BVP and BIE Formulations for Scattering by Fractal Screens

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Abstract

There are various formulations as BVPs or BIEs (boundary integral equations) for screen scattering problems in acoustics, all equivalent when the screen occupies a bounded open Lipschitz subset of the plane. Motivated by applications in electromagnetics and ultrasonics we explore what happens when the screen is less regular, in particular fractal or with fractal boundary. The standard formulations divide into an infinite family of well-posed BVP and equivalent BIE formulations, with infinitely many distinct solutions. We use "limiting geometry" arguments to select physically appropriate solutions, and illustrate numerically the surprising new effects that arise.

Keywords: Fractal, Helmholtz Equation, scattering

1 Introduction

We consider time-harmonic acoustic scattering problems modelled by the Helmholtz equation

$$\Delta u + k^2 u = 0, \tag{1}$$

where $k > 0$. Our focus is on scattering by thin planar screens in $\mathbb{R}^n$ ($n = 2$ or 3), so that the domain in which (1) holds is $D := \mathbb{R}^n \setminus \overline{\Gamma}$, where Γ, the *screen*, is a bounded subset of the hyperplane $\Gamma_\infty := \{\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n : x_n = 0\}$, and the compact set $\overline{\Gamma}$ is its closure. As usual, the complex-valued function u can be interpreted physically as the (total) acoustic pressure field, and we write u as $u = u^i + u^s$, where u^i is the incident field chosen to be the plane wave

$$u^i(\mathbf{x}) = \exp(ik\mathbf{d} \cdot \mathbf{x})$$

where $\mathbf{d}$ is a unit vector, the direction of incidence. The *scattered field* $u^s := u - u^i$ is assumed to satisfy (1) and the standard Sommerfeld radiation condition. For brevity we restrict attention to *sound-hard* boundary conditions, assuming that

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \tag{2}$$

on the screen in some appropriate sense, where $\mathbf{n}$ is the unit normal pointing in the x_n direction. (For generalisations to other incident fields and a treatment of sound soft scattering see [4].)

This is a long-standing scattering problem, its mathematical study dating back at least to [10], and it is well-known [4, 14] that, for arbitrary bounded $\Gamma \subset \Gamma_\infty$, this problem is well-posed (and the solution depends only on the closure $\overline{\Gamma}$) if the boundary condition is understood in the standard weak sense that $u \in W_2^{1,\mathrm{loc}}(D)$ and

$$\int_D (v\Delta u + \nabla v \cdot \nabla u)\, \mathrm{d}x = 0, \quad v \in W_2^{1,\mathrm{comp}}(D). \tag{3}$$

In the standard case that Γ is a (relatively) open subset of Γ_∞ that is Lipschitz or smoother, the alternative, *classical formulation*, dating to the late 40s [1], imposes the boundary conditions (2) in a classical sense, and additionally imposes "edge conditions" requiring locally finite energy, that u and ∇u are square integrable in some neighbourhood of $\partial\Gamma$. Equivalently, one can formulate a BVP for u^s in a Sobolev space setting, seeking $u^s \in W_2^{1,\mathrm{loc}}(D)$ satisfying (1) and the radiation condition, and imposing the boundary condition (2) in a trace sense, requiring that the Neumann traces on Γ_∞, $\partial_{\mathbf{n}}^{\pm} u^s$, satisfy $(\partial_{\mathbf{n}}^{\pm} u^s)|_\Gamma = g \in H^{-1/2}(\Gamma)$, where $g := -(\partial_{\mathbf{n}}^{\pm} u^i)|_\Gamma$ (see, e.g., [11]). Finally, it is well-known (e.g. [11]) that for Lipschitz Γ one can reformulate this BVP as the BIE

$$T[u] = g. \tag{4}$$

In this equation the unknown is the jump across the screen in u, $[u] \in \widetilde{H}^{1/2}(\Gamma)$, and the isomorphism $T : \widetilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is a hypersingular boundary integral operator (BIO). Here $\widetilde{H}^s(\Gamma) \subset H^s(\Gamma_\infty)$, for $s \in \mathbb{R}$, denotes the closure in $H^s(\Gamma_\infty)$ of $C_0^\infty(\Gamma)$. As pointed out in [3], (4) is well-posed for arbitrary open Γ. The total field u is given in terms of $[u]$ by

$$u(\mathbf{x}) = u^i(\mathbf{x}) + \mathcal{D}[u](\mathbf{x}), \quad \mathbf{x} \in D, \tag{5}$$

where $\mathcal{D} : H^{1/2}_{\overline{\Gamma}} \to C^2(D) \cap W^{1,\mathrm{loc}}_2(D)$ is the standard double-layer potential operator, and $H^{1/2}_{\overline{\Gamma}} \subset H^{1/2}(\Gamma_\infty)$ is the closed subspace of those $\phi \in H^{1/2}(\Gamma_\infty)$ that are supported in $\overline{\Gamma}$. We note that $\tilde{H}^{1/2}(\Gamma) \subset H^{1/2}_{\overline{\Gamma}}$, that these spaces coincide if Γ is sufficiently regular, in particular if Γ is a C^0 open set, but that in general these spaces are distinct.

In this paper we explore what happens when the screen Γ is irregular, in particular fractal or with fractal boundary, this motivated by the use of planar screens with precisely these structures as antennae in electromagnetics and ultrasonics (e.g., [8, 9, 12]). We will see in §2 that the standard classical/Sobolev spaces formulations can be ill-posed, or if well-posed have solution different to the standard weak formulation. In §3 we will see that there exists, when the screen is sufficiently irregular, a whole family of well-posed BVP and BIE formulations with infinitely many distinct solutions. In §4 we discuss the selection of a correct solution by taking limits with respect to the geometry. In the last two sections we explore theoretically, illustrated by numerical computations, wave penetration through a zero-surface-area fractal "hole" in a sound hard screen.

We use throughout the notation S° to denote the (relative) interior of $S \subset \Gamma_\infty$. For Borel $S \subset \Gamma_\infty$ we will denote by $m(S)$ the $(n-1)$-dimensional (surface) Lebesgue measure of S, and by $\mathrm{cap}(S)$ the n-dimensional capacity of S defined as in [4]. For $s \in \mathbb{R}$ we will say that S is *s-null* if the only $\phi \in H^s(\Gamma_\infty)$ with $\mathrm{supp}(\phi) \subset S$ is $\phi = 0$. Importantly, it holds that S is $-1/2$-null if and only if $\mathrm{cap}(S) = 0$, and that $\mathrm{cap}(S) > 0$ if $\dim_{\mathrm{H}}(S) > n-2$, while $\mathrm{cap}(S) = 0$ if $\dim_{\mathrm{H}}(S) < n-2$, where $\dim_{\mathrm{H}}(S)$ denotes the Hausdorff dimension of S. For proofs of these statements and other characterisations of s-nullity see [7].

2 Equivalence and well-posedness (or not) of standard formulations

As we have observed above, the weak formulation of the scattering problem, with the boundary condition imposed in the sense (2), is well-posed for every bounded $\Gamma \subset \Gamma_\infty$. The (equivalent) classical and Sobolev space formulations above equation (4), however, are only well-posed for sufficiently regular Γ. Precisely:

Theorem 1 *[4] The classical and Sobolev space problems formulations are well-posed if and only if $\widetilde{H}^{1/2}(\Gamma^\circ) = H^{1/2}_{\overline{\Gamma}}$ and $\partial\Gamma$ is $-1/2$-null. In particular, these formulations are well-posed if Γ is a Lipschitz open set, or if Γ is C^0 except at a countable set of points that has only finitely many limit points, provided also $\partial\Gamma \subset \cup_{j=1}^\infty \partial\Omega_j$, with each $\Omega_j \subset \Gamma_\infty$ a Lipschitz open set. But these formulations are not well-posed if* $\mathrm{cap}(\partial\Gamma) > 0$, *in particular if* $\dim_{\mathrm{H}}(\partial\Gamma) > n-2$. *If these formulations are well-posed then they are equivalent to the weak formulation.*

Figure 1 illustrates this theorem for $n = 3$ with a (non-Lipschitz, indeed non-C^0) example of a screen for which the classical/Sobolev space formulations are well-posed, and an example (the Koch snowflake) where these formulations are not well-posed (because $\dim_{\mathrm{H}}(\partial\Gamma) = \log 4/\log 3 > 1$).

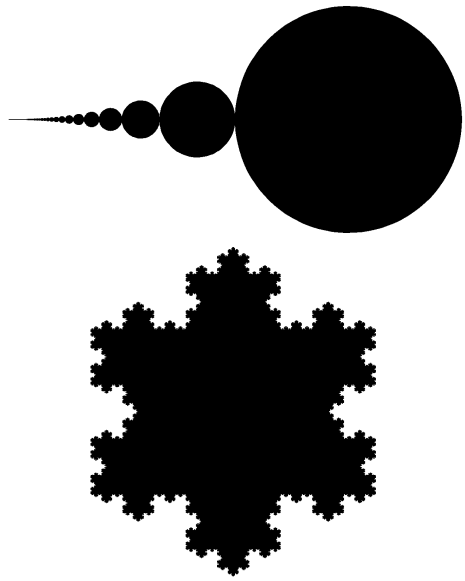

Figure 1: Example sound hard screens for which the classical/Sobolev space formulations are well-posed (top) and not well-posed (bottom).

The solution to the weak formulation always satisfies the classical and Sobolev space formulations, so that when well-posedness of these latter formulations fails it is because the standard edge conditions are insufficient to ensure uniqueness if the screen is sufficiently irregular. But these formulations become well-posed if the standard edge conditions are supplemented by the conditions in the following theorem.

Theorem 2 *[4] The classical and Sobolev space problems formulations, supplemented by the additional requirements that: a)* $[u] \in \widetilde{H}^{1/2}(\Gamma)$*; b)* $[\partial_{\mathbf{n}} u] = 0$*; are well-posed for every bounded open* $\Gamma \subset \Gamma_\infty$.

If $\widetilde{H}^{1/2}(\Gamma) \subsetneqq H^{1/2}_{\overline{\Gamma}}$, which holds in particular if $\dim_{\mathrm{H}}(\overline{\Gamma}^{\circ} \setminus \Gamma^\circ) > n-2$ [2, 4], then the equivalent classical/Sobolev space formulations supplemented by the additional constraints a) and b) are well-posed, but [4] their unique solution is different from the solution to the weak formulation, for almost all incident wave directions **d**. Indeed, we will see in the next section that when $\widetilde{H}^{1/2}(\Gamma) \subsetneqq H^{1/2}_{\overline{\Gamma}}$ there exists an infinite family of well-posed BVPs, intermediate between the weak and classical formulations.

3 An infinite family of BVP and BIE formulations

Recall that $H^{-1/2}(\Gamma_\infty)$ is (a unitary realisation of) the dual space of $H^{1/2}(\Gamma_\infty)$ through the duality pairing $\langle\cdot,\cdot\rangle$ on $H^{-1/2}(\Gamma_\infty) \times H^{1/2}(\Gamma_\infty)$ that extends the $L^2(\Gamma_\infty)$ inner product. Let V be a closed subspace of $H^{1/2}(\Gamma_\infty)$, in particular we will be interested in subspaces satisfying

$$\widetilde{H}^{1/2}(\Gamma^\circ) \subset V \subset H^{1/2}_{\overline{\Gamma}}. \qquad (6)$$

Let $V^a := \{\phi \in H^{-1/2}(\Gamma_\infty) : \langle \phi, \psi\rangle = 0 \text{ for all } \psi \in V\}$ be the annihilator of V, and let $V^* := (V^a)^\perp \subset H^{-1/2}(\Gamma_\infty)$, so that V^* is the natural unitary realisation of the dual space of V through the duality pairing that is the restriction of $\langle\cdot,\cdot\rangle$ to $V^* \times V$ [2]. Let $P : H^{-1/2}(\Gamma_\infty) \to V^*$ be orthogonal projection. Explicitly, $V^* = (\widetilde{H}^{-1/2}((\overline{\Gamma})^c))^\perp$ if $V = H^{1/2}_{\overline{\Gamma}}$, where ^{c} denotes complement in Γ_∞. Similarly, $V^* = (H^{-1/2}_{(\Gamma^\circ)^c})^\perp$ if $V = \widetilde{H}^{1/2}(\Gamma^\circ)$.

We can associate to each $V \subset H^{1/2}_{\overline{\Gamma}}$ a formulation $SN(V)$ of the scattering problem, this a physically sensible mathematical model if V is constrained by (6), and interesting as a numerical approximation when V is finite-dimensional. In this formulation $\mathcal{D}_1$ denotes the set of those $\chi \in C_0^\infty(\Gamma_\infty)$ that are $= 1$ in some neighbourhood of $\overline{\Gamma}$

Scattering Problem SN(V): *Find* $u \in C^2(D) \cap W_2^{1,\mathrm{loc}}(D)$ *such that: i)* (1) *holds in* D*; ii)* $u^s := u - u^i$ *satisfies the Sommerfeld radiation condition; iii)* $[u] \in V$*; iv)* $[\partial_{\mathbf{n}} u] = 0$*; v) the boundary condition* (2) *holds on* Γ *in the sense that* $P(\chi \partial^{\pm}_{\mathbf{n}} u) = 0$*, for every* $\chi \in \mathcal{D}_1$.

The choice of V in $SN(V)$ plays two roles: the larger V is the larger the space in which we constrain $[u]$ to lie, and simultaneously the stronger the sense in which we impose the boundary condition (2). In particular [4]: a) if $V = H^{1/2}_{\overline{\Gamma}}$ then $SN(V)$ is equivalent to the weak formulation with the boundary condition understood in the sense (3); and b), for every V satisfying (6) the boundary condition in the sense v) implies that $(\partial^{\pm}_{\mathbf{n}} u)|_{\Gamma^\circ} = 0$, indeed is equivalent to this condition if $V = \widetilde{H}^{1/2}(\Gamma^\circ)$.

Theorem 3 *[4]* $SN(V)$ *has exactly one solution, and this solution is a solution to the classical/Sobolev space formulation above* (4) *if* V *satisfies* (6)*, indeed* $SN(V)$ *is equivalent to the classical/Sobolev space formulation augmented by the conditions iii) and iv) if* $V = \widetilde{H}^{1/2}(\Gamma^\circ)$*. If* $V = H^{1/2}_{\overline{\Gamma}}$ *then the solution to* $SN(V)$ *coincides with the solution to the weak formulation with boundary condition in the sense* (3)*. If* $\widetilde{H}^{1/2}(\Gamma^\circ) = H^{1/2}_{\overline{\Gamma}}$ *then there is only one formulation* $SN(V)$ *satisfying* (6)*, but if* $\widetilde{H}^{1/2}(\Gamma^\circ) \subsetneqq H^{1/2}_{\overline{\Gamma}}$ *there are infinitely many (with cardinality that of the continuum) distinct formulations, and for almost all incident wave directions these formulations have infinitely many distinct solutions.*

To each $V \subset H^{1/2}_{\overline{\Gamma}}$ we can also associate a unique BIE formulation. To define the associated BIO, choose any bounded open set $\Gamma_\dagger \supset \overline{\Gamma}$, let $T_\dagger : \widetilde{H}^{1/2}(\Gamma_\dagger) \to H^{-1/2}(\Gamma_\dagger)$ be the standard hypersingular BIO on $\widetilde{H}^{1/2}(\Gamma_\dagger) \supset H^{1/2}_{\overline{\Gamma}} \supset V$, and define the hypersingular operator $T_V : V \to V^*$ by

$$T_V \phi = P E T_\dagger \phi, \quad \phi \in V,$$

where $E : H^{-1/2}(\Gamma_\dagger) \to H^{-1/2}(\Gamma_\infty)$ is the operator of minimum norm extension.

Theorem 4 *[4] For every* $V \subset H^{1/2}_{\overline{\Gamma}}$ *the hypersingular operator* $T_V : V \to V^*$ *is an isomorphism. Further,* u *satisfies* $SN(V)$ *if and only if* (5) *holds and* $[u] \in V$ *and satisfies, for some* $\chi \in \mathcal{D}_1$,

$$T_V[u] = -P(\chi \partial^{\pm}_{\mathbf{n}} u^i). \qquad (7)$$

Moreover, (7) *can be written equivalently in variational form as*

$$\langle T_V[u], v\rangle = -\langle \chi \partial^{\pm}_{\mathbf{n}} u^i, v\rangle, \quad v \in V.$$

4 Limiting geometry principles

If $\widetilde{H}^{1/2}(\Gamma^\circ) = H^{1/2}_{\overline{\Gamma}}$ the formulations $SN(V)$ that satisfy (6) collapse to a single formulation, equivalent to the standard weak formulation with boundary condition (3). So there is a single unique solution in this case. We note that $\widetilde{H}^{1/2}(\Gamma^\circ) = H^{1/2}_{\overline{\Gamma}}$ if Γ is a C^0 open set, and also if Γ is C^0 except at countably many points on $\partial\Gamma$, as long as this set has only finitely many limit points [2] (an example is the screen at the top of Figure 1).

On the other hand, if $\widetilde{H}^{1/2}(\Gamma^\circ) \subsetneqq H^{1/2}_{\overline{\Gamma}}$, there are infinitely many distinct solutions to the formulations $SN(V)$ by Theorem 3. We propose to select physically appropriate solutions by thinking of the screen Γ as $\lim_{j\to\infty}\Gamma_j$, with convergence in some appropriate sense and with each bounded $\Gamma_j \subset \Gamma_\infty$ satisfying $\widetilde{H}^{1/2}(\Gamma_j^\circ) = H^{1/2}_{\overline{\Gamma_j}}$. If the (well-defined) solution u_j, for scattering by Γ_j, converges to a limit u which satisfies $SN(V)$ for some V satisfying (6), we will say that $SN(V)$ is the correct formulation for scattering by Γ in this limit. This approach for selecting the correct formulation, which we term a *limiting geometry principle*, seems natural for the many fractal scatterers defined as the limit of a sequence of regular prefractals, and dates back, in the context of potential theory, to Wiener [13].

Given a bounded screen $\Gamma \subset \Gamma_\infty$ there are many different possible approximating sequences Γ_j, and many different senses in which Γ_j may converge to Γ, and correspondingly we expect many different formulations $SN(V)$ to be appropriate as particular limiting geometry solutions (LGSs) [4]. We will focus here on the following particular cases:

Definition 5 (LGS for an Open Screen) *If $\Gamma \subset \Gamma_\infty$ is bounded and open, we call the total field u a* LGS for the open screen Γ *if there exists a sequence $(\Gamma_j)_{j\in\mathbb{N}}$ of open subsets of Γ_∞ such that: (i) $\Gamma_1 \subset \Gamma_2 \subset \ldots$ and $\Gamma = \cup_{j=1}^\infty \Gamma_j$; (ii) for $j \in \mathbb{N}$, $\widetilde{H}^{1/2}(\Gamma_j) = H^{1/2}_{\overline{\Gamma_j}}$, so that the formulations $SN(V)$ satisfying* (6) *collapse to a single formulation with a well-defined unique solution u_j; (iii) for $\mathbf{x} \in D = \mathbb{R}^n \setminus \overline{\Gamma}$, $u(\mathbf{x}) = \lim_{j\to\infty} u_j(\mathbf{x})$.*

Definition 6 (LGS for a Closed Screen) *If $\Gamma \subset \Gamma_\infty$ is compact, call the total field u a* LGS for the closed screen Γ *if there exists a sequence $(\Gamma_j)_{j\in\mathbb{N}}$ of compact subsets of Γ_∞ such that: (i) $\Gamma_1 \supset \Gamma_2 \supset \ldots$ and $\Gamma = \cap_{j=1}^\infty \Gamma_j$; (ii) for $j \in \mathbb{N}$, $\widetilde{H}^{1/2}(\Gamma_j^\circ) = H^{1/2}_{\Gamma_j}$, so that the formulations $SN(V)$ satisfying* (6) *collapse to a single formulation with a well-defined unique solution u_j; (iii) for $\mathbf{x} \in D = \mathbb{R}^n \setminus \Gamma$, $u(\mathbf{x}) = \lim_{j\to\infty} u_j(\mathbf{x})$.*

The following characterises these LGSs in terms of the formulations $SN(V)$.

Theorem 7 *[4] For every bounded open screen Γ there exists a unique LGS u, and this is the unique solution of $SN(V)$ with $V = \widetilde{H}^{1/2}(\Gamma)$. Similarly, for every compact screen Γ there exists a unique LGS u, and this is the unique solution of $SN(V)$ with $V = H^{1/2}_\Gamma$.*

5 What differences between formulations are detectable in the scattered field?

Our first result is concerned with whether the incident field "sees" the screen, i.e., for which Γ and u^i it holds that $u^s \neq 0$.

Theorem 8 *[4] Suppose that u satisfies $SN(V)$ for some $V \subset H^{1/2}_{\overline{\Gamma}}$. Then $u = u^i$ (so that $u^s = 0$) if $\overline{\Gamma}$ is $1/2$-null. If $\overline{\Gamma}$ is not $1/2$-null, i.e., $H^{1/2}_{\overline{\Gamma}} \neq \{0\}$, and $\{0\} \neq V \subset H^{1/2}_{\overline{\Gamma}}$, then $u \neq u^i$ for almost all incident directions $\mathbf{d}$.*

$\overline{\Gamma}$ is $1/2$-null if $m(\overline{\Gamma}) = 0$, and clearly not $1/2$-null if Γ° is non-empty. There exist screens with $\Gamma^\circ = \emptyset$ and $m(\overline{\Gamma}) > 0$ with $\overline{\Gamma}$ not $1/2$-null [4, Example 9.3].

In the interesting case that $\Gamma^\circ = \emptyset$ and $\overline{\Gamma}$ is not 1/2-null, the above result implies that $SN(V_1)$ and $SN(V_2)$ have distinct solutions for almost all incident directions if $V_1 = H^{1/2}_{\overline{\Gamma}}$ and $V_2 = \widetilde{H}^{1/2}(\Gamma^\circ) = \{0\}$. The following theorem implies results of the same flavour for general screens Γ.

Theorem 9 *[4] Suppose that V_1 and V_2 are subspaces satisfying* (6) *and that u_j is the solution to $SN(V_j)$, for $j = 1, 2$. Then $u_1 = u_2$ if $\widetilde{H}^{1/2}(\Gamma^\circ) = H^{1/2}_{\overline{\Gamma}}$, while $u_1 \neq u_2$ for almost all incident directions $\mathbf{d}$ if $\widetilde{H}^{1/2}(\Gamma^\circ) \neq H^{1/2}_{\overline{\Gamma}}$ and $V_1 \neq V_2$. Further, $\widetilde{H}^{1/2}(\Gamma^\circ) \neq H^{1/2}_{\overline{\Gamma}}$ if $\overline{\Gamma} \setminus \overline{\Gamma^\circ}$ is $1/2$-null or if $\overline{\Gamma^\circ} \setminus \Gamma^\circ$ is $-1/2$-null, in particular if $\dim_{\mathrm{H}}(\overline{\Gamma^\circ} \setminus \Gamma^\circ) > n-2$.*

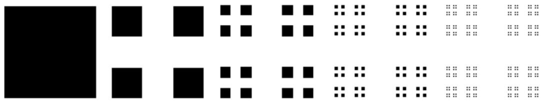

Figure 2: The first five prefractal approximations to the standard two-dimensional middle-third Cantor set (or Cantor dust).

6 Fractal apertures in a sound hard screen: theory and numerical results

As an illustration of the above results (cf. [2, Remark 4.6], [4, Example 9.5]) suppose that $n = 2$ or 3 and let

$$C_j := \left\{(\tilde{\mathbf{x}}, 0) : \tilde{\mathbf{x}} \in E_{j-1}^{n-1}\right\} \subset \Gamma_\infty,$$

with $\mathbb{R} \supset E_0 \supset E_1 \supset \ldots$ the standard recursive sequence generating the "middle-λ" Cantor set, for some $0 < \lambda < 1$ [5, Example 4.5]. Where $\alpha = (1-\lambda)/2 \in (0, 1/2)$, explicitly $E_0 = [0, 1]$, $E_1 = [0, \alpha] \cup [1-\alpha, 1]$, $E_2 = [0, \alpha^2] \cup [\alpha - \alpha^2, \alpha] \cup [1-\alpha, 1-\alpha+\alpha^2] \cup [1-\alpha^2, 1]$, ..., so that $E_j \subset \mathbb{R}$ is the closure of a Lipschitz open set that is the union of 2^j open intervals of length α^j, while $E_j^2 \subset \mathbb{R}^2$ is the closure of a Lipschitz open set that is the union of 4^j squares of side-length α^j. The limit $C := \cap_{j=1}^\infty C_j$ is the middle-λ Cantor set for $n = 2$, the corresponding Cantor dust for $n = 3$, with [5] $\dim_{\mathrm{H}}(C) = 2^{n-2} \log(2)/\log(1/\alpha)$. Figure 2 visualises $E_0^2, \ldots, E_4^2$ (i.e., $C_1, .., C_5$ for $n = 3$) in the classical "middle third" case $\alpha = \lambda = 1/3$.

Let $\Gamma_0 := C_1^\circ$, and, for $j \in \mathbb{N}$, let $\Gamma_j := \Gamma_0 \setminus C_j$, so that Γ_j is a Lipschitz open set. Let $\Gamma := \cup_{j=1}^\infty \Gamma_j = \Gamma_0 \setminus C$. Let u_0 denote the total field for scattering by the screen Γ_0 (just the unit interval for $n = 2$, a unit square for $n = 3$) which we compare with scattering by Γ, which is Γ_0 with the fractal "hole" C removed. Let u_j denote the solution for scattering by Γ_j and u the LGS for the open set Γ in the sense of Definition 5 which, by Theorem 7, is the solution to $SN(V)$ with $V = \widetilde{H}^{1/2}(\Gamma)$. Then $u_j \to u$ as $j \to \infty$ pointwise, and also [4] locally in W_2^1 norm on compact subsets of D.

Whether the "hole" C has an effect, i.e., whether $u \neq u_0$, depends on the dimension and on λ. The total fields u_0 and u are the solutions to the formulations $SN(V_1)$ and $SN(V_2)$, respectively, with $V_1 = H_{\overline{\Gamma}}^{1/2} = H_{\overline{\Gamma_0}}^{1/2} = \widetilde{H}^{1/2}(\Gamma_0)$ and $V_2 = \widetilde{H}^{1/2}(\Gamma)$. Thus, by Theorem 9, for almost all incident wave directions, $u \neq u_0$ if $\dim_{\mathrm{H}}(\Gamma_0 \setminus \Gamma) = \dim_{\mathrm{H}}(C) > n-2$, which holds if $n = 2$ or if $n = 3$ and $\alpha > 1/4$, so the hole has an effect in these cases. More detailed analysis [4] shows that $u = u_0$, i.e., the hole has no effect, if $n = 3$ and $\alpha \leq 1/4$.

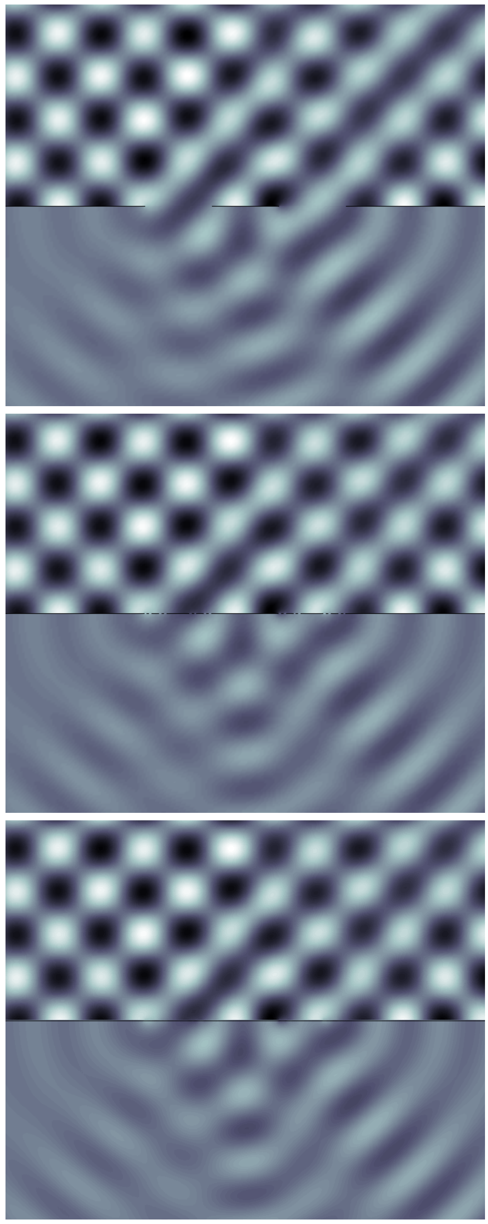

Figure 3: Reflection and transmission by a prefractal Cantor set aperture C_j in a sound hard screen: $j = 2$ (top); $j = 5$ (middle); $j = 8$ (bottom).

Figure 3 shows numerical results for $n = 2$ and $\alpha = \lambda = 1/3$ for a slightly modified problem of scattering by the fractal "hole" or aperture C in an infinite sound hard screen, which can be reduced to a (sound soft) screen scattering problem by a Babinet principle (e.g., [6]). Shown is $\Re u_j$, computed accurately by a BEM for $j = 2, 5, 8$, where u_j is the total field when $\mathbf{d} = (1, -1)/\sqrt{2}$ and the incident field has wavelength 0.3, so that $k = 20\pi/3 \approx 20.94$, and with the fractal hole C replaced by its prefractal approximation C_j. Our theoretical results predict that u_j approaches a limit that is different from the solution for a screen with no hole, i.e., a limit with a finite non-zero scattered field in the lower half-plane. This indeed seems to be the case, even though in this limit the hole has vanishing size: the total length of the components of C_j is $(2/3)^{j-1}$, which tends to zero as $j \to \infty$ and is ≈ 0.059 for $j = 8$.

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