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A well-posed integral equation formulation for three-dimensional rough surface scattering

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We consider the problem of scattering of time-harmonic acoustic waves by an unbounded sound-soft rough surface. Recently, a Brakhage–Werner type integral equation formulation of this problem has been proposed, based on an ansatz as a combined single- and double-layer potential, but replacing the usual fundamental solution of the Helmholtz equation with an appropriate half-space Green's function. Moreover, it has been shown in the three-dimensional case that this integral equation is uniquely solvable in the space $L^2(\Gamma)$ when the scattering surface Γ does not differ too much from a plane. In this paper, we show that this integral equation is uniquely solvable with no restriction on the surface elevation or slope. Moreover, we construct explicit bounds on the inverse of the associated boundary integral operator, as a function of the wave number, the parameter coupling the single- and double-layer potentials, and the maximum surface slope. These bounds show that the norm of the inverse operator is bounded uniformly in the wave number, κ, for $\kappa > 0$, if the coupling parameter η is chosen proportional to the wave number. In the case when Γ is a plane, we show that the choice $\eta = \kappa/2$ is nearly optimal in terms of minimizing the condition number.

Keywords: boundary integral equation method; rough surface scattering; Helmholtz equation; condition number

1. Introduction

This paper is concerned with boundary integral equation methods for scattering by unbounded surfaces. More precisely, we are concerned with what are termed *rough surface scattering problems* in the engineering literature. We use the phrase *rough surface*, as is the practice in this literature, to denote a surface which is a (usually non-local) perturbation of an infinite plane surface such that the whole surface lies within a finite distance of the original plane. In particular, we are concerned with what is the usual case in the engineering literature where

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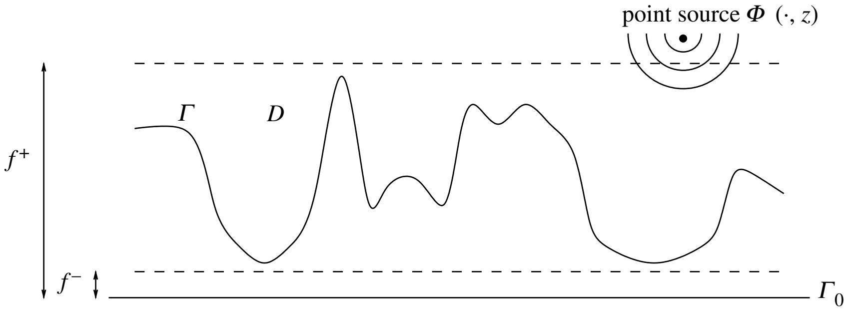

Figure 1. Geometrical setting of the scattering problem.

the scattering surface Γ is the graph of some bounded continuous function $f:\mathbb{R}^2 \to \mathbb{R}$, i.e. (figure 1)

$$\Gamma := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = f(x_1, x_2)\}. \tag{1.1}$$

We will focus on a typical problem of this type, namely acoustic scattering by a rough, sound-soft surface, the acoustic medium of propagation occupying the perturbed half-space,

$$D := \{x = (x_1, x_2, x_3) : x_3 > f(x_1, x_2)\}, \tag{1.2}$$

above the scattering surface Γ. We will suppose throughout that f is a moderately smooth function, i.e. is continuously differentiable with Hölder continuous first derivative (Γ is Lyapunov). Thus, the difficulties in understanding the boundary integral equation formulation will be associated with the unboundedness of Γ rather than its lack of smoothness.

Rough surface scattering problems arise frequently in applications; for example, modelling acoustic and electromagnetic wave propagation over outdoor ground and sea surfaces or, at a very different scale, optical scattering from the surface of materials in nanotechnology. The mathematical and computational modelling of these problems has a large literature (see, e.g. the reviews and monographs by Ogilvy (1991), Voronovich (1998), Saillard & Sentenac (2001), Warnick & Chew (2001), DeSanto (2002) and Elfouhaily & Guerin (2004)). The simulation of these scattering problems, requiring discretizations of sections of three-dimensional surfaces of diameter large compared to the wavelength, is a substantial scientific computing problem for which boundary integral equation methods are very popular, with many effective, specialized numerical algorithms developed (Saillard & Sentenac 2001; Warnick & Chew 2001; Xia *et al.* 2003).

Despite this interest in the application of the BIE method, the associated mathematical and numerical analysis to support these practical computations is largely absent, in the more important three-dimensional case at least. For example, to date there is no integral equation formulation that is known to be uniquely solvable for a general three-dimensional rough surface scattering problem. This lack of a theoretical basis for the BIE method will be addressed in this paper.

However, for the two-dimensional rough surface scattering problem much progress has been made in the last 10 years in terms of deriving well-posed boundary integral equations for a variety of acoustic, electromagnetic and elastic wave problems (e.g. Chandler-Wilde & Zhang 1998; Chandler-Wilde *et al.* 1999; Arens 2002; Zhang & Chandler-Wilde 2003). An important point is that, for general rough surface scattering problems, the usual integral equation formulations for scattering by bounded surfaces, while they have been successfully used for computations, are unattractive from a theoretical point of view since the standard boundary integral operators (e.g. the standard single- and double-layer potential operators) are not bounded operators on any of the usual function spaces when the scattering surface is unbounded. This has important practical consequences, and in particular can be expected to lead to large condition numbers when the standard integral equations are discretized on large sections of rough surfaces.

In the two-dimensional case, alternative integral equations, with bounded integral operators, have been obtained by replacing the standard fundamental solution by the Dirichlet or impedance Green's function for a half-plane that contains the domain D of propagation (see, e.g. Chandler-Wilde *et al.* 1999; Arens 2002; Zhang & Chandler-Wilde 2003). This modification leads to kernels of boundary integral operators that are weakly singular in their asymptotic behaviour at infinity so that the integral operators are bounded on $L^p(\Gamma)$ for $1 \leq p \leq \infty$ and on $BC(\Gamma)$, the space of bounded continuous functions on Γ. In the case of a two-dimensional sound-soft rough surface, Zhang & Chandler-Wilde (2003) followed the approach that was proposed for the sound-soft bounded obstacle, independently by Brakhage & Werner (1965), Leis (1965) and Panich (1965). This approach, to seek the solution to the exterior Dirichlet problem as a linear combination of double and single-layer potentials, will be termed, for brevity, as is common in the literature, the Brakhage–Werner method. It is used in Zhang & Chandler-Wilde (2003), with the twist that the standard fundamental solution is replaced by the Dirichlet Green's function for a half-plane.

The analogous modification has been recently employed by us in the three-dimensional case in Chandler-Wilde *et al.* (2006). Following Zhang & Chandler-Wilde (2003), we derive a Brakhage–Werner-type integral equation, replacing the standard fundamental solution with the Dirichlet Green's function for a half-space that contains D. The complication in the three-dimensional case is that this modification, while it improves the behaviour of the kernels of the integral operators significantly in terms of their behaviour at infinity, as discussed in Chandler-Wilde *et al.* (2006), leads to kernels of the integral operators that are strongly singular rather than weakly singular as in the two-dimensional case, even when the boundary is smooth. As a consequence, the boundary integral operators are no longer well-defined as operators on $BC(\Gamma)$ or $L^\infty(\Gamma)$. In Chandler-Wilde *et al.* (2006) we are able, however, to show the boundedness of the operators on $L^2(\Gamma)$ by expressing each integral operator as a sum of products of convolution and multiplication operators plus a well-behaved remainder, and by showing, through explicit calculations, that the Fourier transform of each convolution kernel is bounded and that each multiplication operator is a multiplication by a bounded function.

To establish existence of solution and well-posedness in the two-dimensional case generalizations of the Riesz theory of compact operators have been developed (see Arens *et al.* (2003) and references therein) but, as discussed in Chandler-Wilde *et al.* (2006), these methods do not appear applicable in the three-dimensional case. In the absence of these tools, we were able in

Chandler-Wilde *et al.* (2006) to prove only a partial result, establishing existence of solution to the integral equation and scattering problem in the case when Γ is sufficiently close to a flat plane. Our tool was to establish existence of solution to the BIE in the special case when Γ is a plane and the integral equation of convolution type, via computation of the Fourier transform of the kernel, and then employ operator perturbation arguments. We mention that existence of solution to the same scattering problem (though formulated rather differently in terms of the function space setting) has recently been established for the case Γ given by (1.1) by variational methods in Chandler-Wilde & Monk (2005), with only the weak assumption that f is bounded.

The results contained in this paper are as follows. We suppose that the rough surface is given by (1.1), with f continuously differentiable with bounded and Hölder continuous first derivative, and restrict attention to the case when the wave number κ is real. We begin by recalling the formulation of the scattering problem in Chandler-Wilde *et al.* (2006), and the Brakhage–Werner type integral equation formulation for solving this scattering problem proposed in Chandler-Wilde *et al.* (2006). This integral equation, in operator form, is

$$(I + K - \mathrm{i}\eta S)\varphi = 2g, \tag{1.3}$$

where I is the identity operator, S and K are single- and double-layer potential operators, defined by (2.8) and (2.9) below, g is the Dirichlet data for the scattered field on Γ, and $\eta > 0$ is the coupling parameter.

Our first main result is to show that

$$A := I + K - \mathrm{i}\eta S, \tag{1.4}$$

is always invertible as an operator on $L^2(\Gamma)$, generalizing the result in Chandler-Wilde *et al.* (2006) for the case when Γ is almost flat. Moreover, we show the explicit bound that

$$\|A^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} < 2 + 2L + 4L^2 + \frac{\kappa}{\eta}(2 + 5L + 3L^{3/2}), \tag{1.5}$$

where

$$L := \sup_{\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^2,\boldsymbol{x}\neq\boldsymbol{y}} \frac{|f(\boldsymbol{x}) - f(\boldsymbol{y})|}{|\boldsymbol{x} - \boldsymbol{y}|} = \sup_{\boldsymbol{x}\in\mathbb{R}^2} |\nabla f(\boldsymbol{x})|, \tag{1.6}$$

is the Lipschitz constant of f (the maximum surface slope). The tools we use to show invertibility and obtain this explicit bound are standard jump relations for layer-potentials (e.g. Colton & Kress 1983) combined with carefully chosen integrations by parts (applications of the divergence theorem) in subsets of D and the region below D. Our techniques are reminiscent of (and inspired by) the somewhat similar arguments used to prove invertibility for second kind boundary integral equations for potential problems in Lipschitz domains (e.g. Verchota 1984; Meyer & Coifman 2000), and of arguments used to obtain *a priori* bounds for solutions to variational formulations of interior problems (Melenk 1995; Cummings & Feng 2006) and rough surface scattering problems (Elschner & Yamamoto 2002; Chandler-Wilde & Monk 2005). In particular, a similarly explicit lower bound for the inf–sup constant of a variational formulation of this rough surface scattering problem is shown in Chandler-Wilde & Monk (2005).

The bound (1.5) is attractive in its explicitness. Indeed, we know of no other rigorous bound on the norm of the inverse of a boundary integral operator for a wave problem which makes explicit the dependence on the wave number and/or the geometry, with the exception of bounds for a very special acoustic scattering problem (scattering by a flat inhomogeneous impedance boundary) in Arens *et al.* (2003) and Chandler-Wilde *et al.* (2004). One consequence of (1.5) is that if η is chosen proportional to the wave number κ, as recommended in the bounded obstacle case in Kress & Spassov (1983), Kress (1985) and Giebermann (1997), then the inverse operator is bounded by an explicit function of the Lipschitz constant of f and, in particular, is bounded independently of the wave number, for $\kappa > 0$.

In §4, we investigate in more detail the optimal choice of the coupling parameter η for the case when Γ is flat (this is analogous to studying the special case of a spherical scatterer in bounded obstacle scattering, as done in Kress & Spassov 1983; Kress 1985; Amini 1990; Giebermann 1997; Buffa & Sauter in press). We show for this special case that the choice $\eta = \kappa/2$ is close to optimal in terms of minimizing the L^2 condition number

$$\operatorname{cond} A := \|A\|_{L^2(\Gamma)\to L^2(\Gamma)} \|A^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}.$$

Indeed, in a sense we make precise, the choice $\eta = \kappa/2$ is asymptotically optimal in the limit $\kappa \to \infty$, leading to a minimal condition number $\operatorname{cond} A$, which is asymptotically proportional to κ. We note that the same choice $\eta = \kappa/2$ is recommended as almost minimizing the condition number for the Brakhage–Werner formulation for scattering by a sound-soft sphere in Kress & Spassov (1983), on the basis of numerical computations at low wave numbers of explicit expressions for the singular values of the operator, and is recommended in Giebermann (1997) based on a study of high- and low-frequency asymptotics of eigenvalues.

We remark that, thanks in large part to the simpler geometry, our results for the case when Γ is flat are more rigorous and complete than the corresponding results for spherical scatterers. For example, our results imply the bound that

$$\|A^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \le \max\left(1, \frac{\kappa}{2\eta}\right), \tag{1.7}$$

with equality holding in the case $\eta \ge \kappa/2$ and in the limit $\kappa \to \infty$. This is precisely the bound on the inverse of an analogous integral operator in the case when Γ is a sphere stated by Giebermann (1997), but the bound in Giebermann (1997) is a conjecture, supported by theoretical investigations, example calculations, and asymptotics. Our bound (1.7) is a theorem.

We note that a rigorous theory for BIE methods for three-dimensional rough surface scattering has been developed previously for two special cases. The first is the case of scattering by a locally perturbed plane, where the unbounded surface coincides with a plane in the exterior of some ball. This case can be reduced to a boundary integral equation on a finite domain, related to the local perturbation; we refer the reader to Willers (1987), Kress & Tran (2000) and Chandler-Wilde & Peplow (2005) and references therein. The second is the case when the surface is a diffraction grating (the function f in (1.1) is bi-periodic) and the incident field is a plane wave. In this case, the boundary integral equation can be reduced to one on a finite part of the surface that is a single period; see Nédélec & Starling (1991) and Dobson & Friedman (1992). In both these cases, reducible to integral equations on finite domains, well-posedness is obtained by compactness arguments, which do not apply to the general rough surface scattering problem and, moreover, do not lead to explicit bounds on the inverses of the boundary integral operators.

We should perhaps emphasize that, since our results assume boundary data in the space $L^2(\Gamma)$, they do not include the interesting case of plane wave incidence, which case is included in the theory that has been developed for the two-dimensional problem (Chandler-Wilde *et al.* 1999; Zhang & Chandler-Wilde 2003). For a partial theoretical justification for BIE methods for three-dimensional rough surface scattering with plane wave incidence, namely a justification, with some provisos, of Green's representation formula, see DeSanto & Martin (1998).

Finally, we remark that a brief summary of some of the results of this paper and those of Chandler-Wilde *et al.* (2006) is published in the proceedings of the fifth UK Boundary Integral Methods Conference (Chandler-Wilde *et al.* 2005).

Notation. Throughout the paper x and y will denote points in $\mathbb{R}^3$ with components $x=(x_1,x_2,x_3)$ and $y=(y_1,y_2,y_3)$. The image of $y\in\mathbb{R}^3$ in the plane $\Gamma_0:=\{x\in\mathbb{R}^3 : x_3=0\}$ will be denoted by $y':=(y_1,y_2,-y_3)$. By $\boldsymbol{x}$, we will denote $(x_1,x_2)\in\mathbb{R}^2$, so that $x=(\boldsymbol{x},x_3)$. Similarly, $\boldsymbol{y}$ denotes (y_1,y_2). The standard scalar product in $\mathbb{R}^2$ is denoted by $\boldsymbol{x}\cdot\boldsymbol{y}$ and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. $BC(\Gamma)$ will denote the set of bounded continuous real- or complex-valued functions on Γ, a Banach space with the norm $\|\cdot\|_{BC(\Gamma)}$ defined by $\|F\|_{BC(\Gamma)}=\sup_{x\in\Gamma}|F(x)|$. For $0<\alpha\le 1$, let $BC^{1,\alpha}(\mathbb{R}^2)$ denote the set of those bounded continuously differentiable functions $F:\mathbb{R}^2\to\mathbb{R}$ that have the property that ∇F is bounded and uniformly Hölder continuous with index α, so that

$$\|F\|_{BC^{1,\alpha}(\mathbb{R}^2)} := \sup_{\boldsymbol{x}\in\mathbb{R}^2}|F(\boldsymbol{x})| + \sup_{\boldsymbol{x}\in\mathbb{R}^2}|\nabla F(\boldsymbol{x})| + \sup_{\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^2,\boldsymbol{x}\neq\boldsymbol{y}}\frac{|\nabla F(\boldsymbol{x})-\nabla F(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^{\alpha}}<\infty.$$

$BC^{1,\alpha}(\mathbb{R}^2)$ is a Banach space under the norm $\|\cdot\|_{BC^{1,\alpha}(\mathbb{R}^2)}$. It is convenient also to have a shorthand for the intersection of the sets $L^2(\Gamma)$ and $BC(\Gamma)$, so we define

$$X:=L^2(\Gamma)\cap BC(\Gamma).$$

Since $L^2(\Gamma)$ and $BC(\Gamma)$ are Banach spaces equipped with their respective norms, so also is X, equipped with the norm $\|\cdot\|_X$ defined by

$$\|F\|_X := \max(\|F\|_{L^2(\Gamma)},\|F\|_{BC(\Gamma)}).$$

2. The rough surface scattering problem

Time-harmonic ($e^{-i\omega t}$ time dependence) acoustic waves are modelled by the Helmholtz equation

$$\Delta u+\kappa^2 u=0. \tag{2.1}$$

In this equation, $\kappa=\omega/c>0$, where c is the speed of sound, is the *wave number*. We consider acoustic wave motion in the *domain of propagation* D defined by (1.2), throughout assuming that $f\in BC^{1,\alpha}(\mathbb{R}^2)$, for some $\alpha\in(0,1]$, and that f is a strictly positive function, so that there exist constants $f^+ > f^- > 0$ with

$$f^- \leq f(\boldsymbol{x}) \leq f^+, \quad \boldsymbol{x} \in \mathbb{R}^2.$$

We denote the boundary of D by Γ, so that Γ is given by (1.1). Whenever we wish to denote explicitly the dependence of the domain on the boundary function f, we will write D^f for D and Γ^f for Γ, so that

$$\Gamma^f = \{x = (x_1, x_2, x_3) : x_3 = f(\boldsymbol{x})\}.$$

We use the notation Γ_h, for $h \in \mathbb{R}$, to denote the plane

$$\Gamma_h := \{x = (x_1, x_2, x_3) : x_3 = h\}.$$

By U_h, we denote the half-space above Γ_h and by S_h the part of D below Γ_h, so that

$$U_h := \{x : x_3 > h\}, \quad S_h := D \setminus \bar{U}_h.$$

We will consider the scattering of an *incident acoustic wave* u^{i} by the surface Γ. For the *total field* $u := u^{\mathrm{i}} + u^{\mathrm{s}}$, which is the sum of the incident field and the *scattered field* u^{s}, we assume on Γ the *Dirichlet* boundary condition

$$u(x) = 0, \quad x \in \Gamma. \tag{2.2}$$

We require that the scattered field is bounded in D, i.e.

$$|u^{\mathrm{s}}(x)| \leq c, \quad x \in D, \tag{2.3}$$

for some constant $c > 0$. We also require that u satisfies the following *limiting absorption principle*: denoting u temporarily by $u^{(\kappa)}$ to indicate its dependence on κ, we suppose that for all sufficiently small $\epsilon > 0$ a solution $u^{(\kappa+\mathrm{i}\epsilon)}$ exists which satisfies (2.1)–(2.3) (with κ replaced by $\kappa + \mathrm{i}\epsilon$) and that, for all $x \in D$,

$$u^{(\kappa+\mathrm{i}\epsilon)}(x) \to u^{(\kappa)}(x), \quad \epsilon \to 0. \tag{2.4}$$

The limiting absorption principle plays the role of a radiation condition, singling out the correct physical solution.

Let

$$\Phi(x, y) := \frac{1}{4\pi} \frac{e^{\mathrm{i}\kappa|x-y|}}{|x-y|}, \quad x, y \in \mathbb{R}^3, x \neq y, \tag{2.5}$$

denote the standard fundamental solution of the Helmholtz equation. In order to get kernels of our boundary integral operators which have faster decay at infinity we will, following Chandler-Wilde *et al.* (2006), replace $\Phi(x, y)$ by an appropriate half-space Green's function for the Helmholtz equation. Specifically, we will work with the function

$$G(x, y) := \Phi(x, y) - \Phi(x, y'), \tag{2.6}$$

with $y' = (y_1, y_2, -y_3)$, which is the Dirichlet Green's function for the half space $\{x : x_3 > 0\}$. The faster decay of $G(x, y)$ compared to $\Phi(x, y)$, as $|x|, |y| \to \infty$ with $x, y \in \Gamma$, is captured in the bound (3.8) in Chandler-Wilde *et al.* (2006), that, for some constant $C>0$,

$$|G(x,y)| \leq \frac{C(1+x_3)(1+y_3)}{|x-y|^2}, \tag{2.7}$$

for all $x, y \in \mathbb{R}^3$ with $x \neq y$ and $x_3, y_3 \geq 0$.

Thus, we will use layer potentials with $\Phi(x,y)$ replaced by $G(x,y)$, so that we define the *single-layer potential operator* by

$$(S\varphi)(x) := 2\int_\Gamma G(x,y)\varphi(y)\mathrm{d}s(y), \quad x \in \Gamma, \tag{2.8}$$

and the *double-layer potential operator* by

$$(K\varphi)(x) := 2\int_\Gamma \frac{\partial G(x,y)}{\partial \nu(y)}\varphi(y)\mathrm{d}s(y), \quad x \in \Gamma, \tag{2.9}$$

where the normal $\nu(y)$ is directed into D. Whenever we wish to denote explicitly the dependence of S and K on the boundary function f, we will write S_f and K_f for S and K, respectively.

Returning to the scattering problem, we wish to develop an analysis that is applicable whenever the incident wave is due to sources of the acoustic field located in some compact set $M \subset D$. Since waves with sources in a bounded set $M \subset \mathbb{R}^3$ can be represented as superpositions of point sources located in the same set, we will concentrate on the case when the incident field is that due to a point source located at some point $z \in D$, i.e. $u^{\mathrm{i}} = \Phi(\cdot, z)$. Thus, as in Chandler-Wilde *et al.* (2006), the following is the specific problem that we will consider in this paper:

Problem 1 (Point source rough surface scattering problem). *Let $u^{\mathrm{i}} = \Phi(\cdot, z)$ be the incident field due to a point source at $z \in D$. Then we seek a scattered field $u^{\mathrm{s}} \in C^2(D) \cap C(\bar{D})$ such that u^{s} is a solution to the Helmholtz equation (2.1) in D, the total field satisfies the sound-soft boundary condition (2.2), and the bound (2.3) and the limiting absorption principle (2.4) hold.*

We will convert this scattering problem to a boundary value problem (BVP). To do this, we will seek the scattered field as the sum of a mirrored point source $\Phi'(\cdot, z) := -\Phi(\cdot, z')$, where z' is the image of z in the flat plane Γ_0, plus some unknown remainder v, i.e. $u^{\mathrm{s}} = v + \Phi'(\cdot, z)$. Note that $\Phi'(\cdot, z)$ is a solution to the scattering problem in the special case that $\Gamma = \Gamma_0$. Using the boundary condition $u^{\mathrm{s}} + \Phi(\cdot, z) = 0$ on $\Gamma = \partial D$ we obtain the boundary condition on v that

$$v(x) = -\{\Phi(x,z) - \Phi(x,z')\} = -G(x,z) =: g(x), \quad x \in \Gamma. \tag{2.10}$$

Clearly, $g \in BC(\Gamma)$ and it follows from (2.7) that $g \in L^2(\Gamma)$, so that $g \in X = L^2(\Gamma) \cap BC(\Gamma)$. Further, by the dominated convergence theorem we see that $\|g_\epsilon - g\|_{L^2(\Gamma)} \to 0$ as $\epsilon \to 0^+$, where g_ϵ is $-G(\cdot, z)$ with κ replaced by $\kappa + \mathrm{i}\epsilon$. Thus, u^{s} satisfies the above scattering problem if and only if v satisfies the following Dirichlet problem, with g given by (2.10) and g_ϵ defined as $-G(\cdot, z)$ with κ replaced by $\kappa + \mathrm{i}\epsilon$.

Problem 2 (BVP). *Given $g \in X$ and $g_\epsilon \in X$, for $\epsilon > 0$, with $\|g_\epsilon - g\|_{L^2(\Gamma)} \to 0$ as $\epsilon \to 0$, find $v \in C^2(D) \cap C(\bar{D})$ which satisfies the Helmholtz equation (2.1) in D, the Dirichlet boundary condition $v = g$ on Γ, the bound (2.3), and the following* *limiting absorption principle: that, for all sufficiently small* $\epsilon > 0$, *there exists* $v_\epsilon \in C^2(D) \cap C(\bar{D})$ *satisfying* $v_\epsilon = g_\epsilon$ *on* Γ, *(2.1) and (2.3), with* κ *replaced by* $\kappa + \mathrm{i}\epsilon$, *such that, for all* $x \in D$, $v_\epsilon(x) \to v(x)$ *as* $\epsilon \to 0$.

In this paper we will, following Chandler-Wilde *et al.* (2006), look for a solution to this BVP as the *combined single- and double-layer potential*

$$v(x) := u_2(x) - \mathrm{i}\eta u_1(x), \quad x \in D, \tag{2.11}$$

with some parameter $\eta > 0$, where for a given function $\varphi \in X$ we define the *single-layer potential*

$$u_1(x) := \int_\Gamma G(x, y)\varphi(y)\mathrm{d}s(y), \quad x \in \mathbb{R}^3, \tag{2.12}$$

and the *double-layer potential*

$$u_2(x) := \int_\Gamma \frac{\partial G(x, y)}{\partial \nu(y)}\varphi(y)\mathrm{d}s(y), \quad x \in \mathbb{R}^3. \tag{2.13}$$

Seeking the solution in this form it is shown in Chandler-Wilde *et al.* (2006), as a consequence of jump relations for the layer-potentials, that, for $g \in X$, the boundary condition $v = g$ on Γ is satisfied if and only if the boundary integral equation

$$A\varphi = 2g, \tag{2.14}$$

holds on Γ, where A is the operator defined by (1.4).

A main result of Chandler-Wilde *et al.* (2006), crucial to the arguments that we will make in §3, is:

Theorem 2.1. *The single- and double-layer potential operators* S *and* K, *defined by (2.8) and (2.9), are bounded operators on* $L^2(\Gamma)$ *and on* X.

In Chandler-Wilde *et al.* (2006), we also showed the following result, establishing that invertibility of A on X and existence of solution to the BVP and scattering problem follow once we show that A is invertible on $L^2(\Gamma)$.

Theorem 2.2. *If* A *is invertible as an operator on* $L^2(\Gamma)$, *then* A *is invertible as an operator on* X. *Moreover, if* A *is invertible on* X, *then the BVP has exactly one solution* v, *defined by (2.11)–(2.13) with* $\varphi \in X$ *given by* $\varphi = 2A^{-1}g$. *Further, for some constant* $c > 0$, *independent of* g,

$$|v(x)| \le c\|g\|_X, \quad x \in \bar{D}.$$

We showed in Chandler-Wilde *et al.* (2006) that A is indeed invertible as an operator on $L^2(\Gamma)$ in the case when Γ is almost flat. In particular, we established the following special case:

Theorem 2.3. *In the case* $\Gamma = \Gamma_h$, *with* $h > 0$, *it holds that* A *is invertible on* $L^2(\Gamma)$ *and that the BVP is uniquely solvable.*

Starting from the above results we will show in §3 that A is invertible on $L^2(\Gamma)$, without restriction on the surface elevation or slope of Γ, establishing the explicit bound (1.5). We will establish this result first of all for the case in which $f \in C^\infty(\mathbb{R}^2)$. We will extend the result to the more general case, in which we only assume that $f \in BC^{1,\alpha}(\mathbb{R}^2)$, by continuity arguments, using the results of continuous dependence of A on Γ established in Chandler-Wilde *et al.* (2006). The continuous dependence result we use is stated precisely in theorem 2.4. In the statement of this theorem, we use the notation T_f for either S or K defined on a surface Γ^f. With the help of the isomorphism

$$I_f : L^2(\Gamma^f) \to L^2(\mathbb{R}^2), \quad (I_f\varphi)(\boldsymbol{y}) = \varphi((\boldsymbol{y}, f(\boldsymbol{y}))), \quad \boldsymbol{y} \in \mathbb{R}^2, \tag{2.15}$$

we associate T_f with the element $\tilde{T}_f = I_f T_f I_f^{-1}$ of the set of bounded linear operators on $L^2(\mathbb{R}^2)$.

Theorem 2.4 (Chandler-Wilde *et al.* 2006). *The single- and double-layer potential operators depend continuously on the boundary Γ of the unbounded domain D in the sense that*

$$\| \tilde{T}_f - \tilde{T}_g \|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \to 0, \tag{2.16}$$

as $g \to f$ in $BC^{1,\alpha}(\mathbb{R}^2)$.

3. Invertibility of A

In this section, our main result is to establish that A is invertible as an operator on $L^2(\Gamma)$, with the explicit bound (1.5) on A^{-1}. Combining this result with theorem 2.2 we also establish the following important corollary.

Theorem 3.1. *A is invertible as an operator on $L^2(\Gamma)$ and as an operator on X. Moreover, the BVP has exactly one solution v, defined by (2.11)–(2.13) with $\varphi \in X$ given by $\varphi = 2A^{-1}g$. Further, for some constant $c>0$, independent of g,*

$$|v(x)| \leq c\|g\|_X, \quad x \in \bar{D}.$$

To establish these results our tools are the theorems from Chandler-Wilde *et al.* (2006) that are stated at the end of §2, certain results from Chandler-Wilde & Monk (2005), and standard properties of layer potentials. We will work with the operators S and K and with their adjoints. We introduce the operator K' defined by

$$(K'\varphi)(x) := 2\int_\Gamma \frac{\partial G(x,y)}{\partial \nu(x)} \varphi(y) \mathrm{d}s(y), \quad x \in \Gamma. \tag{3.1}$$

Our first result is as follows:

Lemma 3.2. *K' is a bounded operator on $L^2(\Gamma)$.*

Proof. The kernel of K' is just the transpose of that of K. Examining the proof of the boundedness of K in Chandler-Wilde *et al.* (2006), we see that it applies word for word to establish that K' is bounded. ■

Of course K' is just the adjoint of K with respect to the bilinear form $(\cdot,\cdot)$ on $L^2(\Gamma) \times L^2(\Gamma)$ defined by

$$(\phi, \psi) = \int_\Gamma \phi(y)\psi(y)\mathrm{d}s(y), \quad \phi, \psi \in L^2(\Gamma).$$

With respect to this bilinear form S and I are both self-adjoint so that the adjoint of A is

$$A' = I + K' - \mathrm{i}\eta S.$$

From standard properties of adjoint operators on Hilbert spaces, we have that A and A' have the same norm, that A is invertible if and only if A is invertible, and that if they are both invertible then

$$\|A^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} = \|A'^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}. \tag{3.2}$$

Thus, we can proceed in the first instance by bounding A'. Our first, key step in this direction is to prove the following lower bound in the case when Γ is smooth.

Lemma 3.3. *Suppose that, in addition to our assumptions throughout on f, it holds that $f \in C^\infty(\mathbb{R}^2)$. Then, for all $\varphi \in L^2(\Gamma)$ there holds*

$$\|A'\varphi\|_{L^2(\Gamma)} \geq B^{-1}\|\varphi\|_{L^2(\Gamma)}, \tag{3.3}$$

where

$$B = B(L, \kappa/\eta) := \frac{1}{2}\left(1 + \left(\frac{3\kappa^2\tilde{L}}{\eta^2}[5\tilde{L} + 6L^2] + 6(\tilde{L} + 3L^2)^2\right)^{1/2}\right), \tag{3.4}$$

and $\tilde{L} := (1 + L^2)^{1/2}$.

Proof. Let $Y \subset L^2(\Gamma)$ denote the set of those $\varphi \in C(\Gamma)$ that are Hölder continuous and compactly supported. Since Y is dense in $L^2(\Gamma)$ and A' is bounded on $L^2(\Gamma)$ it is sufficient to show that (3.3) holds for all $\varphi \in Y$.

So suppose $\varphi \in Y$, let $\tilde{\Gamma} \subset \Gamma$ denote the (bounded) support of φ, and define the single-layer potential u by

$$u(x) := \int_\Gamma G(x, y)\varphi(y)\mathrm{d}s(y), \quad x \in \mathbb{R}^3,$$

and note that

$$u(x) = \int_{\tilde{\Gamma}} G(x, y)\varphi(y)\mathrm{d}s(y) = \int_{\tilde{\Gamma}} \Phi(x, y)\varphi(y)\mathrm{d}s(y) - \int_{\tilde{\Gamma}} \Phi(x, y)\varphi(y)\mathrm{d}s(y),$$

where $\tilde{\Gamma}' := \{y' : y \in \tilde{\Gamma}\}$ denotes the image of $\tilde{\Gamma}$ in Γ_0. From standard properties of the single-layer potential (e.g. Colton & Kress 1983), we have that $u \in C(\mathbb{R}^3) \cap C^2(\mathbb{R}^3 \backslash (\tilde{\Gamma} \cup \tilde{\Gamma}'))$ and that u satisfies the Helmholtz equation in $\mathbb{R}^3 \backslash (\tilde{\Gamma} \cup \tilde{\Gamma}')$. Further, it follows from the bound (2.7), which allows us to estimate $|u|$, and by interior elliptic regularity estimates for solutions of the Helmholtz equation (e.g. Lemma 2.7 in Chandler-Wilde & Zhang 1998), which allow us then to estimate $|\nabla u|$, that

$$u(x) = O(|x|^{-2}), \quad \nabla u(x) = O(|x|^{-2}), \tag{3.5}$$

as $|x| \to \infty$ with $x \in \bar{U}_0$ and $x_3 = O(1)$. Moreover, where $M = \{x : 0 < x_3 < f(\boldsymbol{x})\}$ denotes the region between Γ and Γ_0, we have (Theorem 2.17 in Colton & Kress (1983)) that ∇u can be continuously extended from D to $\bar{D}$ and from M to $\bar{M}$, with limiting values on Γ given by

$$\nabla u_\pm(x) = \int_{\tilde{\Gamma}} \nabla_x G(x, y)\varphi(y)\mathrm{d}s(y) \mp \frac{1}{2}\varphi(x)\nu(x), \quad x \in \Gamma, \tag{3.6}$$

where $\nu(x)$ is the unit normal vector at x, directed into D, and

$$\nabla u_\pm(x) := \lim_{\epsilon \to 0+} \nabla u(x \pm \epsilon\nu(x)), \quad x \in \Gamma.$$

We note from (3.6) that the normal derivative jumps across Γ, with

$$\frac{\partial u_\pm}{\partial \nu}(x) := \nu(x)\cdot\nabla u_\pm(x) = \frac{1}{2}[(K'\varphi)(x) \mp \varphi(x)], \quad x\in\Gamma. \tag{3.7}$$

On the other hand, the tangential part of ∇u is continuous across Γ. We denote this tangential part by $\nabla_T u$, so that

$$\nabla_T u(x) = \nabla u_\pm(x) - \nu(x)\frac{\partial u_\pm}{\partial \nu}(x), \quad x\in\Gamma.$$

Noting that

$$u(x) = \frac{1}{2}S\varphi(x), \quad x\in\Gamma,$$

and defining

$$g := \frac{1}{2}A'\varphi = \frac{1}{2}(I + K' - \mathrm{i}\eta S)\varphi,$$

we see that

$$\frac{\partial u_-}{\partial \nu}(x) - \mathrm{i}\eta u(x) = g(x), \quad x\in\Gamma. \tag{3.8}$$

Further, from (3.7) we see that

$$\frac{\partial u_-}{\partial \nu}(x) - \frac{\partial u_+}{\partial \nu}(x) = \varphi(x), \quad x\in\Gamma. \tag{3.9}$$

Note that to complete the proof we have to show that

$$\|\varphi\|_{L^2(\Gamma)} \leq 2B\|g\|_{L^2(\Gamma)}.$$

We will achieve this by bounding the normal derivatives of u on Γ via applications of the divergence theorem in M and D.

We start with a simple and standard application of the divergence theorem in M. This, and our other application of the divergence theorem in M, are valid since $u\in C^1(\bar{M})\cap C^2(\bar{M}\backslash\tilde{\Gamma})$ and decays rapidly at infinity, as quantified in (3.5), so that u is also in the standard Sobolev space $H^1(M)$. Precisely, these properties of u are enough to justify our applications of the divergence theorem in M by first applying the divergence theorem in the region $\{x=(\boldsymbol{x},x_3) : |\boldsymbol{x}| < C, 0 < x_3 < f(\boldsymbol{x})-\epsilon\}$, for some $C>0$ and sufficiently small $\epsilon>0$, and then letting first $\epsilon\to 0$ and then $C\to\infty$.

Proceeding with our argument, since u satisfies the Helmholtz equation in M, we have that

$$0 = \mathrm{Im}\int_M (\Delta u + \kappa^2 u)\bar{u}\,\mathrm{d}x = \mathrm{Im}\int_M \nabla\cdot(\bar{u}\nabla u)\mathrm{d}x.$$

Applying the divergence theorem, since $u=0$ on Γ_0, we have that

$$\mathrm{Im}\int_\Gamma \bar{u}\frac{\partial u_-}{\partial \nu}\mathrm{d}s = 0. \tag{3.10}$$

Using (3.8) to replace $\partial u_-/\partial\nu$ in the above equation, and applying Cauchy–Schwarz, we see that

$$\eta\|u\|^2_{L^2(\Gamma)} = -\mathrm{Im}\int_\Gamma \bar{u}g\,\mathrm{d}s \leq \|u\|_{L^2(\Gamma)}\|g\|_{L^2(\Gamma)},$$

so that

$$\|u\|_{L^2(\Gamma)} \leq \eta^{-1}\|g\|_{L^2(\Gamma)}. \tag{3.11}$$

Alternatively, from (3.10) we have that

$$\mathrm{Re}\int_\Gamma \mathrm{i}\eta\bar{u}\frac{\partial u_-}{\partial\nu}\,\mathrm{d}s = 0,$$

and, using (3.8) and Cauchy–Schwarz, we see that

$$\left\|\frac{\partial u_-}{\partial\nu}\right\|_{L^2(\Gamma)} \leq \|g\|_{L^2(\Gamma)}. \tag{3.12}$$

It remains to bound the L^2 norm of $\partial u_+/\partial\nu$ in terms of $\|g\|_{L^2(\Gamma)}$. To achieve this goal, we first make a second application of the divergence theorem in M. We have

$$0 = 2\,\mathrm{Re}\int_M \left(\Delta u + \kappa^2 u\right)\frac{\partial\bar{u}}{\partial x_3}\,\mathrm{d}x = \int_M \nabla\cdot\left[e_3(\kappa^2|u|^2 - |\nabla u|^2) + 2\,\mathrm{Re}\left(\frac{\partial\bar{u}}{\partial x_3}\nabla u\right)\right]\mathrm{d}x,$$

where e_3 is the unit vector in the x_3-direction. Applying the divergence theorem, we obtain that

$$\begin{aligned}\int_{\Gamma_0}\left|\frac{\partial u}{\partial x_3}\right|^2\mathrm{d}s &= \int_\Gamma\left\{\nu_3(\kappa^2|u|^2 - |\nabla u_-|^2) + 2\,\mathrm{Re}\left(\frac{\partial\bar{u}_-}{\partial x_3}\frac{\partial u_-}{\partial\nu}\right)\right\}\mathrm{d}s\\ &= \int_\Gamma\left\{\nu_3\left(\kappa^2|u|^2 + \left|\frac{\partial u_-}{\partial\nu}\right|^2 - |\nabla_{\mathrm{T}}u|^2\right) + 2\,\mathrm{Re}\left(e_3\cdot\nabla_{\mathrm{T}}\bar{u}\frac{\partial u_-}{\partial\nu}\right)\right\}\mathrm{d}s,\end{aligned} \tag{3.13}$$

where $\nu_3 := e_3\cdot\nu$ is the vertical component of ν. Since

$$\frac{1}{\tilde{L}} \leq \nu_3(x) \leq 1, \quad |e_3\cdot\nabla_T u(x)| \leq \frac{L}{\tilde{L}}|\nabla_T u(x)|, \quad x\in\Gamma, \tag{3.14}$$

we deduce that

$$\frac{1}{\tilde{L}}\int_\Gamma |\nabla_T u|^2\,\mathrm{d}s \leq \int_\Gamma\left(\kappa^2|u|^2 + \left|\frac{\partial u_-}{\partial\nu}\right|^2\right)\mathrm{d}s + \frac{2L}{\tilde{L}}\int_\Gamma|\nabla_T u|\left|\frac{\partial u_-}{\partial\nu}\right|\mathrm{d}s. \tag{3.15}$$

Applying Cauchy–Schwarz and noting that

$$2ab \leq \epsilon a^2 + \frac{b^2}{\epsilon}, \tag{3.16}$$

for $a, b \geq 0$, $\epsilon > 0$, we see that

$$\frac{1}{\tilde{L}}\int_\Gamma|\nabla_T u|^2\mathrm{d}s \leq \int_\Gamma\left(\kappa^2|u|^2 + \left|\frac{\partial u_-}{\partial\nu}\right|^2\right)\mathrm{d}s + \frac{1}{2\tilde{L}}\int_\Gamma|\nabla_T u|^2\mathrm{d}s + \frac{2L^2}{\tilde{L}}\int_\Gamma\left|\frac{\partial u_-}{\partial\nu}\right|^2\mathrm{d}s. \tag{3.17}$$

Using (3.11) and (3.12), we deduce that

$$\|\nabla_T u\|_{L^2(\Gamma)} \leq \left(2\tilde{L} + \frac{2\tilde{L}\kappa^2}{\eta^2} + 4L^2\right)^{1/2}\|g\|_{L^2(\Gamma)}. \tag{3.18}$$

To complete our argument, we carry out a similar integration by parts in the region $S_H = D \setminus U_H$, for some $H > f^+$, in order to bound $\partial u_+/\partial \boldsymbol{\nu}$ in terms of $\|\nabla_T u\|_{L^2(\Gamma)}$. Arguing exactly as we did to obtain (3.13), we find that

$$\int_{\Gamma_H} \left\{ \kappa^2 |u|^2 + \left|\frac{\partial u}{\partial x_3}\right|^2 - |\nabla_{\boldsymbol{x}} u|^2 \right\} \mathrm{d}s$$

$$= \int_{\Gamma} \left\{ \nu_3 \left(\kappa^2 |u|^2 + \left|\frac{\partial u_+}{\partial \boldsymbol{\nu}}\right|^2 - |\nabla_T u|^2 \right) + 2\,\mathrm{Re}\left(e_3 \cdot \nabla_T \bar{u} \frac{\partial u_+}{\partial \boldsymbol{\nu}} \right) \right\} \mathrm{d}s, \qquad (3.19)$$

where $\nabla_{\boldsymbol{x}} = (\partial/\partial x_1, \partial/\partial x_2)$ denotes the gradient operator on Γ_H.

To bound the left-hand side of equation (3.19), we first construct an explicit representation for u in a half-space above Γ containing Γ_H. Define the Fourier transform operator on $L^2(\mathbb{R}^2)$ by

$$(\mathcal{F}\ell)(\boldsymbol{k}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}} \ell(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}, \quad \boldsymbol{k} \in \mathbb{R}^2. \qquad (3.20)$$

Pick h such that $f_+ < h < H$, let $\psi_h := u|_{\Gamma_h}$ denote the restriction of u to Γ_h and, identifying Γ_h with $\mathbb{R}^2$, let $\hat{\psi}_h := \mathcal{F}\psi_h$ denote the Fourier transform of u on Γ_h, well-defined as an element of $L^2(\mathbb{R}^2)$ since $u \in L^2(\Gamma_h)$. Note further that $\psi_h \in C^2(\Gamma_h)$ and that, by the same interior elliptic estimates we used to deduce the bound (3.5) on ∇u, it follows that the second-order partial derivatives of u decay at least as rapidly as $|x|^{-2}$ on Γ_h. Thus $\psi_h \in H^2(\Gamma_h)$, so that, by Cauchy–Schwarz, we see that $\hat{\psi}_h \in L^1(\mathbb{R}^2)$ with

$$\left\{ \int_{\mathbb{R}^2} |\hat{\psi}_h(\boldsymbol{k})| \mathrm{d}\boldsymbol{k} \right\}^2 \le \beta \int_{\mathbb{R}^2} |\hat{\psi}_h(\boldsymbol{k})|^2 (1 + \boldsymbol{k}^2)^2 \,\mathrm{d}\boldsymbol{k} = \beta \|\psi_h\|^2_{H^2(\Gamma_h)},$$

where

$$\beta := \int_{\mathbb{R}^2} (1 + \boldsymbol{k}^2)^{-2} \,\mathrm{d}\boldsymbol{k}.$$

Now define

$$v(x) := \frac{1}{2\pi} \int_{\mathbb{R}^2} \exp(\mathrm{i}[(x_3 - h)\sqrt{\kappa^2 - \boldsymbol{k}^2} + \boldsymbol{x}\cdot\boldsymbol{k}]) \hat{\psi}_h(\boldsymbol{k}) \mathrm{d}\boldsymbol{k}, \qquad (3.21)$$

for $x = (\boldsymbol{x}, x_3) \in \bar{U}_h$, where $\sqrt{\kappa^2 - \boldsymbol{k}^2} = \mathrm{i}\sqrt{\boldsymbol{k}^2 - \kappa^2}$ for $|\boldsymbol{k}| > \kappa$. We obtain our representation for u by proving that u coincides with v in $\bar{U}_h$.

To see this, we note first that u restricted to $\bar{U}_h$ satisfies Problem 2 in the case that we set $\Gamma = \Gamma_h$, $g = \psi_h$, and define g_ϵ to be the restriction of u to Γ_h when u is defined by (2.11) but with κ replaced by $\kappa + \mathrm{i}\epsilon$ in the definition of the Dirichlet Green's function G. (We note that $g_\epsilon \in L^2(\Gamma_h)$ and converges in norm to g as $\epsilon \to 0$, by the dominated convergence theorem, since the bound (2.7) holds with the same constant C when κ is replaced by $\kappa + \mathrm{i}\epsilon$; see Chandler-Wilde *et al.* (2006).) But it is straightforward to show that v satisfies the same BVP; in particular, since $\hat{\psi}_h \in L^1(\mathbb{R}^2)$, $v \in C(\bar{U})$ follows by the dominated convergence theorem, and it holds that $v = \psi_h$ on Γ_h since the right-hand side of (3.21) is just the inverse Fourier transform of $\hat{\psi}_h$ when $x_3 = h$. Thus, by theorem 2.3, it follows that $u(x) = v(x)$, $x \in \bar{U}_h$.

Having shown this explicit formula for u in U_h, a plane-wave spectrum representation for u, we can apply lemma 2.2 in Chandler-Wilde & Monk (2005) to deduce that

$$\int_{\Gamma_H}\left\{\kappa^2|u|^2+\left|\frac{\partial u}{\partial x_3}\right|^2-|\nabla_{\boldsymbol{x}}u|^2\right\}\mathrm{d}s\le 2\kappa\,\mathrm{Im}\int_{\Gamma_H}\bar{u}\frac{\partial u}{\partial x_3}\mathrm{d}s. \tag{3.22}$$

To proceed further we make a further application of the divergence theorem in S_H to obtain that

$$\mathrm{Im}\int_{\Gamma_H}\bar{u}\frac{\partial u}{\partial x_3}\mathrm{d}s=\mathrm{Im}\int_{\Gamma}\bar{u}\frac{\partial u_+}{\partial \nu}\mathrm{d}s, \tag{3.23}$$

by arguing exactly as we did to get (3.10). Combining (3.19), (3.22) and (3.23), and noting (3.14), we see that

$$\frac{1}{\tilde{L}}\int_{\Gamma}\left|\frac{\partial u_+}{\partial \nu}\right|^2\mathrm{d}s\le\int_{\Gamma}\nu_3\left|\frac{\partial u_+}{\partial \nu}\right|^2\mathrm{d}s\le\int_{\Gamma}|\nabla_T u|^2\mathrm{d}s+\frac{2L}{\tilde{L}}\int_{\Gamma}|\nabla_T u|\left|\frac{\partial u_+}{\partial \nu}\right|\mathrm{d}s+2\kappa\int_{\Gamma}|u|\left|\frac{\partial u_+}{\partial \nu}\right|\mathrm{d}s.$$

Applying Cauchy–Schwarz and then applying (3.16) twice, first with $\epsilon=3$, $a=L\|\nabla_T u\|_{L^2(\Gamma)}$ and $b=\|\partial u_+/\partial\nu\|_{L^2(\Gamma)}$r and then with $\epsilon=3\tilde{L}$, $a=\kappa\|u\|_{L^2(\Gamma)}$ and $b=\|\partial u_+/\partial\nu\|_{L^2(\Gamma)}$, it follows that

$$\frac{1}{3\tilde{L}}\int_{\Gamma}\left|\frac{\partial u_+}{\partial \nu}\right|^2\mathrm{d}s\le\left(1+\frac{3L^2}{\tilde{L}}\right)\int_{\Gamma}|\nabla_T u|^2\,\mathrm{d}s+3\kappa^2\tilde{L}\int_{\Gamma}|u|^2\,\mathrm{d}s. \tag{3.24}$$

Bounding the right-hand side using (3.11) and (3.18), we find that

$$\left\|\frac{\partial u_+}{\partial \nu}\right\|_{L^2(\Gamma)}\le\left(\frac{3\kappa^2\tilde{L}}{\eta^2}[5\tilde{L}+6L^2]+6(\tilde{L}+3L^2)^2\right)^{1/2}\|g\|_{L^2(\Gamma)}. \tag{3.25}$$

Putting this together with (3.9) and (3.12), we conclude that

$$\|\varphi\|_{L^2(\Gamma)}\le 2B\|g\|_{L^2(\Gamma)}, \tag{3.26}$$

where B is defined by (3.4), concluding the proof. ■

The lemma we have just proved is the major part of showing that A' is invertible, that is, of establishing the following theorem.

Theorem 3.4. *A' and A are invertible on $L^2(\Gamma)$, with*

$$\|A^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}=\|A'^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}\le B, \tag{3.27}$$

where $B=B(L,\kappa/\eta)$ is defined by (3.4).

Proof. We have remarked already that A and A' are invertible together, and, that if they are both invertible, then their norms are equal.

We show first that A and A' are invertible in the case when $f\in C^\infty(\mathbb{R}^2)$. Note that, if $f\in C^\infty(\mathbb{R}^2)$, it follows from lemma 3.2 that the bound (3.27) holds if A' is invertible. To prove that A and A' are invertible, define f_ϵ, for $0\le\epsilon\le 1$, by

$$f_\epsilon(\boldsymbol{x})=\epsilon f(\boldsymbol{x})+(1-\epsilon)f^+,\quad \boldsymbol{x}\in\mathbb{R}^2.$$

Then $f_1=f$, so that $\Gamma^{f_1}=\Gamma^f$, while Γ^{f_0} is the flat plane Γ_{f^+}. Denoting A and A' by A_f and A'_f, to indicate their dependence on f, we associate A_f and A'_f with the elements $\tilde{A}_f = I_f A_f I_f^{-1}$ and $\tilde{A}'_f = I_f A'_f I_f^{-1}$ of the space $BL(L^2(\mathbb{R}^2))$ of bounded linear operators on $L^2(\mathbb{R}^2)$. Here, I_f is the isomorphism defined by (2.15). Note that I_{f_ϵ} and $I_{f_\epsilon}^{-1}$ are bounded, uniformly in ϵ, for $0 \le \epsilon \le 1$, i.e.

$$c_\epsilon := \|I_{f_\epsilon}\| \, \|I_{f_\epsilon}^{-1}\| \le C_I, \quad 0 \le \epsilon \le 1,$$

for some constant $C_I \ge 1$. We note that A_{f_0} is invertible by theorem 2.3. We will show now by a simple homotopy argument that $A_{f_1} = A_f$ is also invertible.

Note that, for $0 \le \epsilon \le 1$, the Lipschitz constant of f_ϵ is not larger than L, the Lipschitz constant of f. Thus, if A_{f_ϵ} is invertible, the bound (3.27) holds on $A_{f_\epsilon}^{-1}$ with $B = B(L, \kappa/\eta)$, so that

$$\|\tilde{A}_{f_\epsilon}^{-1}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \le c_\epsilon \|A_{f_\epsilon}^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \le C_I B. \tag{3.28}$$

Since the mapping

$$[0,1] \to BC^{1,\alpha}(\mathbb{R}^2), \quad \epsilon \mapsto f_\epsilon, \tag{3.29}$$

is continuous, it follows from theorem 2.4 that the mapping

$$[0,1] \to BL(L^2(\mathbb{R}^2)), \quad \epsilon \mapsto \tilde{A}_{f_\epsilon}, \tag{3.30}$$

is continuous, in fact uniformly continuous since $[0,1]$ is compact. Thus, there exists $N \in \mathbb{N}$ such that

$$\|\hat{A}_j - \hat{A}_{j-1}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} < (C_I B)^{-1}, \tag{3.31}$$

for $j = 1, \dots, N$, where $\hat{A}_j$ is an abbreviation for $\tilde{A}_{f_\epsilon}$ when $\epsilon = j/N$. But, by standard operator perturbation results, if $\hat{A}_{j-1}$ is invertible, so that the bound (3.28) applies to $\hat{A}_{j-1}$, then (3.31) ensures that $\hat{A}_j$ is also invertible. Since $\hat{A}_0 = \tilde{A}_{f_0}$ is invertible, by induction, $\tilde{A}_f = \hat{A}_N$ is invertible, so A_f is invertible.

We have shown that A and A' are invertible whenever $f \in C^\infty(\mathbb{R}^2)$. We finish the proof by using this result and the explicit bound (3.27) to show invertibility in the more general case when we have only that $f \in BC^{1,\alpha}(\mathbb{R}^2)$, for some $\alpha \in (0,1]$. Choose a non-negative function $\chi \in C^\infty(\mathbb{R}^2)$ with the property that $\chi(\boldsymbol{x}) = 0$, $|\boldsymbol{x}| > 1$, and

$$\int_{\mathbb{R}^2} \chi(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = 1.$$

Define $\chi_\epsilon \in C^\infty(\mathbb{R}^2)$, for $\epsilon > 0$, by $\chi_\epsilon(\boldsymbol{x}) := \epsilon^{-2}\chi(\boldsymbol{x}/\epsilon)$, $\boldsymbol{x} \in \mathbb{R}^2$, so that $\chi_\epsilon(\boldsymbol{x}) = 0$, $|\boldsymbol{x}| > \epsilon$, and

$$\int_{\mathbb{R}^2} \chi_\epsilon(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = 1, \quad \epsilon > 0.$$

Next, define $f_\epsilon \in C^\infty(\mathbb{R}^2)$, for $\epsilon > 0$, by

$$f_\epsilon(\boldsymbol{x}) := \int_{\mathbb{R}^2} \chi_\epsilon(\boldsymbol{x} - \boldsymbol{y}) f(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} = \int_{\mathbb{R}^2} f(\boldsymbol{x} - \boldsymbol{y}) \chi_\epsilon(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}, \quad \boldsymbol{x} \in \mathbb{R}^2,$$

which implies that

$$\nabla f_\epsilon(\boldsymbol{x}) = \int_{\mathbb{R}^2} \chi_\epsilon(\boldsymbol{x} - \boldsymbol{y}) \nabla f(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}, \quad \boldsymbol{x} \in \mathbb{R}^2.$$

Since $f \in BC^{1,\alpha}(\mathbb{R}^2)$ and $f^- < f(\boldsymbol{x}) < f^+$, it is a straightforward calculation to see that the same is true for f_ϵ, for every $\epsilon > 0$. In particular, where $E := \|f\|_{BC^{1,\alpha}(\mathbb{R}^2)}$, it holds that

$$|\nabla f_\epsilon(\boldsymbol{x}) - \nabla f_\epsilon(\boldsymbol{y})| \leq E|\boldsymbol{x}-\boldsymbol{y}|^\alpha, \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2.$$

Further, choosing β with $0 < \beta < \alpha$ and setting $\gamma = \alpha - \beta$, we have that

$$|\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y}) - (\nabla f_\epsilon(\boldsymbol{x}) - \nabla f_\epsilon(\boldsymbol{y}))| \leq 2E \min(\epsilon^\alpha, |\boldsymbol{x}-\boldsymbol{y}|^\alpha) \leq 2E\epsilon^\gamma |\boldsymbol{x}-\boldsymbol{y}|^\beta.$$

With the help of this inequality it follows that

$$\|f - f_\epsilon\|_{BC^{1,\beta}(\mathbb{R}^2)} \to 0, \quad \epsilon \to 0, \tag{3.32}$$

and then from theorem 2.4 that

$$\|\tilde{A}_f - \tilde{A}_{f_\epsilon}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \to 0, \quad \epsilon \to 0. \tag{3.33}$$

Further, since $f_\epsilon \in C^\infty(\mathbb{R}^2)$, A_{f_ϵ} is invertible. It is a straightforward calculation to see that the Lipschitz constant of f_ϵ is not greater than that of f. Thus,

$$c_\epsilon := \|I_{f_\epsilon}\| \, \|I_{f_\epsilon}^{-1}\| \leq C_I, \quad 0 \leq \epsilon \leq 1,$$

for some constant $C_I \geq 1$, and the bound (3.27) holds on $A_{f_\epsilon}^{-1}$ with $B = B(L, \kappa/\eta)$, i.e.

$$\|A_{f_\epsilon}^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq B, \quad \epsilon > 0, \tag{3.34}$$

so that

$$\|\tilde{A}_{f_\epsilon}^{-1}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \leq c_\epsilon \|A_{f_\epsilon}^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C_I B, \quad \epsilon > 0.$$

Choosing ϵ such that $\|\tilde{A}_f - \tilde{A}_{f_\epsilon}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} < (C_I B)^{-1}$, we deduce from standard operator perturbation results that $\tilde{A}_f$, and hence A_f and A'_f, are invertible. Also, by (3.33),

$$\|A_f^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} = \|I_f^{-1}\tilde{A}_f^{-1} I_f\|_{L^2(\Gamma)\to L^2(\Gamma)} = \lim_{\epsilon\to 0} \|I_f^{-1}\tilde{A}_{f_\epsilon}^{-1} I_f\|_{L^2(\Gamma)\to L^2(\Gamma)}.$$

Further, (3.32) implies that $\|I_{f_\epsilon}^{-1} I_f\| \to 1$ and $\|I_f^{-1} I_{f_\epsilon}\| \to 1$, as $\epsilon \to 0$, so that, using (3.34),

$$\|A_f^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} = \lim_{\epsilon\to 0} \|I_f^{-1} I_{f_\epsilon} A_{f_\epsilon}^{-1} I_{f_\epsilon}^{-1} I_f\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq \limsup_{\epsilon\to 0} \|I_f^{-1} I_{f_\epsilon}\| B \|I_{f_\epsilon}^{-1} I_f\| = B,$$

i.e. (3.27) holds. ■

In the special case when Γ is flat, i.e. $\Gamma = \Gamma_h$, for some $h > 0$, theorem 3.4 predicts that

$$\|A^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq \frac{1}{2}\left[1 + \left(\frac{15\kappa^2}{\eta^2} + 6\right)^{1/2}\right]. \tag{3.35}$$

We will consider this special case with more precise tools in §4, showing that

$$\|A^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq \max\left(1, \frac{\kappa}{2\eta}\right), \tag{3.36}$$

with equality holding in the case $\eta \geq \kappa/2$ and in the limit $\kappa h \to \infty$. (Interestingly, this is precisely the bound conjectured by Giebermann (1997) for the inverse of an analogous integral operator in the case when Γ is a sphere, based on a study of asymptotics of eigenvalues.) It is encouraging that the bound (3.35), while necessarily larger than the sharp bound (3.36), is larger by at most a factor $(1+\sqrt{66})/2 < 5$, for all η/κ. This gives hope that the bound (3.27) is fairly sharp in the general case when Γ is not flat.

In §4, our calculations will lead to the conclusion that, when Γ is flat, the choice of coupling parameter $\eta = \kappa/2$ is almost optimal in terms of minimizing the condition number of A. Using the triangle inequality that $(a^2+b^2)^{1/2} \leq a+b$, for $a, b \geq 0$, we can simplify the bound (3.27) on A^{-1} when $\eta = \kappa/2$, obtaining that

$$\begin{aligned}\|A^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} &\leq \frac{1}{2}\left(1 + (66\tilde{L}^2 + 108\tilde{L}L^2 + 54L^4)^{1/2}\right) < \frac{1}{2}(1 + 9\tilde{L} + 8L^2)\\ &\leq \frac{1}{2}(10 + 9L + 8L^2) \leq 5(1+L)^2. \end{aligned} \tag{3.37}$$

Finally, we note that, again using the triangle inequality, we can simplify (at the cost of a little sharpness) the bound (3.27) in the general case. From (3.27) it follows by the triangle inequality that

$$\begin{aligned}\|A^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} &\leq \frac{1}{2}\left(1 + \sqrt{6}\tilde{L} + 3\sqrt{6}L^2\right) + \frac{\kappa}{2\eta}\left(\sqrt{15}\tilde{L} + 3\sqrt{2}L\sqrt{\tilde{L}}\right)\\ &\leq \frac{1}{2}\left(1 + \sqrt{6} + \sqrt{6}L + 3\sqrt{6}L^2\right)\\ &\quad + \frac{\kappa}{2\eta}\left(\sqrt{15} + (\sqrt{15} + 3\sqrt{2})L + 3\sqrt{2}L^{3/2}\right),\end{aligned}$$

from which the bound (1.5) follows.

4. Minimizing the condition number when Γ is flat

In this section, we consider the special case when Γ is flat, i.e. $\Gamma = \Gamma_h$ for some $h > 0$, aiming to compute the condition number of $A = I + K - \mathrm{i}\eta S$ explicitly, and then to use the explicit results we obtain to select η so as to approximately minimize A.

In the case $\Gamma = \Gamma_h$ the kernels of K and S only depend on the difference $\boldsymbol{x} - \boldsymbol{y}$ and thus, identifying Γ_h with $\mathbb{R}^2$, the operators are convolution operators on $L^2(\mathbb{R}^2)$. Explicitly (see Chandler-Wilde *et al.* 2006), we can write the kernel of the double-layer potential operator as $P_h(\boldsymbol{x} - \boldsymbol{y})$, where $P_h(\boldsymbol{y}) := p_h(|\boldsymbol{y}|)$ and

$$p_h(r) := -\frac{\mathrm{i}\kappa h}{\pi} \frac{\mathrm{e}^{\mathrm{i}\kappa\sqrt{r^2+4h^2}}}{r^2+4h^2} + \frac{h}{\pi} \frac{\mathrm{e}^{\mathrm{i}\kappa\sqrt{r^2+4h^2}}}{(r^2+4h^2)^{3/2}}, \quad r > 0.$$

The kernel of the single-layer potential operator is $Q_h(\boldsymbol{x} - \boldsymbol{y})$, where $Q_h(\boldsymbol{y}) := q_h(|\boldsymbol{y}|)$ and

$$q_h(r) := \frac{1}{2\pi}\left\{\frac{\mathrm{e}^{\mathrm{i}\kappa r}}{r} - \frac{\mathrm{e}^{\mathrm{i}\kappa\sqrt{r^2+4h^2}}}{\sqrt{r^2+4h^2}}\right\}, \quad r > 0.$$

Hence, the integral equation (1.3) reduces to the convolution integral equation

$$\varphi(\boldsymbol{x}) + \int_{\mathbb{R}^2} R_h(\boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{d}\boldsymbol{y} = 2g(\boldsymbol{x}), \quad \boldsymbol{x}\in\mathbb{R}^2, \tag{4.1}$$

where $R_h := P_h - \mathrm{i}\eta Q_h$. Defining the Fourier transform operator $\mathcal{F}$ by (3.20), equation (4.1) can be rewritten, using standard results on convolution operators, as

$$\varphi + 2\pi\mathcal{F}^{-1}((\mathcal{F}R_h)(\mathcal{F}\varphi)) = 2g.$$

For more details see Chandler-Wilde *et al.* (2006). Thus, in the special case $\Gamma = \Gamma_h$,

$$A\psi = \mathcal{F}^{-1}(M_h(\mathcal{F}\psi)), \quad \psi \in L^2(\mathbb{R}^2), \tag{4.2}$$

where $M_h := 1 + 2\pi\mathcal{F}R_h$.

In Chandler-Wilde *et al.* (2006), we have computed the function M_h explicitly, finding that $M_k(\boldsymbol{k}) = K(|\boldsymbol{k}|)$, for almost all $\boldsymbol{k}\in\mathbb{R}^2$, where

$$K(k) := F(h\sqrt{k^2-\kappa^2}), \quad k \geq 0,$$

with $\sqrt{k^2-\kappa^2} = -\mathrm{i}\sqrt{\kappa^2-k^2}$ for $\kappa > k$ and

$$F(z) := 1 + \mathrm{e}^{-2z} - \frac{\mathrm{i}h\eta}{z}(1-\mathrm{e}^{-2z}). \tag{4.3}$$

Since $\mathcal{F}$ is an isometric isomorphism on $L^2(\mathbb{R}^2)$, we see that

$$\|A\|_{L^2(\Gamma)\to L^2(\Gamma)} = \operatorname{ess}\sup_{\boldsymbol{k}\in\mathbb{R}^2} |M_h(\boldsymbol{k})| = \sup_{k\geq 0}|K(k)|, \tag{4.4}$$

and

$$\|A^{-1}\|^{-1}_{L^2(\Gamma)\to L^2(\Gamma)} = \operatorname{ess}\inf_{\boldsymbol{k}\in\mathbb{R}^2} |M_h(\boldsymbol{k})| = \inf_{k\geq 0}|K(k)|, \tag{4.5}$$

so that

$$\operatorname{cond} A = \frac{\sup_{k\geq 0}|K(k)|}{\inf_{k\geq 0}|K(k)|}. \tag{4.6}$$

Now, as k increases from 0 to κ to ∞, $h\sqrt{k^2-\kappa^2}$ moves in the complex plane from $-\mathrm{i}\kappa h$ to 0 to ∞. Thus, defining

$$G(t) := \begin{cases} |F(t)|^2, & t \geq 0, \\ |F(\mathrm{i}t)|^2, & -\kappa h \leq t < 0, \end{cases}$$

we see that

$$\left(\sup_{k\geq 0}|K(k)|\right)^2 = \sup_{t\geq -\kappa h} G(t), \quad \left(\inf_{k\geq 0}|K(k)|\right)^2 = \inf_{t\geq -\kappa h} G(t). \tag{4.7}$$

Moreover, for $t \geq 0$,

$$G(t) = (1 + \mathrm{e}^{-2t})^2 + \frac{h^2\eta^2}{t^2}(1-\mathrm{e}^{-2t})^2,$$

which is decreasing on $t\geq 0$, from the value $G(0)=4+4h^2\eta^2$ to the value $\lim_{t\to\infty} G(t)=1$. For $-\kappa h\leq t\leq 0$,

$$G(t) = (1+\cos 2t - \frac{h\eta}{t}(1-\cos 2t))^2 + (\sin 2t + \frac{h\eta}{t}\sin 2t)^2$$
$$= 4\cos^2 t + 4h^2\eta^2\frac{\sin^2 t}{t^2}.$$

Since $|\sin t|\leq |t|$, $t<0$, we see that $G(t)\leq G(0)$ for $-\kappa h\leq t<0$ so that, applying (4.4) and (4.7),

$$\|A\|_{L^2(\Gamma)\to L^2(\Gamma)} = \left(\sup_{t\geq -\kappa h} G(t)\right)^{1/2} = 2\sqrt{1+h^2\eta^2}.$$

If $\eta\geq\kappa/2$, then also $4h^2\eta^2/t^2\geq 1$ for $-\kappa h\leq t\leq 0$, so that, from (4.5) and (4.7),

$$\|A^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} = \left(\inf_{t\geq -\kappa h} G(t)\right)^{-1/2} = 1.$$

Thus,

$$\text{cond } A = 2\sqrt{1+h^2\eta^2}, \quad \eta\geq\kappa/2. \tag{4.8}$$

If $\eta<\kappa/2$ and $\kappa h\leq\pi/2$, we see that

$$\|A^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} = \left(\inf_{t\geq -\kappa h} G(t)\right)^{-1/2}$$
$$= \max\left(1, \frac{1}{2}\left(\cos^2\kappa h + \eta^2\frac{\sin^2\kappa h}{\kappa^2}\right)^{-1/2}\right) \leq \frac{\kappa}{2\eta}, \tag{4.9}$$

so that

$$\text{cond } A = 2\max\left(1, \frac{1}{2}\left(\cos^2\kappa h + \eta^2\frac{\sin^2\kappa h}{\kappa^2}\right)^{-1/2}\right)\sqrt{1+h^2\eta^2}, \tag{4.10}$$

and so

$$2\sqrt{1+h^2\eta^2}\leq \text{cond } A\leq \frac{\kappa}{\eta}\sqrt{1+h^2\eta^2}. \tag{4.11}$$

Since, for $\eta<\kappa/2$ and $-\kappa h\leq t<0$ it holds that

$$G(t) = 4\left(1-\frac{h^2\eta^2}{t^2}\right)\cos^2 t + \frac{4h^2\eta^2}{t^2}\geq\frac{4\eta^2}{\kappa^2}, \tag{4.12}$$

the bounds (4.9) and (4.11) hold whenever $\eta<\kappa/2$. However, if $\kappa h\geq\pi/2$ we can sometimes sharpen the lower part of the bound in (4.11). Precisely, since $\cos t$ vanishes at some point in $[\tilde{t}, \kappa h]$, where $\tilde{t} := \max(\pi/2, \kappa h-\pi)$, it follows from (4.12) that

$$\inf_{-\kappa h\leq t\leq 0} G(t)\leq\frac{4h^2\eta^2}{\tilde{t}^2} = \frac{4\eta^2(H(kh))^2}{\kappa^2},$$

where

$$H(s) := \frac{s}{\max(\pi/2, s-\pi)}, \quad s \geq \pi/2.$$

Thus, for $\eta < \kappa/2$ and $\kappa h \geq \pi/2$,

$$\max\left(1, \frac{\kappa}{2H(\kappa h)\eta}\right) \leq \|A^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq \frac{\kappa}{2\eta},$$

so that

$$\max\left(2, \frac{\kappa}{H(\kappa h)\eta}\right)\sqrt{1+h^2\eta^2} \leq \text{cond } A \leq \frac{\kappa}{\eta}\sqrt{1+h^2\eta^2}. \tag{4.13}$$

Note that, in the above inequality, the ratio of the upper to the lower bound on cond A is

$$\min\left(\frac{\kappa}{2\eta}, H(\kappa h)\right) \leq H(\kappa h) \leq \sup_{s \geq \pi/2} H(s) = 3.$$

Further, the lower and upper bounds are asymptotically equal in either of the limits $\eta \to \kappa/2$ or $\kappa h \to \infty$.

Having computed cond A exactly for $\eta \geq \kappa/2$ and for $\eta < \kappa/2$ with $\kappa h < \pi/2$, and having achieved fairly sharp upper and lower bounds in the other cases, we turn to selecting η to approximately minimize the condition number. From (4.8) and (4.11) we see that cond $A = T(\eta)$ for $\eta \geq \kappa/2$, and that cond $A \leq T(\eta)$ for $\eta < \kappa/2$, where

$$T(\eta) := \begin{cases} 2\sqrt{1+h^2\eta^2}, & \eta \geq \kappa/2, \\ \kappa\sqrt{h^2+\eta^{-2}}, & \eta < \kappa/2. \end{cases}$$

The choice $\eta = \kappa/2$ minimizes $T(\eta)$ on $\eta > 0$. We shall see that this choice comes close to also minimizing cond A. For $s > 0$, let $C(s)$ denote the value of cond A, when $\eta = s$. Then, for $\kappa h < \pi/2$, from (4.8) and (4.11),

$$\frac{C(\kappa/2)}{C(\eta)} \leq \sqrt{1+\kappa^2h^2/4} \leq \sqrt{1+\pi^2/16}, \quad \eta > 0.$$

For $\kappa h \geq \pi/2$ and $\eta < \kappa/2$, from (4.8) and (4.13),

$$\frac{C(\kappa/2)}{C(\eta)} \leq \frac{H(\kappa h)\sqrt{4+h^2\kappa^2}}{\sqrt{\kappa^2/\eta^2+h^2\kappa^2}} \leq H(\kappa h),$$

and this bound holds also for $\eta \geq \kappa/2$ by (4.8). Thus, if we extend the definition of H from $[\pi/2, \infty)$ to $(0, \infty)$ by setting $H(s) := \sqrt{1+\pi^2/16} \approx 1.27$ for $0 < s < \pi/2$, we see that

$$\frac{C(\kappa/2)}{\inf_{\eta>0} C(\eta)} \leq H(\kappa h) \leq 3. \tag{4.14}$$

In particular, since $H(\pi/2) = 1$ and $H(s) \to 1$ as $s \to \infty$, this bound shows that the choice $\eta = \kappa/2$ is optimal when $\kappa h = \pi/2$ and in the limit $\kappa h \to \infty$.

Although, for all values of κ and h, $\eta = \kappa/2$ is almost optimal in terms of minimizing cond A, and is exactly optimal in the limit $\kappa h \to \infty$, it should be noted that a range of values of η give almost as small a condition number. Precisely, from (4.8) and (4.11) we see that, for every $\Omega > 1$,

$$C(\eta) \leq \Omega\, C(\kappa/2),$$

if

$$\Omega^{-1}\left(1 + \frac{\kappa^2 h^2(\Omega^2 - 1)}{4\Omega^2}\right)^{-1/2} \leq \frac{2\eta}{\kappa} \leq \Omega\left(1 + \frac{4(\Omega^2 - 1)}{\Omega^2 \kappa^2 h^2}\right)^{1/2}.$$

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