamma)} \leq \left(\frac{3\kappa^2 \tilde{L}}{\eta^2} [5\tilde{L} + 6L^2] + 6(\tilde{L} + 3L^2)^2 \right)^{1/2} ||g||_{L^2(\Gamma)}. \tag{1.38}$$

Putting this together with (1.28), we see that

$$||\varphi||_{L^2(\Gamma)} \leq \left|\left| \frac{\partial u_+}{\partial \nu} \right|\right|_{L^2(\Gamma)} + \left|\left| \frac{\partial u_-}{\partial \nu} \right|\right|_{L^2(\Gamma)} \leq 2B||g||_{L^2(\Gamma)}, \tag{1.39}$$

where B is defined by (1.5), concluding the proof.

Lemma 3 is the major step in showing that A' is invertible, that is, of establishing the following theorem. For further details see [6].

THEOREM 4. *A' and A are invertible on $L^2(\Gamma)$, with*

$$||A^{-1}||_{L^2(\Gamma) \to L^2(\Gamma)} = ||{A'}^{-1}||_{L^2(\Gamma) \to L^2(\Gamma)} \leq B, \tag{1.40}$$

where $B = B(L, \kappa/\eta)$ is defined by (1.5).

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