Vertex Splitting and Connectivity Augmentation in Hypergraphs

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by

Benjamin Colin Cosh

Goldsmiths’ College, University of London
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Abstract

We consider problems of splitting and connectivity augmentation in hypergraphs. In a hypergraph $G = (V + s, E)$, to split two edges $su, sv$, is to replace them with a single edge $uv$. We are interested in doing this in such a way as to preserve a defined level of connectivity in $V$. The splitting technique is often used as a way of adding new edges into a graph or hypergraph, so as to augment the connectivity to some prescribed level. We begin by providing a short history of work done in this area. Then several preliminary results are given in a general form so that they may be used to tackle several problems.

We then analyse the hypergraphs $G = (V + s, E)$ for which there is no split preserving the local-edge-connectivity present in $V$. We provide two structural theorems, one of which implies a slight extension to Mader’s classical splitting theorem. We also provide a characterisation of the hypergraphs for which there is no such “good” split and a splitting result concerned with a specialisation of the local-connectivity function.

We then use our splitting results to provide an upper bound on the smallest number of size-two edges we must add to any given hypergraph to ensure that in the resulting hypergraph we have $\lambda(x, y) \geq r(x, y)$ for all $x, y \in V$, where $r$ is an integer valued, symmetric requirement function on $V^2$. This is the so-called “local-edge-connectivity augmentation problem” for hypergraphs. We also provide an extension to a Theorem of Szigeti, about augmenting to satisfy a requirement $r$, but using hyperedges. Next, in a result born of collaborative work with Zoltán Király from Budapest, we show that the local-connectivity augmentation problem is NP-complete for hypergraphs.

Lastly we concern ourselves with an augmentation problem that includes a locational constraint. The premise is that we are given a hypergraph $H = (V, E)$ with a bipartition $P = \{P_1, P_2\}$ of $V$ and asked to augment it with size-two edges, so that the result is $k$-edge-connected, and has no new edge contained in some $P_i$. We consider the splitting technique and describe the obstacles that prevent us forming “good” splits. From this we deduce results about which hypergraphs have a complete $Pk$-split. This leads to a minimax result on the optimal number of edges required and a polynomial algorithm to provide an optimal augmentation.
Dedication

This work is dedicated to my parents. Thank you for giving me the opportunity to do this, and for your love and support during the journey. I owe you more than I could ever pay, I can’t tell you how proud I am to be your son and, even though writing a book full of squiggles is perhaps a funny way to show it, I love you both very much.

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Chapter 1

Introduction and History

1.1 Introduction

What follows is the result of attempts to find a way into a combinatorial optimisation problem. Suppose that you are designing a communications network. Is it strong enough for you? Does it have enough built in redundancy to withstand any faults that may appear? If not, it will be necessary to “improve” the construction to meet your requirements. We call this improvement process augmentation and in practice (both mathematically and in real life) we will aim to make the augmentation as easy as possible. By easy, the business world would generally mean cheap and this can put constraints on the strengthening process. For instance, if we were trying to augment a computer network with additional cabling it could be more expensive to run a new link from the ground floor to a server on the ninth floor, than to connect two machines in the same office. However, we might not be interested in how much an augmentation costs - “Money is no object!” Similarly, it may be the case that all we have to use is a bundle of identical wires. In situations like this, we are only concerned with adding the smallest number of new cables. All of these problems fit the same general question.

*Given a starting network, what is the best way to improve it to meet a given strength condition?*

The concepts of graph theory provide an orderly way to deal with this type of question. We must specify a type of network and a mathematical measure of strength. We must also explicitly state the method of augmentation and what the word “best” means. Considerable effort has been made in the class of problems generated using permutations of the following collection of possibilities.
Networks

- Undirected graphs
- Directed graphs
- Mixed graphs
- Hypergraphs

Measures of Strength

- Edge Connectivity
- Vertex Connectivity

We say a graph is $n$-(vertex)-connected if the removal of fewer than $n$ vertices does not disconnect the graph. We also consider edge-connectivity, which is concerned with how many edges we must remove to disconnect. Menger’s classical theorem and its analogues for digraphs and for edge connectivity, (see [7], [15]), link this property with the number of disjoint paths between pairs of vertices. Loosely speaking, if a graph is $n$-connected there are at least $n$ disjoint routes between every pair of vertices. This makes connectivity an ideal measure of strength to use. It is possible, for instance, to begin with a graph and a requirement function that specifies the required number of disjoint routes between each and every pair of vertices. (For example, in one building, we may need ten disjoint routes between every pair of terminals in each office, but only require two routes if a pair of terminals are on different floors.) Questions of this type can be precisely formulated and tackled with the same methods as those used for less complex arrangements.

**Remark**: In general, vertex connectivity versions are much harder to tackle than their edge analogues. An intuitive reason for this is that much more damage is caused by the removal of a vertex. For instance, consider a graph with three vertices \( \{x, y, z\} \) and ten parallel edges from \( x \) to \( y \) and a further ten parallel edges from \( y \) to \( z \). To disconnect this graph we must remove at least ten edges. However, the removal of the single vertex \( y \) would be enough to break up the network. This difference is reflected by the fact that there are many more concrete results on the edge connectivity side of the problem.

**Method of Augmentation**

We are interested in the addition of new edges. These may be all directed or undirected, (or be a mixture of both in the case of mixed graphs). When dealing
with hypergraphs we can consider the addition of size-two edges, and/or larger hyperedges. Additional questions can be formed by restricting how many edges we may add at a given vertex, or between pairs of vertices.

**Definitions of Best**

The most common questions ask for the smallest number of new edges to satisfy the requirement. However, there are weighted edge versions, where the goal is to find an augmenting set of minimum weight/cost. Also, in a version of the hypergraph problem we consider adding a collection of hyperedges of minimum value - where we define the value of a collection of hyperedges as the sum of the sizes of the hyperedges in the set.

Using combinations of the above, we generate a whole class of problems, that are usually stated in a manner similar to the following.

*Given an undirected graph, what is the smallest number of new edges we must add to ensure that the resulting graph is k-edge-connected?*

We refer to this as the **undirected global edge-connectivity augmentation problem** - “global” because we are trying to ensure that there are at least \( k \) edge disjoint paths between every pair of vertices. This is a special case of the more general “local” problem, which specifies for each pair of vertices \( u,v \) a requirement function \( r(u,v) \), for the number of paths needed between \( u \) and \( v \).

A good deal of work has been done in this field. In this chapter we present a history of these problems and finish with an outline of the new results presented in this thesis. The historical survey will look at both undirected and directed graphs - discussing both edge and vertex connectivity versions of the problem. We then move on to some more recent questions involving mixed graphs and hypergraphs. We also consider the technique called **splitting** and its use in augmentation problems.

For this section we require a few pieces of notation. A graph \( G = (V,E) \) (or digraph) consists of two disjoint sets - vertices, \( V \), and edges, \( E \) - together with an incidence relation which associates a subset of \( V \) of size two, to each edge \( e \in E \). In a digraph the associated subset is an ordered pair - thus giving the edge a direction. In undirected graphs the degree of a vertex \( v \), denoted by \( d(v) \), is the number of edges incident with \( v \). The degree of a set, \( X \subseteq V \), denoted by \( d(X) \), is the number of edges with exactly one endpoint in \( X \). In digraphs we refer to the in-degree and out-degree of vertices and subsets of \( V \), denoting them by \( d_{in} \) and \( d_{out} \) respectively. A subpartition, \( \{X_1,X_2,\ldots,X_t\} \) of \( V \) is a collection of pairwise disjoint subsets of \( V \). We use \( \lambda(u,v) \) to denote the number of “different” paths between vertices \( u \) and \( v \). When working with an edge-connectivity problem, for “different” read edge-disjoint and when dealing
with vertex connectivity, we are refering to *internally disjoint* paths.

## 1.2 Undirected Graphs

### Edge Connectivity

This is the version of the graph augmentation question that has been dealt with most successfully. It is also the problem that is most closely associated with the hypergraph questions we tackle in this thesis.

**General Problem:** Let $G = (V, E)$ be an undirected graph and let $r$ be a function, $r : V^2 \to \mathbb{N}$. Find a smallest possible set $F$ of new edges such that $\lambda(u, v) \geq r(u, v)$ for all $u, v \in V$ holds in $G' = (V, E \cup F)$.

In [30], (1970), H. Frank and W. Chou solved this problem for the special case when the starting graph has no edges - that is, they dealt with designing a “survivable network” from scratch. However, if we briefly return to our real world scenario, it is as important to be able to make effective use of an existing network. Communications systems are very expensive to throw away completely, so we need methods that enable us to efficiently improve on what we’ve got.

In 1976, two papers appeared, independently, solving the general problem for an arbitrary starting graph, when $r \equiv 2$. They are [18] by Eswaran and Tarjan and [57] by Plesnik, and both give a linear time algorithm.

The $k$-edge-connected problem - that is, finding the minimum number of new edges required to satisfy $r \equiv k$ for any positive integer $k$ - became the new focus of interest. In [59], (1983), Ueno, Kajitami and Wada solve this problem, when the starting graph is a tree. For an arbitrary starting graph, the $k = 1$ case, is easily solved (by adding $c - 1$ new edges, where $c$ is the number of components in the starting graph) and the $k = 2$ case is in [18],[57], mentioned above. The case, $r \equiv k$, for any positive integer $k \geq 2$, and any starting graph was solved by two pairs of authors.

Watanabe and Nakamura of Hiroshima University, Japan produced several (internal) research reports on the problem and in 1984 submitted an article claiming a solution. However, revisions were necessary and it was not until 1987 that [60] was printed. At the same time, Cai and Sun of Tianjin University, China were working along similar lines. In 1986 they spoke at a symposium in California, (see [9]), and in 1987 they submitted [8], which appeared in 1989.

They provide efficient algorithms and prove - both in a somewhat complicated manner - the following result.
Theorem 1.2.1 Given an undirected graph $G = (V,E)$ and an integer $k \geq 2$, let $\gamma$ be the smallest number of new edges we must add to $G$ to make it $k$-edge-connected. Then

$$\gamma = \max \left\lceil \frac{\sum (k - d(X_i))}{2} \right\rceil$$

where the maximum is taken over all subpartitions $\{X_1, X_2, \ldots, X_t\}$ of $V$. \hfill \square

A graph is $k$-edge-connected when every set $\emptyset \neq X \subset V$ has $d(X) \geq k$. When trying to augment a given graph, any set $X$ that starts with $d(X) < k$, must have at least $k - d(X)$ new edges added with exactly one endpoint in $X$. If we consider a collection of disjoint subsets of $V$ and note that any new edge could contribute to the deficiency of at most two of these sets, we see that $\max \left\lceil \frac{\sum (k - d(X_i))}{2} \right\rceil$ is a lower bound for the minimum size of a good augmentation. The algorithms in [60] and [8] prove that this bound is attainable. After this, improvements were made to the algorithm. For examples of new approaches see [56] and [33].

We point out that Theorem 1.2.1 is not true for $k = 1$. To see this consider a starting graph with four vertices and no edges. The bound in Theorem 1.2.1 would suggest an augmentation with two edges, but this is clearly not enough.

The next steps were taken by András Frank. In [22], he considers numerous versions of the problem. In the undirected edge-connectivity case, he provides a new, easier to handle, proof of Theorem 1.2.1 and goes on to generalize this with the following degree constrained case.

Theorem 1.2.2 Let $G = (V,E)$ be an undirected graph and $k \geq 2$ be an integer. Let $f \leq g$ be two non-negative integer valued functions on $V$. Then $G$ can be made $k$-edge-connected by adding a set $F$ of precisely $\gamma$ new edges so that $f(v) \leq d_F(v) \leq g(v)$ holds for every $v \in V$ if and only if

(a) $2\gamma \leq g(V)$,

(b) $k - d(X) \leq g(X)$ for all $\emptyset \subset X \subset V$ and

(c) $\sum (k - d(X_i)) + f(X_0) \leq 2\gamma$ for every partition $\{X_0, X_1, X_2, \ldots, X_t\}$ of $V$ where $X_0$ may be empty. \hfill \square

In the above, the functions $f$ and $g$ are defined on single vertices. They are extended to set functions by defining $f(X) := \sum_{x \in X} f(x)$ and $g(X) := \sum_{x \in X} g(x)$.

Frank goes on to solve the local connectivity problem completely. Here we consider a requirement function $r : V^2 \to \mathbb{N}$. Using this function, we define a set function $r : 2^V \to \mathbb{N}$, by putting $r(\emptyset) := 0 := r(V)$, and $r(X) := \max\{r(x,y) :
\( x \in X \) and \( y \in V - X \). (This is a slight abuse of notation, but as one function is for sets and the other for pairs of vertices, it is always clear which function we are using.) It turns out that a similar bound to that in Theorem 1.2.1 is almost always attainable. However, there is a difficulty that evolves from the fact that the global case with \( k = 1 \) is different to those with \( k \geq 2 \). The trouble is caused by so-called marginal components, introduced by Frank, in [22].

Let \( C \subset V \) be the vertex set of a component of \( G \). We say that component is **marginal** (with respect to the requirement function \( r \)) when \( r(C) - d(C) \leq 1 \) and \( r(X) - d(X) = 0 \) for every proper subset \( X \subset C \). Frank shows that we can eliminate these marginal components one at a time. In essence, they require at most one edge each and then the remainder of the graph can be handled as before. The following two results are from [22].

**Theorem 1.2.3** Given a graph \( G = (V, E) \) and a requirement function \( r \), let \( \gamma(G) \) denote the minimum number of edges we must add to \( G \) to ensure that \( \lambda(u, v) \geq r(u, v) \) for all \( u, v \in V \). If \( C \) is a marginal component of \( G \), then \( \gamma(G) = \gamma(G - C) + (r(C) - d(C)) \).

**Theorem 1.2.4** Let \( G = (V, E) \) be a graph with no marginal components and \( r \) be a requirement function. We can add a set, \( F \), of \( \gamma \) new edges and satisfy \( r \) if and only if

\[
\sum (r(X_i) - d(X_i)) \leq 2\gamma
\]

for all subpartitions \( \{X_1, X_2, \ldots, X_t\} \) of \( V \).

Frank goes on to generalise this result with degree constrained cases similar to those considered in the global case.

More recently, attention has been focussed on special cases and related questions. In [11], Cheng and Jordán consider the **Successive Augmentation Problem**. In this problem, from a starting graph \( G \) and a sequence of requirement functions \( r_1, r_2, \ldots \), we try to find a sequence of graphs \( G = G_0, G_1, G_2, \ldots \) such that each \( G_i \) is a subgraph of \( G_j \) for any \( i < j \), and each \( G_i \) is an optimal augmentation of \( G \) with respect to \( r_i \). Cheng and Jordán provide the next theorem, for requirement functions \( r_i : V^2 \rightarrow \mathbb{N} \), that satisfy the following.

\[
r_{i+1}(u, v) - 1 = r_i(u, v) \geq 2, \text{ for any } u, v \in V \text{ and for any } 1 \leq i \leq k - 1. \quad (\ast)
\]

**Theorem 1.2.5** Let \( G = (V, E) \) be an undirected graph and \( r_i \) for \( i = 1, 2, \ldots \), be a sequence of requirement functions for which (\ast) holds. Then there exists an infinite sequence \( G = G_0, G_1, G_2, \ldots \) of increasing supergraphs of \( G \), such that each \( G_i \) is an optimal augmentation of \( G \) with respect to \( r_i \).
In [43], Jordán considers two special cases of the graph $k$-edge-connectivity problem. In the *Increasing Edge-Connectivity by Reinforcing Edges* problem we may only add edges parallel to edges existing in the start graph $G$. In the *Simplicity Preserving Edge-Connectivity Augmentation Problem* (SPEA), we begin with a simple graph, and try to find the smallest number of edges we can add to $G$ to form a $k$-edge-connected simple graph. Jordán shows that the associated decision problems are NP-complete in both cases. Also, in [2], Jordán collaborates with Bang-Jensen, and they provide an $O(n^4)$ algorithm that finds an optimal solution for SPEA, on any graph with $n$ vertices, for any fixed $k$.

Writing $\gamma_k(G)$ for the smallest number of new edges we must add to $G$, to ensure that the result is $k$-edge-connected, and $\gamma_{Sk}(G)$ for the smallest number of new edges we must add to $G$, to ensure that the result is $k$-edge-connected and simple, Bang-Jensen and Jordán showed the following in [2].

**Theorem 1.2.6** If $\gamma_k(G) \geq 3k^4/2$ for some graph $G$, then $\gamma_{Sk}(G) = \gamma_k(G)$. Furthermore, for any starting graph $G$, and any target connectivity, $k \geq 2$, we have that $\gamma_{Sk}(G) \leq \gamma_k(G) + 2k^2 + 1$. \hfill $\square$

To close this section on edge-connectivity augmentation of graphs, we refer to [3], by Bang-Jensen, Gabow, Jordán and Szigeti. This paper is concerned with the problem of augmentating a graph, $G = (V,E)$, such that the result is $k$-edge-connected, and no new edge is contained in one element of a partition, $\{P_1, P_2, \ldots, P_r\}$, of $V$. Bang-Jensen et al. give a straightforward lower bound, $\Phi$, for the minimum size of such an augmenting set. They go on to show that many graphs attain this bound, characterise those that do not and show that these can be augmented with a set of size $\Phi+1$. They also describe a polynomial algorithm that performs an optimal augmentation on any starting graph.

**Vertex Connectivity**

As has been mentioned previously, the vertex connectivity versions are more difficult than their edge counterparts. The statement of the general problem is similar to that for edge-connectivity, but now $\lambda(u,v)$ means the number of internally (vertex) disjoint paths between $u$ and $v$. We denote the set of neighbours of a set of vertices $X$, by $\Gamma(X)$.

**General Problem** : Let $G = (V,E)$ be an undirected graph and $r$ be a function, $r : V^2 \to \mathbb{N}$. Find a smallest possible set $F$ of new edges such that $\lambda(u,v) \geq r(u,v)$ for all $u,v \in V$ holds in $G' = (V,E \cup F)$.

In [38], (1962), Harary found a solution for when the starting graph has no edges and $r \equiv k$ for a positive integer $k$. When we begin with an arbitrary
starting graph, the case $r \equiv 1$ is the same as for 1-edge-connectivity. However, things quickly get more involved.

In [18] Eswaran and Tarjan, and in [57] Plesnik, alongside the 2-edge-connectivity results are solutions to the 2-vertex-connectivity problem. They showed the following.

**Theorem 1.2.7** An undirected graph can be made 2-connected by adding at most $\gamma$ new edges if and only if

(a) $\sum(2 - |\Gamma(X_i)|) \leq 2\gamma$ for every subpartition $\{X_1, X_2, \ldots X_t\}$ of $V$ where $|X_i| \leq |V| - 3$, and

(b) $G - v$ has at most $\gamma + 1$ components for every vertex $v \in V$. $\Box$

Watanabe and Nakamura solved the $r \equiv 3$ case. (See [61] and [62]). The result is similar to that for 2-connectivity, except here we consider $\sum(3 - |\Gamma(X_i)|)$ over subpartitions with $|X_i| \leq |V| - 4$ and we require at most $\gamma + 1$ components in $G - X$ for every two element subset $X$ of $V$. For these two cases, Hsu and Ramachandran provide linear time augmentation algorithms in [40]. In [39], Hsu solves the case when we have $r \equiv 4$ and the starting graph is 3-connected.

The next steps in this area have been taken by Jordán. In [47], [49] and [44], he considers the problem of augmenting a $k$-connected graph to one that is $(k + 1)$-connected. To successfully increase the connectivity in this way, we must add at least one new neighbour to each set $X \subseteq V$ with $|\Gamma(X)| = k$ and $|V - X| \geq k + 1$. Here, we call such an $X$ tight. So, if $t(G)$ is the maximum number of pairwise disjoint tight sets in $G$, we must add at least $\lceil t(G)/2 \rceil$ new edges. Also, we assume that $G$ is not $(k + 1)$-connected. That is, there exists a set, $K$, of $k$ vertices that disconnects $G$. We call $K$ a minimal cut, use $\omega_K$ to denote the number of components in $G - K$, and $\omega(G)$ to denote the max $\{\omega_K\}$ over all minimal cuts in $G$. To make $G$, $(k + 1)$-connected, we must link all the components of $G - K$ for any minimal cut. Hence, any good augmenting set must have at least $\omega(G) - 1$ edges. So if we let $M(G) := \max\{\omega(G) - 1, [t(G)/2]\}$, $M(G)$ is a lower bound for $\gamma(G)$, the minimum number of edges we must add to $G$ to make it $(k + 1)$-connected.

The works of Eswaran and Tarjan, Plesnik, and Watanabe and Nakamura mentioned above show that for $k = 1$ and 2, we have $M(G) = \gamma(G)$. However for larger values of $k$ this result fails. Consider the complete bipartite graph $K_{k,k}$, which is $k$-connected. In this case, $\gamma = 2k - 2$ and $M = k$, that is, $\gamma = M + (k - 2)$. In [47] and [49], Jordán shows, in the following result, that this is an extreme situation.
Theorem 1.2.8  Let $G$ be a $k$-connected graph and $\gamma(G)$ be the minimum number of edges we must add to $G$ to make it $(k+1)$-connected. Then for $k = 1$, $\gamma(G) = M(G)$, and for $k \geq 2$, $M(G) \leq \gamma(G) \leq M(G) + k - 2$. 

Jordán also provides a polynomial algorithm to perform an augmentation with at most $\gamma(G) + k - 2$ edges. Recently, in [44], Jordán has shown that a slightly modified version of the algorithm in [47] and [49] will produce an augmentation that is at most $\lceil (k - 1)/2 \rceil$ over the optimum.

The results of [47] and [49] are also contained in his PhD thesis, [48]. Therein, he goes on to show that the decision problem associated with the case of a local vertex connectivity requirement function, $r$, (as stated at the beginning of this section) is NP-complete. He uses a transformation from a digraph augmentation problem shown to be NP-complete in [22]. (See Problem B in the next section.)

1.3 Directed Graphs

Edge Connectivity

As in previous cases, the first results found the minimum number of edges required to make an empty digraph $k$-edge-connected. This problem was solved by Fulkerson and Shapley in [32], (1971).

In [18], Eswaran and Tarjan solved the problem for an arbitrary starting digraph, with requirement function, $r \equiv 1$. That is, they found the minimum number of edges required to make an arbitrary digraph strongly connected. In the case where we start with a directed tree and wish to make a $k$-edge-connected digraph, the minimum number of additional edges was found by Kajitani and Ueno in [50].

In his 1992 paper, [22], alongside the results for undirected graphs already mentioned, Frank also considered digraph problems. He settled the question of how to find the smallest number of edges we must add to an arbitrary digraph to ensure at least $k$ edge-disjoint, directed paths between every pair of vertices.

Theorem 1.3.1  Let $G = (V,E)$ be a digraph and $k$ a positive integer. Then $G$ can be made $k$-edge-connected by adding at most $\gamma$ new edges if and only if

$$\sum (k - d_{in}(X_i)) \leq \gamma \text{ and } \sum (k - d_{out}(X_i)) \leq \gamma$$

for every subpartition $\{X_1, X_2, \ldots, X_t\}$ of $V$. 

Frank goes on to provide generalisations involving degree constrained augmentations, similar to those for undirected edge connectivity. The last result we
mention from [22], concerns the decision problem associated with the general requirement function, \( r : V^2 \rightarrow \mathbb{N} \), version. Frank shows that this is NP-complete, even if \( r \) only takes values 0 and 1.

**Problem A :** Let \( G = (V, E) \) be a digraph, \( s \) be a specified vertex of \( G \), \( T \) be a specified subset of \( V \) and \( \gamma \) be a positive integer. Decide if it is possible to add at most \( \gamma \) new edges to \( G \) so as to have a path from \( s \) to every element of \( T \).

**Problem B :** Let \( G = (V, E) \) be a digraph, \( R \) be a specified subset of \( V \) and \( \gamma \) be a positive integer. Decide if it is possible to add at most \( \gamma \) new edges to \( G \) so as to have a path from every vertex in \( R \) to any other vertex of \( R \).

Frank shows that these are both NP-complete. Firstly, he uses the fact that SET COVER is NP-complete to deal with Problem A. Then he shows that, if Problem B is solvable in polynomial time, then so is Problem A.

Frank joined with Bang-Jensen and Jackson to produce [4], (1995). This paper was primarily concerned with augmenting mixed graphs. (See Section 1.5.) However their results implied solutions to digraph questions. The next two theorems are from [4], the first dealing with the case when the starting digraph is di-Eulerian. That is, \( d_{\text{in}}(v) = d_{\text{out}}(v) \) for every \( v \in V \).

**Theorem 1.3.2** Let \( G = (V, E) \) be a di-Eulerian digraph. (That is, \( d_{\text{in}}(v) = d_{\text{out}}(v) \) for all \( v \in V \).) Let \( r \) be a symmetric requirement function for \( G \) and \( \gamma \) be a positive integer. Then we can augment \( G \) with \( \gamma \) new edges and ensure that in the resulting digraph, \( \lambda(u, v) \geq r(u, v) \) for all \( u, v \in V \) if and only if
\[
\sum (r(X_i) - d_{\text{in}}(X_i)) \leq \gamma \quad \text{and} \quad \sum (r(X_i) - d_{\text{out}}(X_i)) \leq \gamma
\]
for every subpartition of \( V \). \( \square \)

**Theorem 1.3.3** Let \( G = (V, E) \) be a digraph and \( T \) be the set \( \{ v \in V : d_{\text{in}}(v) \neq d_{\text{out}}(v) \} \). Let \( k \) and \( \gamma \) be positive integers. Then we can augment \( G \) with \( \gamma \) new edges and ensure that \( \lambda(u, v) \geq k \) for every \( u, v \in T \) if and only if
\[
\sum (k - d_{\text{in}}(X_i)) \leq \gamma \quad \text{and} \quad \sum (k - d_{\text{out}}(X_i)) \leq \gamma
\]
holds for every subpartition \( \{X_1, X_2, \ldots, X_t\} \) of \( V \) which has \( X_i \cap T \) and \( T - X_i \neq \emptyset \) for all \( i = 1, 2, \ldots, t \). \( \square \)

Bang-Jensen et al. go on to generalise the above results to deal with a special kind of local connectivity requirement function. This will be discussed in the next section.

Another special case of the \( k \)-augmentation question was solved in [28] by Frank and Jordan. In a digraph \( G = (V, E) \), with two subsets \( S, T \subset V \), we say \( G \)}
is \( k \)-edge-connected from \( S \) to \( T \), when \( \lambda(s,t) \geq k \) for all \( s \in S, t \in T \). Frank and Jordán find the minimum number of edges with tails in \( S \) and heads in \( T \), required to make any digraph \( k \)-edge-connected from \( S \) to \( T \). Their approach does not give rise to a combinatorial algorithm, but Enni provides one for the \( k = 1 \) case in [16].

In their Successive Augmentation paper (mentioned above), [11], Cheng and Jordán also consider digraphs and provide the following result.

**Theorem 1.3.4** Let \( t \) be a positive integer and let \( G = (V,E) \) be a digraph. Then there exist sets of edges \( F_1, F_2, \ldots, F_t \) on \( V \), with \( F_i \subset F_j \) for \( 1 \leq i < j \leq t \), such that \( G_j = (V,E \cup F_j) \) is an optimal \( j \)-edge-connected augmentation of \( G \) for all \( j = 1, 2, \ldots, t \).

Finally, we mention new work by Gabow and Jordán. In [34], they consider the bipartition constrained digraph augmentation problem. That is, starting with a digraph \( G = (V,E) \) and a bipartition \( \{P_1, P_2\} \) of \( V \), what is the smallest number of new edges we must add to make a \( k \)-edge-connected digraph, with no new edge having both ends in either \( P_i \)? In similar fashion to the graph case of [3], they provide a lower bound \( \Phi \) and then show that the minimum lies between \( \Phi \) and \( \Phi + k \).

**Vertex Connectivity**

When we only require that there is one path between every pair of vertices, \( (r \equiv 1) \), the distinction between edge-disjoint paths and internally (vertex) disjoint paths is irrelevant. So the solution to this problem with requirement function identically 1, follows from Eswaran and Tarjan’s edge result in [18]. Then, in [54], Masuzawa, Hagihara, Wada and Tokura, solve the \( k \)-connectivity problem when the starting digraph is a rooted directed tree - that is, a directed tree in which every vertex but one has in-degree equal to one and the exceptional vertex, the root, has in-degree zero.

In [28] (see also [48],[49]) Frank and Jordán solve the \( r \equiv k \) case. In a digraph, \( G = (V,E) \), they define a one-way pair \( (X,Y) \), as a pair of disjoint, non-empty subsets of \( V \), with no edge from \( X \) to \( Y \). For any pair \( (X,Y) \) of disjoint non-empty subsets of \( V \), they define \( h(X,Y) := |V - (X \cup Y)| \). Also, a family \( \mathcal{F} \) of pairs of disjoint subsets of \( V \) is called independent, if for any two members of \( \mathcal{F} \), \( (X,Y), (X',Y') \), at least one of \( X \cap X' \) and \( Y \cap Y' \) is empty. The next theorem is from [28].

**Theorem 1.3.5** A digraph \( G = (V,E) \) can be made \( k \)-connected by adding at most \( \gamma \) new edges if and only if

\[
\sum (k - h(X,Y) : (X,Y) \in \mathcal{F}) \leq \gamma
\]
holds for all independent families of one-way pairs, $F$. □

The proof is does not give rise to a combinatorial algorithm. However, the technique allows the solution of several extensions of this case. Moreover, in [29], the same authors refine some of the results of [28], to produce a combinatorial algorithm which is polynomial for any fixed $k$.

1.4 Mixed Graphs

A mixed graph has both directed and undirected edges. It is possible to augment such a network using either directed or undirected edges, or indeed both. In [36], Gusfield tackles the following problem. *Given a mixed graph, find the minimum number of new directed edges we must add to ensure that in the resulting mixed graph, every edge belongs to a cycle that contains no “backwards” edge.* He solves this problem, and that where we wish to add just undirected edges and provides a polynomial algorithm for both cases.

More familiar types of mixed graph augmentation problems are considered by Bang-Jensen, Frank and Jackson in [4]. Following the line of earlier graph and digraph results, they solve several versions of the augmentation problem. Before stating some of their results we need a little more notation.

A mixed graph, $M$ is considered as the union of a directed graph $D = (V, A)$ and an undirected graph $G = (V, E)$. Let $d_{inM}(X) := d_{inD}(X) + d_{G}(X)$ and $d_{outM}(X) := d_{outD}(X) + d_{G}(X)$. Then let $\beta_{M}(X) := \min\{d_{inM}(X), d_{outM}(X)\}$. A vertex $v \in V$ is *di-Eulerian* if $d_{inD}(v) = d_{outD}(v)$ and $M$ is *di-Eulerian* if every vertex of $M$ is di-Eulerian. We use $T(D)$ to denote the set of non-di-Eulerian vertices, $\{x \in V : d_{inD}(x) \neq d_{outD}(x)\}$. Let $k$ be a positive integer and $r$ be a non-negative integer-valued requirement function defined for every pair of vertices in $V$. We say $r$ is a *$k$-general requirement function* if it satisfies

(a) $r(x, y) = r(y, x) \leq k$ for every $x, y \in V$, and

(b) $r(x, y) = k$ for every $x, y \in T(D)$.

As before, when dealing with a requirement function $r$, we define $r(\emptyset) := 0 =: r(V)$, and for $X \subset V$, $r(X) := \max\{r(x, y) : x \in X$ and $y \in Y\}$. Using this notation we can read the following results from [4]. We are trying to increase edge-connectivity - so $\lambda(x, y)$ is denoting the number of edge disjoint paths from $x$ to $y$ - with no backwards edges. The first theorem tells us the minimum number of directed edges required to satisfy $r$. 

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Theorem 1.4.1 Let \( M = (V, A \cup E) \) be a mixed graph, \( k, \gamma \) be positive integers and \( r \) be a \( k \)-general requirement function. \( M \) can be extended to a mixed graph in which \( \lambda(x, y) \geq r(x, y) \) for all \( x, y \in V \), by adding \( \gamma \) new directed edges, if and only if
\[
\sum (r(X_i) - d_{inM}(X_i)) \leq \gamma \quad \text{and} \quad \sum (r(X_i) - d_{outM}(X_i)) \leq \gamma
\]
both hold for every subpartition \( \{X_1, X_2, \ldots, X_t\} \) of \( V \).

We note that Theorems 1.3.2 and 1.3.3, (directed graphs, edge connectivity section) appear in [4] as corollaries to Theorem 1.4.1.

Before trying to augment with undirected edges, it is necessary to remove any marginal components. These are defined for mixed graphs in the same way as for undirected graphs. Also, the method used by Frank in [22], to remove them “one at a time” (see Theorem 1.2.3) can also be used in the mixed case. This means that we need only consider how to best augment mixed graphs with no marginal components.

Theorem 1.4.2 Let \( M = (V, A \cup E) \) be a mixed graph, \( k \geq 2, \gamma \geq 0 \) be positive integers and \( r \) be a \( k \)-general requirement function, such that \( M \) has no marginal components with respect to \( r \). \( M \) can be extended to a mixed graph in which \( \lambda(x, y) \geq r(x, y) \) for all \( x, y \in V \), by adding \( \gamma \) new undirected edges, if and only if
\[
\sum (r(X_i) - \beta_M(X_i)) \leq 2\gamma
\]
for every subpartition \( \{X_1, X_2, \ldots, X_t\} \) of \( V \).

1.5 Hypergraphs

An ordinary, undirected graph consists of vertices and edges, where each edge is a subset of the vertex set, containing exactly two elements. In a hypergraph \( H = (V, E) \), edges are also subsets of the vertex set, but with no restriction on cardinality. That is, each edge can “contain” any number of vertices. It is possible to have multiple hyperedges - that is, two or more edges corresponding to the same subset of \( V \). This means that care must be taken when identifying an edge with the vertex subset to which it corresponds. The size of an edge \( e \in E \) (\( e \subseteq V \)) is just \( |e| \).

Hypergraphs retain many graph-like properties and hypergraph questions can often be tackled with graph methods. However, we must be very careful before assuming that because something is true for graphs, it will be true for hypergraphs also. We see that this care is necessary in our results on splitting later.
on. For further examples of hypergraph versions of well known graph results, we refer the reader to [7] and [63].

Of interest here, is the idea of a path in a hypergraph. We retain the notion of a journey from one vertex to another, by way of an alternating sequence of vertices and edges. In this case though, from any particular vertex, each incident edge can give any number of possible “one-step” neighbours, as opposed to the one offered by a graph edge. We define a path in a hypergraph as a sequence of distinct vertices and edges \( \{v_1, e_1, v_2, e_2, \ldots, v_p\} \) such that for each \( i = 1, 2, \ldots, p - 1 \), \( v_i, v_{i+1} \subseteq e_i \). We use \( \lambda(x, y) \) to denote the size of the largest possible family of edge disjoint paths from \( x \) to \( y \), that is, the edge-connectivity between \( x \) and \( y \). In [63] there is a hypergraph reworking of the edge version of Menger’s Theorem which allows us to approach hypergraph edge-connectivity augmentation problems with the methods used for graphs.

We consider the case where our starting network is a hypergraph, \( H = (V, E) \) and we wish to increase its edge-connectivity. We point out that one motivation for looking at edge-connectivity augmentation in hypergraphs, is that it seems to be a kind of “stepping-stone” between edge-connectivity and vertex-connectivity in graphs. Removing a hyperedge does more damage than removing an ordinary graph-edge, but (maybe) not as much damage as removing a vertex. In particular, we can consider the Konig representation of a hypergraph, (see [63]). This is a bipartite graph \( K = (X \cup Y, A) \), where \( X = V, Y = E \) and \( xy \in A \) if and only if \( x \in y \) in \( H \). It is not difficult to derive links between the connectivity in \( H \) and that in \( K \).

We could pose a question such as this.

What is the smallest number of edges we must add to a hypergraph \( H = (V, E) \), to ensure that in the resulting hypergraph, \( H^+ \), \( \lambda_{H^+}(x, y) \geq r(x, y) \), for all \( x, y \in V \) where \( r \) is a symmetric requirement function defined for every pair of vertices?

As in previous sections, from the requirement function, we define a set function, \( r \) where \( r(\emptyset) := 0 =: r(V) \) and \( r(X) := \max\{r(x, y) : x \in X, y \in V - X\} \). We also use a similar extension of the \( \lambda \) function, defining \( \lambda(\emptyset) := 0 =: \lambda(V) \) and \( \lambda(X) := \max\{\lambda(x, y) : x \in X, y \in V - X\} \). If we are permitted to add any (hyper)edge, the task of producing a good augmentation is not difficult. For instance, we could add \( r_{\text{max}} \) copies of an edge \( e = V \), where \( r_{\text{max}} = \max\{r(X) - \lambda(X)\} \) over all proper subsets of \( V \). This process is certainly polynomial in the size of the input and the problem is, therefore, somewhat uninteresting.

However, in [58], Szigeti finds the smallest “valued” edge set we must add to an arbitrary starting hypergraph. He defines the value of a set of edges \( F \) as \( \text{val}(F) := \sum(|e| : e \in F) \).
Theorem 1.5.1 Let $H = (V, E)$ be a hypergraph and $r : V^2 \rightarrow \mathbb{N}$ be a symmetric requirement function. The minimum value of a set of edges whose addition to $H$ creates $H^+$ in which $\lambda_{H^+}(x, y) \geq r(x, y)$ for all $x, y \in V$ is given by
\[
\max \{ \sum_{X_i \in S} (r(X_i) - d_H(X_i)) \}
\]
where the maximum is taken over all subpartitions, $S$ of $V$. 

Work has also been done on augmenting hypergraphs with edges of size-two. In [10], Cheng considers the problem of increasing the global edge-connectivity by one. Adapting the decomposition theories of Cunningham, (see [12], [13] and [14]), Cheng finds the minimum number of size-two edges that must be added to a $(k - 1)$-edge-connected hypergraph to make it $k$-edge-connected, and gives a polynomial algorithm to perform the augmentation.

In [1], Bang-Jensen and Jackson generalise this, by considering the case when the starting hypergraph has arbitrary connectivity. For a hypergraph $H = (V, E)$ let $p_k(H)$ be the maximum value taken by $\sum (k - d(X_i))$ over all sub-partitions $\{X_1, X_2, . . . X_t\}$ of $V$ and let $c_k(H)$ denote the maximum number of components we can form by the removal of at most $k - 1$ edges from $H$. Using this notation, the following theorem appears in [1]. The proof is constructive, and leads to an efficient algorithm to find an optimal augmentation.

Theorem 1.5.2 Let $H = (V, E)$ be a hypergraph and $k$ be a positive integer. Then the minimum number of new size two edges we must add to $H$ to create a $k$-edge-connected hypergraph is 
\[
\max \{ \lceil p_k(H)/2 \rceil, c_k(H) - 1 \}.
\]

This theorem is generalised by Benczúr and Frank in [6]. They consider the case in which there is a set $T \subset V$, and we wish to augment and satisfy the requirement function, $r : V^2 \rightarrow \mathbb{N}$ where,
\[
r(x, y) = \begin{cases} 
    k & \text{if } x, y \in T, \\
    0 & \text{otherwise}.
\end{cases}
\]

That is we want to make the resulting hypergraph $k$-edge-connected in $T$. We say that a set $X \subset V$ separates $T$ when neither $X \cap T$, nor $T - X$ is empty. We say a subpartition separates $T$, when every set in the subpartition separates $T$. We state Benczúr and Frank’s result as it is in their paper (along with Theorem 1.5.2, this shows the two common ways of stating a min-max result). Let $c_T(H)$ denote the maximum number of components intersecting $T$, we can form by the removal of at most $k - 1$ edges from $H$. 

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Theorem 1.5.3  A hypergraph $H = (V, E)$ can be made $k$-edge-connected in a set $T \subset V$, by adding at most $\gamma$ new size-two edges, if and only if,

$$\sum_{X_i \in S} (k - d_H(X_i)) \leq 2\gamma \text{ for every subpartition } S \text{ of } V, \text{ separating } T,$$

and

$$c_T(H) - 1 \leq \gamma.$$

\[ \square \]

A Note on Sub- and SuperModular Functions

The proofs of many of the results mentioned so far rely heavily on the deep theory of supermodular and submodular functions. Frank has achieved a great deal of success using this method, and many of his graph and hypergraph theorems are special cases of results on covering symmetric supermodular and submodular functions. We refer the interested reader to [26] and [27] for surveys by Frank. Of the papers already discussed, [28] by Frank and Jordán, [58] by Szigeti and Jordán’s PhD thesis, [48], are good examples of work that discuss the more general “covering function” ideas first, before specialising them to splitting and augmentation problems.

1.6 Splitting

Many of the results mentioned above use a technique called splitting. To split a pair of edges $su, sv$ (in a network $G$) is to remove them and add a new edge $uv$. Usually we denote the network obtained in this way by $G_{uv}$. This provides a neat way to add edges to a network. Suppose we begin with a network $G_1 = (V, E_1)$. We add a new vertex $s$ and edges from $s$ to $V$, giving us a network $G = (V + s, E)$. When adding edges, we initially ensure that we have enough to satisfy our requirement function $r$, and then remove them one at a time until we reach the point where no more can be removed without “violating” $r$. We now try to split these edges away from $s$ in such a way as to preserve the extra connectivity we have engineered for $V$ in $G$. That is, we try to find “good” splits. This method of augmentation is used extensively in many of the augmentation problems already mentioned.

All of the cases that use this method, have one question in common. Can we form a good split in $G$? Lovász was the first to answer this question in a specific case, in [51]. Mader generalized this in [53] and later authors have provided their own splitting theorems to fit with many versions of the augmentation problem.
Lovász begins with an undirected graph, $G = (V + s, E)$, that has at least $k$ edge disjoint paths between every pair of vertices in $V$, that is, is $k$-edge-connected in $V$. He tried to find splits that preserved $k$-edge-connectivity in $V$.

**Theorem 1.6.1** Let $G = (V + s, E)$ be an undirected graph, that is $k$-edge-connected in $V$, and has $d(s) > 0$ even. Then for every edge $su$ there is an edge $sv$ such that $G_{uv}$ is $k$-edge-connected in $V$. 

Lovász proves this for Eulerian graphs in [51] and for non-Eulerian graphs in [52]. (Frank points out in [22], that in [52], Lovász mistakenly omits the caveat that in the non-Eulerian case, it is still necessary to have $d(s)$ even, but that otherwise his proof is correct.) For a shorter proof than Lovász’ own, see [22].

Mader extended this to the more general case where we wish to split preserving local edge-connectivity in $V$. The following result is from Mader’s paper, [53].

**Theorem 1.6.2** Let $G = (V + s, E)$ be a connected, undirected graph, with no cut-edge incident to $s$ and $d(s) \neq 3$. Then there exists a pair of edges $su, sv$ such that there is no cut-edge incident with $s$ in $G_{uv}$ and, $\lambda_{G_{uv}}(x, y) = \lambda_{G}(x, y)$ for all $x, y \in V$.

This powerful result was the cornerstone of Frank’s local edge-connectivity augmentation result, Theorem 1.2.4, (see [22]) and in [23] he provides a much simpler proof than Mader’s own. Mader also produced a version of his Theorem for directed splitting and Frank uses this in [22] to solve digraph augmentation problems.

Many of the previously mentioned augmentation papers provide their own splitting theorems, specifically designed to tackle the special cases under consideration. We mention here the results most relevant to the work presented later in this thesis.

Firstly, we look at the partition constrained problem in [3]. The splitting question involves a graph $G = (V + s, E)$ which is $k$-edge-connected in $V$ for an integer $k \geq 2$, has $d(s)$ even and a partition $P = \{P_1, \ldots, P_r\}$ of $V$. We look for a $P_k$-split. That is, a pair of edges $su, sv$ such that $G_{uv}$ is $k$-edge-connected in $V$, and $u$ and $v$ do not belong to the same member of the partition. Initially we hope to find a single split, and then a complete split, in which we can split all the edges from $s$. In the case when $k$ is even, Lovász Theorem extends naturally and the following result from [3] is not too difficult. For subsets $X, Y \subset V$, we use $d(X, Y)$ to denote the number of edges with one end in $X$ and the other in $Y$. 

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Theorem 1.6.3 Let $G = (V + s, E)$ be a graph that is $k$-edge-connected in $V$, with $k \geq 2$ even, $d(s)$ even, and a partition $P = \{P_1, \ldots, P_r\}$ of $V$. Then there is a complete $P_k$-split at $s$ if and only if $d(s, P_i) \leq d(s)/2$ for all $1 \leq i \leq r$. \hfill \Box

When $k$ is odd, Bang-Jensen et al. show that if $d(s) \geq 6$ and $d(s, P_i) \leq d(s)/2$ for all $1 \leq i \leq r$, there is a $P_k$-split. There can be a problem, however, when $d(s) = 4$. The structure that prevents a $P_k$-split in this case comes in one of the two forms defined below.

Let $C = \{A_1, A_2, B_1, B_2\}$ be a partition of $V$. Then $C$ is called a $C_4$-obstacle when it satisfies the following. (Note that the properties describe how the partition $C$ interacts with the partition $P$.)

(a) $d(X) = k$ for all $X \in C$,
(b) $d(X, Y) = 0$ for $(X, Y) \in \{(A_1, A_2), (B_1, B_2)\}$,
(c) for some $1 \leq i \leq r$, $N(s) \cap Z = N(s) \cap P_i$ for $Z \in \{(A_1 \cup A_2), (B_1 \cup B_2)\}$, and
(d) for the same $i$, $d(s, P_i) = d(s)/2$.

The second obstacle can appear when $d(s) = 6$, and ensures that you can perform only one $P_k$-split.

Let $C' = \{A_1, A_2, B_1, B_2, C_1, C_2\}$ be a partition of $V$. Then $C'$ is called a $C_6$-obstacle when it satisfies the following properties - relating to both $C'$ and $P = \{P_1, \ldots, P_r\}$.

(a) $d(X) = k$ for all $X \in C'$,
(b) $d(X, Y) = (k - 1)/2$ for all $(X, Y) \in \{(A_1, B_1), (B_1, C_1), (C_1, A_2), (A_2, B_2), (B_2, C_2), (C_2, A_1)\}$,
(c) $d(s, X) = 1$ for all $X \in C$, and
(d) for three distinct indices $1 \leq a, b, c \leq r$, $N(s) \cap Z = N(s) \cap P_i$ where $(Z, i) \in \{(A_1 \cup A_2, a), (B_1 \cup B_2, b), (C_1 \cup C_2, c)\}$.

Using these definitions, the following is the main splitting result in [3].

Theorem 1.6.4 Let $G = (V + s, E)$ be a graph that is $k$-edge-connected in $V$, with $k \geq 3$ odd, $d(s)$ even, and a partition $P = \{P_1, \ldots, P_r\}$ of $V$. Then there is a complete $P_k$-split at $s$ if and only if $d(s, P_i) \leq d(s)/2$ for all $1 \leq i \leq r$ and $G$ contains neither a $C_4$-obstacle nor a $C_6$-obstacle. \hfill \Box

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The structures of the $C_4$- and $C_6$-obstacles, provide the basis for the characterisation of the graphs which cannot be augmented with the lower bound $\Phi$ mentioned in Section 1.2.

Hypergraphs present their own problems. Unlike the graph and digraph cases solved by Lovász and Mader, there are non-trivial hypergraphs when it is not possible to make a “good” split. In [1], Bang-Jensen and Jackson deal with hypergraphs $G = (V + s, E)$ that are $k$-edge-connected in $V$, and look for $k$-splits - that is, those that preserve $k$-edge-connectivity in $V$. They characterise the hypergraphs in which there is no $k$-split.

**Theorem 1.6.5** Let $G = (V + s, E)$ be a hypergraph that is $k$-edge-connected in $V$ and has $d(s)$ even. If $st \in E$ is an edge with $t \in V$, then exactly one of the following holds.

(a) There is a $k$-split using $st$, or

(b) $d(s) \geq 4$ and there exists a set $A \subset E$ of $k - 1$ edges such that,

\begin{enumerate}
  \item $G - s - A$ has $d(s)$ components,
  \item $s$ has one neighbour in each component of $G - s - A$, and
  \item each $e \in A$ intersects every component of $G - s - A$.
\end{enumerate}

In [1], this theorem is extended to the following result about complete $k$-splits, which is used in their augmentation result. In Chapter 6 we adapt it for our own purposes.

**Theorem 1.6.6** Let $G = (V + s, E)$ be a hypergraph that is $k$-edge-connected in $V$ and has $d(s)$ even. Then exactly one of the following holds.

(a) There is a complete $k$-split at $s$, or

(b) there is a set $A \subset E$ of $k - 1$ edges such that $G - s - A$ has at least $d(s)/2 + 2$ components.

This Theorem is proved by showing a stronger result about constructing the longest sequence of $k$-splits from $s$. The authors use Theorem 1.6.6 to derive both their augmentation result (Theorem 1.5.2) and the longest-sequence-of-splits result in their polytime algorithm.

We end our historical survey with mention of a new approach to finding good splits introduced by Jordán in [42]. He is concerned with finding $k$-splits that must satisfy some additional property, $P$ - we shall call them $Pk$-splits. The new method is to form a “non-admissibility” graph, $B(s)$, on a copy of $N_G(s)$...
by joining two vertices with an edge if and only if they do not form a $k$-split from $s$. Then on another copy of $N_G(s)$, we form a demand graph, $D(s)$ by joining two vertices if and only if they satisfy $\mathcal{P}$. Then we have a $\mathcal{P}k$-split if and only if $D(s)$ is not a subgraph of $B(s)$.

In [42], Jordán analyses the structure of $B(s)$ and compares it to the demand graphs generated by several different $\mathcal{P}$'s. In this way he provides new proofs for some of the partition constrained splitting results from [3] and for results on planarity-preserving splits due to Nagamochi and Eades, [55]. Also, he tackles a new problem on simultaneous splitting and augmentation. Let $G = (V + s, E + F)$ be $k$-edge-connected in $V$ and let $H = (V + s, K + F)$ be $l$-edge-connected in $V$. Let $F$ be the set of edges incident with $s$, and be such that $|F|$ is even, and let $d(s) := d_G(s) = d_H(s)$. We say edges $su, sv$ form a $kl$-split, if they are a $k$-split in $G$ and an $l$-split in $H$. Jordán proves the following theorem and uses it in solving the related augmentation problem.

**Theorem 1.6.7** If $d(s) \geq 6$, then there is a $kl$-split from $s$. If $k$ and $l$ are both even, there is a complete $kl$-split from $s$. □

### 1.7 New Results in this Thesis

**Chapter 2** includes most of the basic notation we shall require and then provides general hypergraph versions of several standard lemmas, that have been used in previously considered versions of the splitting and augmentation problems in papers such as [22] and [1].

**Chapter 3** considers the problem of finding splits from $s$ in a hypergraph $G = (V + s, E)$, that preserve the local connectivity in $V$. We call such a split a $\lambda$-split. By describing the systems of so-called dangerous sets present in a hypergraph that has no $\lambda$-split, we prove two theorems about the edges incident with $s$. We go on to fully describe the structure of those hypergraphs with no $\lambda$-split. Although the structure evolves naturally from that given by Bang-Jensen and Jackson (see [1]) in Theorem 1.6.5, it is more complicated. There is a new “nesting” property that applies to the sets involved, and to the edges underlying them. We close the chapter with a special case of the no-split theorem that deals with splits preserving a function $\lambda_k(x, y) = \min\{\lambda(x, y), k\}$.

**Chapter 4** uses the two theorems about the edges incident with $s$, from Chapter 3, to prove two augmentation results. The first provides lower and upper bounds for the local-connectivity hypergraph augmentation problem. That is, we are given a hypergraph, $H = (V, E)$ and a requirement function $r : V^2 \to \mathbb{N}$ and asked to find the smallest number of size-two edges we must add to $H$ to ensure that in the result, $H^+$ we have $\lambda_{H^+}(x, y) \geq r(x, y)$ for all $x, y \in V$. Our
upper bound is a function of the largest edge present in $H$, and we show that it is sharp, in the sense that there are hypergraphs which require this many edges for a good augmentation. Also, the proof is constructive - describing how to perform an augmentation that achieves the upper bound. We then prove an extension of Szigeti's Theorem 1.5.1, from [58], about augmenting with a set of hyperedges of minimum value. We show that there is a minimum value augmenting set, with at most one edge having size bigger than two.

Chapter 5 considers a natural extension of the problem solved by Benczúr and Frank in [6]. Instead of finding the smallest set of new size-two edges to make $G$, $k$-edge-connected in some set $T \subset V$, we consider a subpartition, $T_1, T_2, \ldots, T_r$ of $V$, and the problem of augmenting to make $\lambda(x, y) \geq k$ whenever $x, y$ are both in a single $T_i$. We solve the problem for $k = 1$, and then describe joint work by the present author and Zoltán Király, which shows that for $k \geq 2$, the associated decision problem is NP-complete.

Chapter 6 uses the “non-admissibility” graph technique introduced by Jordán, [42], to tackle the problem of finding $k$-splits that satisfy a bipartition constraint. This is a hypergraph version of the partition constrained problem solved by Bang-Jensen et al. in [3]. That is, we are given a hypergraph $G = (V + s, E)$ and a bipartition $P_1, P_2$ of $V$, and we look for $su, sv$ that form $k$-splits and have $u \in P_1$ and $v \in P_2$ or vice versa. We adapt ideas from both [1] and [3] to describe the obstacles that prevent splitting, and give a theorem characterising those hypergraphs with a complete $P_k$-split.

Chapter 7 considers the augmentation problem related to the splitting questions in Chapter 6. We are given a hypergraph $H = (V, E)$ with a bipartition of $V$ and asked to find the smallest number of size-two edges, none of which may lie in a single set $P_i$, that we must add to make the result $k$-edge-connected. We solve the problem completely, giving the two possible values for the size of a minimum augmenting set, and characterise which hypergraphs take which. We finish with a polynomial algorithm to perform an optimal augmentation.
Chapter 2

Notation and Preliminary Results

2.1 The Basics

We must begin with a few definitions. A hypergraph is a pair of disjoint sets called vertices and edges, together with an incidence relation that associates a subset of the vertices with each edge. In this paper we shall identify an edge with its corresponding subset of the vertex set. We denote an arbitrary hypergraph by $H = (U, E)$, where $U$ is the vertex set and $E$ is the edge set. (We use $U$ instead of the more common $V$, because later on we will be talking about hypergraphs with a special vertex, for which we shall write $G = (V + s, E)$.)

We assume throughout that our hypergraphs contain no loops.

The size of an edge $e$ is $|e|$. If an edge has size two, say $\{x, y\}$ we use the usual graph notation $xy$. Let $e \in E$, $x \in U$ and $X \subseteq U$. We say that $e$ is incident with $x$ when $x \in e$ and $e$ intersects $X$, when $e \cap X$ is non-empty.

For $x \in U$ we use $d(x)$ to denote the number of edges which are incident with $x$. For $X \subseteq U$, $\Delta(X)$ denotes the set of edges which intersect both $X$ and $U - X$ and we write $d(X) := |\Delta(X)|$.

For $X, Y \subseteq U$, following the line of Bang-Jensen and Jackson in [1], we write the following.

- $d_1(X, Y)$ is the number of edges which intersect $X - Y$ and $Y - X$ and do not intersect either of the sets $X \cap Y$ and $U - (X \cup Y)$.
- $d_2(X, Y)$ is the number of edges which intersect $X - Y$, $Y - X$ and exactly one of $X \cap Y$ and $U - (X \cup Y)$. 

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• \( d_3(X, Y) \) is the number of edges which intersect \( U - (X \cup Y) \) and \( X \cap Y \) and do not intersect either of the sets \( X - Y \) and \( Y - X \).

• \( d_4(X, Y) \) is the number of edges which intersect \( X \cap Y, U - (X \cup Y) \) and exactly one of \( X - Y \) and \( Y - X \).

The following is from [1].

**Proposition 2.1.1 (Bang-Jensen and Jackson)** For an arbitrary hypergraph \( H = (U, E) \), the following equalities both hold for all \( X, Y \subseteq U \).

(a) \( d(X) + d(Y) = d(X \cap Y) + d(X \cup Y) + 2d_1(X, Y) + d_2(X, Y) \).

(b) \( d(X) + d(Y) = d(X - Y) + d(Y - X) + 2d_3(X, Y) + d_4(X, Y) \).

**Proof:** This is seen easily, by checking that each edge contributes the same to both sides of the two equalities.

Throughout, we make comparisons between hypergraphs, for instance, \( H \) and \( H' \). When we do this we use a subscript to make clear which hypergraph is being considered. For example, we may write \( d_H(X) \) to mean the degree of set \( X \) in \( H \), or \( d_{1,H'}(X,Y) \) to mean “\( d_1(X,Y) \) in \( H' \).”

As for graphs, we define a **path** in a hypergraph as an alternating sequence of distinct vertices and edges \( v_1, e_1, v_2, e_2, \ldots, e_{p-1}, v_p \) such that \( \{v_i, v_{i+1}\} \subseteq e_i \) for all \( i = 1, \ldots, p - 1 \). A \( (u,v) \)-**path** has end vertices \( u \) and \( v \). A hypergraph is **connected** if there is a \( (u,v) \)-path for every pair of vertices \( u, v \). We say a set \( X \subseteq U \) **separates** vertices \( u \) and \( v \) if, \( u \in X \) and \( v \in U - X \) or if \( v \in X \) and \( u \in U - X \). We use \( \lambda_H(x, y) \) to denote the maximum number of edge-disjoint \( (x, y) \)-paths in \( H \). When there is no confusion about which hypergraph we are referring to, we drop the subscript.

### 2.2 Routing Functions

When augmenting connectivity, a well used technique is to add a vertex \( s \). We will call \( s \) **special** if it has degree at least four, and is only incident with size two edges. This will give us a hypergraph \( G = (V + s, E) \). In such a hypergraph, we call the neighbours of \( s \), **s-neighbours** and for a subset \( X \subseteq V \), we use \( \delta(X) \) to denote the set of edges intersecting both \( X \) and \( V - X \). Note that this means \( \delta(X) \) does not include any edges from \( X \) to \( s \). When dealing with this type of hypergraph, we need to compare the degree function with either a requirement function, the local connectivity function, or some specialisation
thereof. These functions all have the property that we define the set function by taking a maximum of values of the ‘ordered pairs function.’

Let \( r : V^2 \to \mathbb{N} \) be a symmetric function. We often use such a function to define our connectivity requirement, and when doing so, from this point forward we automatically, create the set function \( r : 2^V \to \mathbb{N} \), by putting \( r(\emptyset) := 0 =: r(V) \) and \( r(X) := \max\{r(x, y) : x \in X, y \in V - X\} \) for all \( X \subseteq V \). Although we are using \( r \) to represent two functions, it will always be clear (by the context) to which we are referring. We extensively use \( \lambda \), the local connectivity function, in this way, writing \( \lambda(X) := \max\{\lambda(x, y) : x \in X, y \in V - X\} \) for all \( X \subseteq V \). The following Proposition was proved for \( \lambda \) in graphs by Frank in [22]. The proof he used therein can be used for any symmetric function \( r \) as above, and is reproduced below for completeness.

**Proposition 2.2.1** Let \( G = (V + s, E) \) be a hypergraph and \( r : V^2 \to \mathbb{N} \) be a symmetric requirement function. For \( X, Y \subseteq V \) at least one of the following holds.

\[
\begin{align*}
(a) & \quad r(X) + r(Y) \leq r(X \cap Y) + r(X \cup Y) \\
(b) & \quad r(X) + r(Y) \leq r(X - Y) + r(Y - X)
\end{align*}
\]

**Proof:** Suppose that \( r(X) = r(x, x') \) and that \( r(Y) = r(y, y') \) where \( X \) separates \( x \) and \( x' \) and \( Y \) separates \( y \) and \( y' \). Assume first that one of the two pairs, say \( \{x, x'\} \), is separated by both \( X \) and \( Y \). By taking the complement of \( Y \) if necessary, we can assume that \( x \in X - Y \) and \( x' \in Y - X \). (If \( Y \) is replaced by \( V - Y \), the two inequalities transform into each other.) If \( y \) and \( y' \) are separated by \( X \), then \( r(Y) = r(X) \leq \min(r(X - Y), r(Y - X)) \) and \( (b) \) follows.

If \( y, y' \) are not separated by \( X \), then either one of \( y, y' \), say \( y \), is in \( Y \cap X \) and \( y' \in X - Y \) or else one of \( y, y' \), say \( y \), is in \( Y - X \) and \( y' \in V - (X \cup Y) \). In the first case \( r(X - Y) \geq r(Y) \) and \( r(Y - X) \geq r(X) \) and \( (b) \) follows. In the second, \( r(Y - X) \geq r(Y) \) and \( r(X - Y) \geq r(X) \) and \( (b) \) follows again.

Finally, assume that neither are \( x, x' \) separated by \( Y \) nor \( y, y' \) separated by \( X \). Again, we can assume \( x \notin Y \). Then \( x' \in V - (X \cup Y) \). Now either one of \( y \) and \( y' \), say \( y \), is in \( Y - X \) and \( y' \in V - (X \cup Y) \). In the first case \( r(X) \leq r(X \cup Y) \) and \( r(Y) \leq r(X \cap Y) \) from which \( (a) \) follows. In the second \( r(X) \leq r(X - Y) \) and \( r(Y) \leq r(Y - X) \) and \( (b) \) follows.

We call \( r \) a **routing function on** \( G \), when \( \lambda_G(x, y) \geq r(x, y) \) for all \( x, y \in V \). We use the following two propositions throughout this thesis, as automatic equivalences to the definition of a routing function.
Proposition 2.2.2 Let $G = (V + s, E)$ be a hypergraph, such that $s$ is only incident with size-two edges. Let $r : V^2 \to \mathbb{N}$ be a symmetric requirement function. Then $r$ is a routing function if and only if $d(X) \geq r(X)$ for all $\emptyset \subset X \subset V$.

Proof: For every set $X$, there exists $x \in X, y \in V - X$ such that $r(X) = r(x, y)$. Then if $r$ is a routing function, we have $r(X) = r(x, y) \leq \lambda(x, y) \leq d(X)$, as required. To see the other direction, we need the hypergraph version of Menger’s edge connectivity theorem. (See [63], or for graph versions, [7], [15].) This says that there are $t$ edge-disjoint $(x, y)$-paths, if and only if, $d(X) \geq t$ for all sets $X$ separating $x$ and $y$. A consequence of this, is for all pairs $x, y$, there is a set $X$, separating $x, y$ such that $d(X) = \lambda(x, y)$. (Actually, we could include pairs that have $s$ as one member, but we are only ever concerned with subsets of $V$.) We assume that $X \subset V$, by replacing $X$ with $V + s - X$ if necessary.

Now, if we have $d(X) \geq r(X)$ for all $X \subset V$, then for all $x, y$ we have a set $X$ such that $r(x, y) \leq r(X) \leq d(X) = \lambda(x, y)$, and hence, $\lambda(x, y) \geq r(x, y)$ as required. \qed

Proposition 2.2.3 Let $G = (V + s, E)$ be a hypergraph, with $s$ only incident with size two edges, and let $r : V^2 \to \mathbb{N}$ be a symmetric function. Then $r$ is a routing function if and only if $d(X) \geq \lambda(X) \geq r(X)$ for all $\emptyset \subset X \subset V$.

Proof: One direction follows from Proposition 2.2.2. To see the other, suppose that $r$ is a routing function and for $X \subset V$ let $x, y$ be vertices such that $r(X) = r(x, y)$. Then we have $r(X) = r(x, y) \leq \lambda(x, y) \leq \lambda(X) \leq d(X)$. \qed

When $r$ is a routing function, we define $s(X) := d(X) - r(X)$. Then $s(X) \geq 0$ for all $X$. Combining Propositions 2.1.1 and 2.2.1 gives us the following. (When necessary, we use a subscript to indicate which hypergraph we are considering. For instance, when dealing with $G$ and $G'$ we will use $s_G(X)$ and $s_{G'}(X)$.)

Proposition 2.2.4 In a hypergraph $G = (V + s, E)$, for $X, Y \subset V$ at least one of the following holds.

(a) $s(X) + s(Y) \geq s(X \cap Y) + s(X \cup Y) + 2d_1(X, Y) + d_2(X, Y)$.

(b) $s(X) + s(Y) \geq s(X - Y) + s(Y - X) + 2d_3(X, Y) + d_4(X, Y)$. \qed
2.3 Splitting, Dangerous Sets and Tight Sets

The process of splitting is the main tool in our augmentation process and an interesting (and difficult) subject in its own right. In a hypergraph $G = (V + s, E)$, to split a pair of edges $su, sv$ is to replace them with a new edge $uv$. We denote the hypergraph created by this operation by $G_{uv}$. If $u = v$ we define $G_{uv} := G - \{su, sv\}$ so that $G_{uv}$ has no loops. We are interested in when we can split edges from $s$ and maintain a certain level of connectivity in $V$. We describe this level using a routing function, $r$. We say a pair of edges $su, sv$ are $r$-splittable when $\lambda_{G_{uv}}(x, y) \geq r(x, y)$ for all $x, y \in V$ - or equivalently, when $d_{G_{uv}}(X) \geq r(X)$ for all $X \subseteq V$. We call a split that preserves $r$ in this way an $r$-split.

In a hypergraph $G = (V + s, E)$ with routing function $r$, we say a set, $X \subset V$, is $r$-tight when $d_{G}(X) = r(X)$. We say $X$ is $r$-dangerous when $d_{G_{uv}}(X) \leq r(X) + 1$. Dangerous sets are the basic objects that prevent us from splitting.

**Lemma 2.3.1** Let $G = (V + s, E)$ be a hypergraph , with $s$ only incident with edges of size two and a routing function $r$. A pair of edges $su, sv$ form an $r$-split if and only if there is no $r$-dangerous set, $X \subset V$, containing both $u$ and $v$.

**Proof:** Suppose there is such an $X$. Then

$$\lambda_{G_{uv}}(X) \leq d_{G_{uv}}(X) = d_{G}(X) - 2 \leq r(X) - 1 < r(X).$$

That is, the $su, sv$ do not form an $r$-split.

Conversely, suppose that $su, sv$ are not $r$-splittable. Then there is a pair of vertices $x$ and $y$ for which $\lambda_{G_{uv}}(x, y) < r(x, y)$. Also, there is a set $X \subseteq V$ separating $x$ and $y$ for which $d_{G_{uv}}(X) = \lambda_{G_{uv}}(x, y)$, and so $d_{G_{uv}}(X) = \lambda_{G_{uv}}(x, y) < r(x, y) \leq r(X) \leq d_{G}(X)$. Therefore $u, v \in X$. So, we have $d_{G}(X) - 2 = d_{G_{uv}}(X) = \lambda_{G_{uv}}(x, y) \leq r(x, y) - 1 \leq r(X) - 1$. That is $X$ is an $r$-dangerous set containing $u$ and $v$. \qed

We will examine the structure of the families of dangerous sets within hypergraphs with no splits at $s$ later. For now, we give the following lemma, which was used, in a graph version, by Frank in [22].

**Lemma 2.3.2** Let $G = (V + s, E)$ be a hypergraph with $s$ only incident with size two edges and $r$ a routing function on $G$. Let $X \subset V$ be $r$-dangerous. Then $d_1(s, X) \leq d_1(s, V - X) + 1$. Further, if $d_G(s)$ is even we have $d_1(s, X) \leq d_1(s, V - X)$. 

Proof: Put \( a = d_1(s, X) \) and \( b = d_1(s, V - X) \). Then we have

\[
\begin{align*}
    r(V - X) &= r(X) \\&\geq d(X) - 1 \\
&= d(V - X) - b + a - 1 \\
&\geq r(V - X) - b + a - 1
\end{align*}
\]

which implies \( a \leq b + 1 \), as required. Also, if \( d(s) = a + b \) is even, we must have \( a \leq b \).

\[
\square
\]

2.4 Contraction

Let \( Z \subseteq U \). We use \( H/Z \) to denote the hypergraph obtained from \( H - Z \) by adding a new vertex \( z^* \) and edges \( e - Z + z^* \) for all \( e \in E \) that intersect both \( Z \) and \( U - Z \). We call this operation contracting the set \( Z \). Often, we write \( H' = (U', E') = H/Z \) where \( U' = U - Z + z^* \), and \( E' = \{ e \in E : e \cap Z = \emptyset \} \cup \{ e - Z + z^* : e \in E \text{ and } e \text{ intersects both } Z \text{ and } U - Z \} \). We carry this ‘dash’ notation through to subsets of the vertex set as follows. Given \( A' \subseteq U' \), we say that \( A' \) is the reduction of the set \( A \subseteq U \) where

\[
A = \begin{cases} 
    A' & \text{if } z^* \notin A' \\
    A' - z^* + Z & \text{if } z^* \in A'
\end{cases}
\]

We say \( A \) is the inflation of \( A' \). Note that this means that the empty set inflates and reduces to itself.

We will need to know what happens when we contract a \( \lambda \)-tight set. Note that \( \lambda \) is itself a routing function \( (\lambda_G(x, y) \geq \lambda_G(x, y) \text{ for all } x, y \in V) \) and so a \( \lambda \)-tight set, \( X \subseteq V \), is one with \( d_G(X) = \lambda_G(X) \).

Lemma 2.4.1 Let \( G = (V + s, E) \) be a hypergraph where \( s \) is only incident with edges of size two, \( Z \subseteq V \) be \( \lambda \)-tight and \( Y \) be a set of vertices with \( Y \subseteq V - Z \) or \( Z \subseteq Y \subseteq V \). Then \( d_G(Y) = d_{G'}(Y') \) and

\[
\lambda'(u, v) = \begin{cases} 
    \lambda(u, v) & \text{if } u, v \notin Z \\
    \max\{\lambda(u, z) : z \in Z\} & \text{if } v = z^*
\end{cases}
\]

Furthermore if \( Y \) is as before, \( \lambda_G(Y) = \lambda_{G'}(Y') \).

Proof: Suppose that \( d_G(Z) = k \) and that \( \Delta_G(Z) = \{e_1, e_2, \ldots, e_k\} \), so that in \( G' \) we have a new vertex \( z^* \) which has \( \Delta_{G'}(z^*) = \{e'_1, e'_2, \ldots, e'_k\} \), where \( e'_i = e_i - Z + z^* \).
Every edge in \( E(G') \) has a unique associate in \( E(G) \). Any edge which does not intersect \( Z \) exists as the same set of vertices in both \( G \) and \( G' \), so we associate such edges with themselves. The edge \( e'_i \in \Delta_G(z^*) \) is associated with \( e_i \in \Delta_G(Z) \). In particular, every edge in \( \Delta_G(Y') \) has an associate in \( \Delta_G(Y) \) and there is no edge in \( \Delta_G(Y) \) which is wholly contained in \( Z \) because \( Y \subseteq V - Z \) or \( Z \subseteq Y \subseteq V \). Therefore \( d_G(Y) = d_G(Y') \).

We now show how paths in \( G \) and \( G' \) correspond. It is straightforward to associate a path in \( G \) which does not intersect \( Z \), with a unique path in \( G' \). The vertices of the path all exist unchanged in \( G' \) and the edges each have a unique associate. Similarly, given a path in \( G' \) which does not contain the vertex \( z^* \) we can find its unique associate in \( G \). We must also, of course, consider paths that intersect \( Z \) or include \( z^* \).

The tightness of \( Z \) implies that \( d_G(Z) = \lambda_G(Z) = \lambda_G(z_1,w) = k \) say, for some \( z_1 \in Z \) and \( w \in V - Z \). So we can find, in \( G \), a family of edge-disjoint paths \( \{P_1,P_2,\ldots,P_k\} \) from \( z_1 \) to \( w \). Each of these paths must include exactly one edge \( e_i \in \Delta_G(Z) \), so we will assume that \( P_i \) contains the edge \( e_i \). We will now define \( Q_i \) to be the ‘half path’ from \( z_1 \) to the edge \( e_i \). That is, if \( P_i = z_1,f_1,v_1,\ldots,f_a = e_i,v_a,f_{a+1},\ldots,w \) we define \( Q_i := z_1,f_1,v_1,\ldots,v_{a-1},e_i \).

We will use \( -Q_j,Q_i \) to represent the sequence from edge \( e_j \) in to \( z_1 \) and back out to edge \( e_i \). We can then associate such a pair with the sequence \( e'_i, z^*, e'_j \) in \( G' \). Because the \( P_i \)'s are edge disjoint, so is the family \( \{Q_1,Q_2,\ldots,Q_k\} \).

Now, suppose \( A \) is a path in \( G \) which intersects \( Z \), but does not terminate in \( Z \). Say \( A = u,f_1,v_1,f_2,\ldots,f_m,v_m = v \). Then somewhere in \( A \) there is a sequence \( e_{a_1} = f_i,v_i,f_{i+1},\ldots,f_{i+j} = e_{a_2} \), where \( e_{a_1}, e_{a_2} \in \Delta_G(Z) \). Then we associate this with the path \( A' = u,f_1,v_1,\ldots,f_{i-1},v_{i-1},e'_{a_1}, z^*, e'_{a_2}, v_{i+1}, f_{i+j+1}, \ldots,v \) in \( G' \). Similarly a path in \( G' \) including \( z^* \), must include a sequence \( e'_1, z^*, e'_2 \) which is replaced in \( G \) with the pair \( -Q_{b_1}, Q_{b_2} \). Using the correspondence of \( -Q_i \) to \( e'_i, z^* \) it is not difficult to extend this argument to paths that terminate in \( Z \).

Using these associations we can prove the second part of the result. Suppose \( u',v' \in V' \), and that \( \lambda'(u',v') = t \). We have two cases.

**Case 1:** \( u',v' \neq z^* \). Then we can find, using the associations above, \( t \) \((u,v)\)-paths in \( G \). Also, we can find no more, because if we could we could associate the other way, finding more than \( t \) \((u',v')\)-paths in \( G' \). So \( \lambda'(u',v') = \lambda(u,v) \).

**Case 2:** \( v' = z^* \). Then we can find \( t \) \((u,z_1)\)-paths in \( G \). If there were \( z_2 \in Z \) with \( \lambda(u,z_2) = t_2 > t \), we could associate the other way to find \( t_2 \) \((u',z^*)\)-paths in \( G' \). So we must have \( t = \max\{\lambda(u,z) : z \in Z\} \) as required.

Finally, suppose \( \lambda_G(Y) = \lambda_G(y_1,x) = n \). Then we can find \( n \) edge disjoint \((y_1,x)\)-paths in \( G \). Also because \( Y \subseteq V - Z \) or \( Z \subseteq Y \subseteq V \), at most one of \( y_1, x \) is in \( Z \). Then using the associations above we can find \( n \) paths in \( G' \) from
y₁ to x, if neither is in Z, or y₁ to z⁺ (or vice versa), if necessary. These paths must also be edge disjoint and so \( \lambda_{G'}(Y') \geq \lambda_G(Y) \). It is straightforward to apply the associations the other way, to see that \( \lambda_G(Y) \geq \lambda_{G'}(Y') \) also holds. Therefore we must have equality. 

When we have a routing function \( r \) and we contract a \( \lambda \)-tight set, we also need to “contract” \( r \). We do this in such a way as to ensure that if \( r \) is a function determined by properties of the hypergraph, then \( r' \) is determined by the new hypergraph \( G' \). For instance, if we are taking \( r \) to be \( \lambda \), we need to define \( r' \) so that it is equivalent to \( \lambda' \) - the local connectivity in the new hypergraph.

Let \( G = (V + s, E) \) be a hypergraph with \( s \) only incident with edges of size two. Let \( r \) be a routing function on \( G \) and \( Z \subseteq V \) be \( \lambda \)-tight. We define, \( r' \) the contraction of \( r \) as follows

\[
r'(x, y) := \begin{cases} r(x, y) & \text{if } x, y \notin Z \\
\max\{r(x, z) : z \in Z\} & \text{if } y = z⁺ 
\end{cases}
\]

**Proposition 2.4.2** Let \( G = (V + s, E) \) be a hypergraph, \( Z \subseteq V \) a \( \lambda \)-tight set, and \( r \) a routing function on \( G \). Then the contraction \( r' \) is such that \( r'(X') = r(X) \) for all sets \( X \) with \( X \subseteq V - Z \) or \( Z \subseteq X \subseteq V \). Furthermore, \( r' \) is a routing function on \( G' = G/Z \).

**Proof:** Firstly, we show that for \( X \subseteq V - Z \) or \( Z \subseteq X \subseteq V \), we must have \( r(X) = r'(X') \). For any such \( X \), \( r'(X') = r'(x, y) = r(a, b) \) for some \( x \in X' \), \( y \in V - X' \), \( a \in X \) and \( b \in V - X \). But \( r(a, b) \leq r(X) \) and so \( r'(X') \leq r(X) \).

We also have \( r(X) = r(x, y) = r'(a, b) \). This is obvious when \( x, y \notin Z \). If \( y \in Z \) say, then \( r(X) = r(x, y) \) must be equal to \( \max\{r(x, z) : z \in Z\} \) = \( r'(x, z⁺) \). So \( r(X) \leq r'(X') \) and hence \( r(X) = r'(X') \).

But then for any \( X' \subseteq V' \), by Lemma 2.4.1, we have \( d'(X') = d(X) \geq \lambda(X) \geq r(X) = r'(X') \), which implies that \( r' \) is indeed a routing function on \( G' \).

The next two lemmas are very important in this thesis. We use them when considering both \( \lambda \)-splits and \( k \)-splits - that is, when \( r = \lambda \) and \( r = k \), for a constant \( k \in \mathbb{Z} \). Even though the results are stated in terms of contracting \( r \)-tight sets, we do use Lemma 2.4.1 and Proposition 2.4.2. This is okay because for a routing function, we have \( d(X) \geq \lambda(X) \geq r(X) \) and hence, an \( r \)-tight set is automatically \( \lambda \)-tight.

**Lemma 2.4.3** Let \( G = (V + s, E) \) be a hypergraph where \( s \) is only incident with edges of size two and \( r \) a routing function on \( G \). Let \( Z \subseteq V \) be \( r \)-tight and \( G' = G/Z \). Let \( Y \subseteq V \) be such that \( Y \subseteq V - Z \) or \( Z \subseteq Y \subseteq V \). Then
(a) \( Y' \) is \( r'\)-dangerous in \( G' \) if and only if \( Y \) is \( r\)-dangerous in \( G \).

(b) \( Y' \) is \( r'\)-tight in \( G' \) if and only if \( Y \) is \( r\)-tight in \( G \).

**Proof:** If \( Y \) is \( r\)-dangerous in \( G \) then using Lemma 2.4.1 and Proposition 2.4.2 we have 

\[
d_{G'}(Y') = d_G(Y) \leq r(Y) + 1 = r'(Y') + 1
\]

and so \( Y' \) is \( r\)-dangerous in \( G' \). We can see that the other direction holds by interchanging \( Y \) and \( Y' \).

Similarly for (b), if \( Y \) is \( r\)-tight in \( G \) then 

\[
d_{G'}(Y') = d_G(Y) = r(Y) = r'(Y').
\]

Thus \( Y' \) is \( r\)-tight in \( G' \) and the other direction, again, holds by a similar argument with \( Y \) and \( Y' \) interchanged. \( \square \)

The next Lemma is basically a corollary of the previous.

**Lemma 2.4.4** Let \( G = (V + s, E) \) be a hypergraph where \( s \) is only incident with edges of size two, let \( r \) be a routing function on \( G \) and let \( Z \) be an \( r\)-tight subset of \( V \). A pair of edges \( su, sv \) form an \( r\)-split in \( G \) if and only if the corresponding pair \( su', sv' \) is an \( r'\)-split in \( G' = G/Z \).

**Proof:** Suppose that \( su, sv \) are not an \( r\)-split. Then by Lemma 2.3.1, there is an \( r\)-dangerous set \( X \subseteq V \) containing \( u \) and \( v \). If \( X \cap Z = \emptyset \), then by Lemma 2.4.3, \( X' \) is \( r'\)-dangerous in \( G' \) and \( su', sv' \) are not \( r'\)-splittable.

Now suppose \( X \cap Z \neq \emptyset \) and let \( Y := X \cup Z \). If \( Y \) is \( r\)-dangerous in \( G \), then by Lemma 2.4.3, \( Y' \) is \( r'\)-dangerous in \( G' \) and \( su', sv' \) are not \( r'\)-splittable. If \( Y \) is not \( r\)-dangerous then \( s_G(X \cup Z) \geq 2 \). Then Proposition 2.2.4 (a) cannot hold, because if it did we would have

\[
1 + 0 \geq s_G(X) + s_G(Z) \geq s_G(X \cap Z) + s_G(X \cup Z) \geq 0 + 2.
\]

So we must have 2.2.4 (b) and so

\[
1 + 0 \geq s_G(X) + s_G(Z) \geq s_G(X - Z) + s_G(Z - X) + 2d_{3, G}(X, Z) + d_{4, G}(X, Z).
\]

This implies that \( d_{3, G}(X, Z) = 0 \) and \( s_G(X - Z) \leq 1 \). This means that \( X - Z \) is an \( r\)-dangerous set which contains \( u \) and \( v \). Then Lemma 2.4.3 implies that \( (X - Z)' \) is \( r'\)-dangerous in \( G' \) and so \( su', sv' \) are not \( r'\)-splittable in \( G' \).

The other direction is easier. Suppose \( su', sv' \) are not \( r'\)-splittable in \( G' \). Then there is a set \( X' \subseteq V' \) which is \( r'\)-dangerous in \( G' \). Lemma 2.4.3 implies that
$X$ is $r$-dangerous in $G$, and contains $u$ and $v$, and thus $su, sv$ is not $r$-splittable in $G$. \hfill \Box

We will also need the following straightforward inclusion laws.

**Proposition 2.4.5** Let $G = (V + s, E)$ be a hypergraph and $Z \subset V$ be a $\lambda$-tight set. Let $G' = G/Z$ and $A', B'$ be subsets of $V'$ such that $A' \supseteq B'$. Then in $V$, $A \supseteq B$. \hfill \Box

**Proposition 2.4.6** Let $G = (V + s, E)$ be a hypergraph and $Z \subset V$ be a $\lambda$-tight set. Let $G' = G/Z$ and $A', B', C'$ be subsets of $V'$ such that $A' \cap B' = C'$. Then in $V$, $A \cap B = C$. \hfill \Box
Chapter 3

Splitting to Preserve $\lambda$

3.1 Introduction

In this chapter, we are concerned with finding (or not finding) $\lambda$-splits in a hypergraph, $G = (V + s, E)$. Recall that $\lambda$ is a routing function. This means that we can use the results in Chapter 2 and we can deal with, $\lambda$-tight and $\lambda$-dangerous sets.

From this point forward, when dealing with a hypergraph $G = (V + s, E)$ where $s$ is only incident with edges of size two, we will denote by $N(s) = \{x_1, x_2, \ldots, x_t\}$ the set of $s$-neighbours in $G$. For $1 \leq i \leq t$ we also define $N^i(s) := \{x_i, x_{i+1}, \ldots, x_t\}$. Sometimes, so that it is clear which hypergraph we are dealing with, we will use $N_G(s)$ and $N^i_G(s)$. In many of the lemmas that follow, we work with hypergraphs for which “every tight subset of $V$ is a singleton.” When this is the case, we suppose that the labelling of the $s$-neighbours has been chosen by degree, so that we have $d(x_1) \leq d(x_2) \leq \ldots \leq d(x_t)$. In our main theorems we work with minimum counterexamples, in which every tight set must be a singleton and this ordering will follow during the proofs.

3.2 Fans

Suppose that there is no $\lambda$-split at $s$ in $G$. Then Lemma 2.3.1 implies that every pair of vertices in $N_G(s)$ must lie in some $\lambda$-dangerous subset of $V$. For $x_i \in N_G(s)$ we define $L_i$, a fan centred on $x_i$, as a family of $\lambda$-dangerous sets, each containing $x_i$ and covering $N^i(s)$, such that $|L_i|$ is as small as possible.

The following results establish some properties of these fans, which we will use
in the main results of this chapter. We will need the following short Proposition used by Frank in the graph case.

**Proposition 3.2.1** In a hypergraph $G = (V + s, E)$ with every $\lambda$-tight set a singleton, $\lambda(x, y) = \min\{d(x), d(y)\}$, for all $x, y \in V$.

**Proof:** By Menger’s theorem, for all $x, y \in V$ there exists some set $X \subseteq V$ which separates $x$ and $y$ and has $d(X) = \lambda(x, y)$. That is, given a pair of vertices, there is a $\lambda$-tight set which separates them. But every $\lambda$-tight set is a singleton and so, that set must either be $\{x\}$ or $\{y\}$. Thus $\lambda(x, y) = \min\{d(x), d(y)\}$.

**Lemma 3.2.2** Let $G = (V + s, E)$ be a hypergraph with $s$ only incident with edges of size two and every $\lambda$-tight set a singleton. Suppose there is no $\lambda$-split at $s$ and let $L_i$ be a fan centred on $x_i$. If $|L_i| \geq 2$, then for all distinct $X, Y \in L_i$, we have $|X - Y| = |Y - X| = d_3(X, Y) = 1$ and $d_4(X, Y) = 0$.

**Proof:** Proposition 3.2.1 tells us that $\lambda(x, y)$ is easy to find in this sort of hypergraph. We use this fact to prove the following.

**Claim 3.2.2.1** $\lambda(X - x_i) \geq \lambda(X)$ for each $X \subseteq V$ with $|N^i(s) \cap X| \geq 2$.

**Proof:** Let $x_j \in N^i(s) \cap (X - x_i)$. Then $j > i$ so $d(x_j) \geq d(x_i)$. Also suppose $\lambda(X) = \lambda(u, v)$ for some $u \in X, v \in V - X$. If $u \neq x_i$ then $\lambda(X - x_i) \geq \lambda(u, v) = \lambda(X)$ as required. If $u = x_i$ then

\[
\lambda(X) = \lambda(x_i, v) = \min\{d(x_i), d(v)\} \\
\leq \min\{d(x_j), d(v)\} \\
= \lambda(x_j, v) \\
\leq \lambda(X - x_i).
\]

The minimality of $|L_i|$ implies that each $X \in L_i$ contains some element of $N(s)$ not in any other member of $L_i$. Also for $X, Y \in L, X \cup Y$ cannot be $\lambda$-dangerous. So we have $s(X \cup Y) \geq 2$.

Suppose that Proposition 2.2.4(b) does not hold for $X, Y \in L_i$. Then Proposition 2.2.4(a) does, and we have

\[1 + 1 \geq s(X) + s(Y) \geq s(X \cap Y) + s(X \cup Y) \geq 0 + 2.\]
Thus we have equality throughout and \( X \cap Y \) is \( \lambda \)-tight which means that \( X \cap Y = \{x_i\} \). Therefore \( X - Y = X - x_i \) and \( Y - X = Y - x_i \) and so, by Claim \( 3.2.2.1 \), \( \lambda(X) \leq \lambda(X - Y) \) and \( \lambda(Y) \leq \lambda(Y - X) \). Therefore, \( \lambda(X) + \lambda(Y) \leq \lambda(X - Y) + \lambda(Y - X) \) and by the definition of \( s(X) \), Proposition 2.2.4(b) holds, that is

\[
s(X) + s(Y) \geq s(X - Y) + s(Y - X) + 2d_3(X, Y) + d_4(X, Y).
\]

Because \( X \) and \( Y \) are both \( \lambda \)-dangerous, \( x_i \in X \cap Y \) and \( x_i \) is an \( s \)-neighbour we have \( d_3(X, Y) \geq 1 \). Using 2.2.4(b) it follows that,

\[
1 + 1 \geq s(X) + s(Y) \geq s(X - Y) + s(Y - X) + 2d_3(X, Y) + d_4(X, Y) \geq 0 + 0 + 2 + 0
\]

and so we have equality throughout. Thus \( X - Y \) and \( Y - X \) are \( \lambda \)-tight, and therefore singleton sets, and \( d_3(X, Y) = 1, d_4(X, Y) = 0 \).

\[\square\square\]

The next result tells us what most fans look like.

**Lemma 3.2.3** Let \( G = (V + s, E) \) be a hypergraph with \( s \) only incident with edges of size two and every \( \lambda \)-tight set a singleton. Suppose there is no \( \lambda \)-split at \( s \) and let \( L_i \) be a fan centred on \( x_i \in N(s) \). If \( |L_i| \geq 2 \), then we can find \( M_i \subset V \) such that \( N(s) \cap M_i = \{x_i\} \), \( L_i = \{X_{ij} = M_i + x_j : x_j \in N(s) \text{ and } j > i\} \) and \( |L_i| = |N(s)| - i \).

**Proof:** Let \( M_i = \bigcap_{X \in L_i} X \). By Lemma 3.2.2, every pair of sets \( X, Y \in L_i \) has \( |X - Y| = |Y - X| = 1 \) and \( N(s) \cap (X \cap Y) = \{x_i\} \). This implies that \( X_{ij} = M_i + x_j \) for each \( X_{ij} \in L_i \), as required. This means that in \( L_i \), we have exactly one set for each \( x_j \) with \( j > i \). Thus, \( |L_i| = |N(s)| - i \). \[\square\]

From this point forward, whenever we have a fan \( L_i \) with \( |L_i| \geq 2 \) we will automatically use Lemma 3.2.3 to define its \( M_i \), which we will call the **heart** of \( L_i \).

The next lemma tells us a little more about how the fan sits in the hypergraph.

**Lemma 3.2.4** Let \( G = (V + s, E) \) be a hypergraph where \( s \) is only incident with edges of size two and every \( \lambda \)-tight set a singleton. Suppose there is no \( \lambda \)-split at \( s \) and let \( L_i \) be the fan centred on \( x_i \). If \( |L_i| \geq 3 \) then every edge in \( \delta(M_i) \) is incident with every \( x_j \) with \( j > i \).

**Proof:** We are assuming that \( |L_i| \geq 3 \) and so, by Lemma 3.2.3, we can find \( M_i \), the heart of the fan. Let \( e \in \delta(M_i) \).
Suppose $e$ is not incident with any $x_j \in N^i(s)$. Choose $X,Y \in L_i$. Then
$X = M_i + x_j$, and $Y = M_i + x_{j_1}$, for some $i < j_1, j_2 \leq t$, and so $e$ does not intersect either $X - Y = x_{j_1}$ or $Y - X = x_{j_2}$. By Lemma 3.2.2, $d_3(X,Y) = 1$ and we have an edge $sz_i$, hence $e$ does not intersect $V - (X \cup Y)$. So $e$ just intersects $X \cap Y = M_i$. That is $e$ is not in $\delta(M_i)$, which is a contradiction. So $e$ is incident with some vertex in $N^i(s)$.

Suppose $e$ is incident with $u \in N^i(s)$ and choose $z,y \in N^i(s)$ such that $u \neq z \neq y$. (This is possible because $|L_i| \geq 3$.) Suppose that $z \in Z, y \in Y$ where $Z,Y \in L_i$. Then, because $e$ is incident with $z \in Z \cap Y$ and $u \in V - (Z \cup Y)$, Lemma 3.2.2 implies that $e$ must intersect both $Z - Y$ and $Y - Z$. That is, $e$ must be incident with both $z$ and $y$. Thus $e$ is incident with every $x_j \in N^i(s)$ with $j > i$.

\[\square\]

### 3.3 Multiple Edges and Maximum Sizes

We can now prove the first main result of this chapter. We are trying to characterise the hypergraphs in the form $G = (V + s, E)$ where there is no $\lambda$-split at $s$. When performing splits it is natural to first split away any multiple edges and it turns out that if any $s$-neighbour has more than one edge to $s$, there is a $\lambda$-split.

**Theorem 3.3.1** Let $G = (V + s, E)$ be a hypergraph where $s$ is only incident with edges of size two, and $d(s) \geq 3$. If $d(s) > |N(s)|$ there is a $\lambda$-split at $s$.

**Proof:** We argue by contradiction using a minimum counterexample. Suppose the statement is not true. Let $G$ have $|V| + |E|$ as small as possible such that $d(s) > |N(s)|$ and there is no $\lambda$-split at $s$. Then every $\lambda$-tight set $X \subseteq V$ has one element. This is because, contracting any larger $\lambda$-tight set would create a smaller hypergraph, which by Lemma 2.4.4, has no $\lambda$-split at $s$, and still has multiple edges to $s$.

Let $L_1$ be a fan centred on $x_1$. If $|L_1| = 1$, then there exists $X$ such that every $s$-neighbour is a member of $X$ and $X$ is dangerous. Then $d_1(s,X) = d(s) \geq 3 > 0 + 1 = d_1(s,V - X) + 1$, which contradicts Lemma 2.3.2. Therefore, $|L_1| \geq 2$.

**Claim 3.3.1.1** $|L_1| \geq 3$.

**Proof:** Suppose $|L_1| = 2$. That is, $L_1 = \{X,Y\}$. Then by Lemma 3.2.2, $|X - Y| = |Y - X| = 1$ and $d_3(X,Y) = 1$. Therefore $s$ has just three neighbours, $x_1, x_2, x_3$. Also, because $d(s) > |N(s)|$, we can assume (wlog) that $d_1(s,x_3) \geq 2$. 39
Therefore any set \( W \subseteq V \) containing both \( x_2 \) and \( x_3 \) has \( d_1(s, W) \geq 3 > 1 + 1 = d_1(s, V - W) + 1 \). So Lemma 2.3.2 implies that \( W \) is not \( \lambda \)-dangerous and so by Lemma 2.3.1 we can split a pair of edges \( \{sx_2, sx_3\} \). This is a contradiction. So \( |L_1| \geq 3 \). By Lemma 3.2.3, \( |N(s)| \geq 4 \) and because \( d(s) > |N(s)| \), we have \( d(s) \geq 5 \).

Let \( M_1 \) be the heart of \( L_1 \). By Lemma 3.2.3, \( L_1 = \{M_1 + x_j : 1 \leq i < j \leq t\} \).

There are two cases to consider. In each we find a smaller counterexample and thus contradict the minimality of \( G \).

**Case 1:** \( \delta(M_1) \) is empty.

Consider two distinct sets \( X, Y \in L_1 \). Because \( |L_1| \geq 3 \), Lemma 3.2.2 implies that \( d_3(X, Y) = d_1(s, x_1) = 1 \). Therefore there is exactly one edge \( sx_1 \) incident with \( M_1 \). Let \( G_1 \) be the hypergraph \( G - sx_1 \). We show that \( G_1 \) is a smaller counterexample to the theorem.

In \( G \) there is only one edge \( sx_1 \), so in \( G_1 \) we have \( d_{G_1}(s) > |N_{G_1}(s)| \). Therefore, there is a vertex in \( N_{G_1}(s) \) with parallel edges to \( s \). Suppose \( X \subseteq V \) is a set which contains at least two \( s \)-neighbours from \( V - M_1 \), and is \( \lambda \)-dangerous in \( G \).

**Claim 3.3.1.2** \( X \) is \( \lambda \)-dangerous in \( G_1 \).

**Proof:** Firstly, note that to create \( G_1 \) from \( G \) we have removed one edge. Therefore \( d_{G_1}(X) \leq d_G(X) \).

Firstly, suppose that \( \lambda_G(X) = \lambda_G(x, y) \geq 2 \) for some \( x \in X, y \in V - X \). Then either \( \{x, y\} \subseteq V - M_1 \) or \( \{x, y\} \subseteq M_1 \), because for a pair of vertices \( a \in V - M_1 \) and \( b \in M_1 \) we have \( \lambda_G(a, b) \leq 1 \). Also, there can be no \( (x, y) \)-path which uses the edge \( sx_1 \), since \( \delta(M_1) \) is empty. So \( \lambda_G(X) = \lambda_G(x, y) = \lambda_{G_1}(x, y) = \lambda_{G_1}(X) \). Therefore \( d_{G_1}(X) \leq d_{G}(X) \leq \lambda_{G}(X) + 1 = \lambda_{G_1}(X) + 1 \) and so \( X \) is \( \lambda \)-dangerous in \( G_1 \).

Now suppose that \( \lambda_G(X) \leq 1 \). Recall that \( X \) is \( \lambda \)-dangerous and contains \( x_i, x_j \in N(s) \) where \( 1 \neq i \neq j \neq 1 \). Lemma 2.3.2 implies that \( X \) cannot contain all the vertices in \( N(s) \), that is \( (V - X) \cap N(s) \) is non-empty. Therefore \( \lambda_G(X) \) cannot be zero, because there is at least one path from the \( s \)-neighbours in \( X \), to those not in \( X \).

If \( (V - X) \cap N(s) = \{x_1\} \), then in \( G \), by Claim 3.3.1.1 we have \( d_1(s, X) \geq 4 > 2 = d_1(s, V - X) + 1 \) contradicting Lemma 2.3.2. So \( V - X \) must contain some other \( x_k \in N(s) \) with \( k \neq 1, i \) or \( j \). But then \( \lambda_{G_1}(X) \geq 1 \) because there is a path from \( x_i \in X \) to \( x_k \in V - X \) through \( s \). So we have \( 1 \geq \lambda_G(X) \geq \lambda_{G_1}(X) \geq 1 \), and therefore equality holds throughout and \( \lambda_{G_1}(X) = \lambda_G(X) \). Thus, again, we
have $d_G(X) \leq d_G(X) \leq \lambda_G(X) + 1 = \lambda_G(X) + 1$ and so $X$ is still $\lambda$-dangerous in $G_1$ as required. 

Now, by Lemma 2.3.1 for each pair $x_i, x_j \in N(s)$ with $i,j \geq 2$ there is a set containing $x_i$ and $x_j$, which is $\lambda$-dangerous in $G$. Therefore in $G_1$, for every pair of $s$-neighbours we can find a $\lambda$-dangerous set. It follows that there is no $\lambda$-split in $G_1$ and thus $G_1$ is a smaller counterexample, contradicting the minimality of $G$.

This completes Case 1.

\textbf{Case 2:} $\delta(M_1)$ is not empty.

We now choose an edge $e \in \delta(M_1)$ and consider the hypergraph $G_2 = G - e$. Note that in $G_2$ we must still have $d_{G_2}(s) > |N_{G_2}(s)|$, so if $G_2$ has no $\lambda$-split, it must be a smaller counterexample.

\textbf{Claim 3.3.1.3} There is no $\lambda$-split in $G_2$.

\textbf{Proof:} For every pair, $u,v \in N(s)$, by Lemma 2.3.1 there is a set $X \subseteq V$ which is $\lambda$-dangerous in $G$. Therefore $d_G(X) \leq \lambda_G(X) + 1$. Lemma 2.3.2 implies that $X$ cannot contain all $x_i$ with $i \geq 2$. Thus, $e$ intersects both $X$ and $V - X$. Therefore $d_{G_2}(X) = d_G(X) - 1$. Also, because we’ve only removed one edge, $\lambda_G(X) - 1 \leq \lambda_{G_2}(X)$. Hence we have

$$d_{G_2}(X) = d_G(X) - 1 \leq \lambda_G(X) \leq \lambda_{G_2}(X) + 1.$$ 

Thus, $X$ is $\lambda$-dangerous in $G_2$ and by Lemma 2.3.1, there is no $\lambda$-split in $G_2$. 

Therefore $G_2$ is a smaller counterexample. This contradicts the minimality of $G$ and completes Case 2.

In both cases we have found a smaller counterexample. Therefore, our original assumption was false and the theorem holds.

\textbf{Corollary 3.3.2} Let $G = (V + s, E)$ be a hypergraph where $s$ is only incident with edges of size two. If there is no $\lambda$-split at $s$, then $d(s) = |N(s)|$.

Theorem 3.3.1 also provides the following very slight extension of Mader’s Theorem for graphs.

\textbf{Theorem 3.3.3} Let $G = (V + s, E)$ be a connected undirected graph, with $d(s) \geq 2$. Then, if $d(s) \neq 3$, there is a $\lambda$-split at $s$. Further, if $d(s) = 3$ and $s$ has only two neighbours, there is a $\lambda$-split at $s$. 

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Proof: The first part is Mader’s classical theorem (see [53], or for a shorter proof, [23]). The second is an immediate corollary of Theorem 3.3.1.

In Chapter 4, we use the next result to provide an upper bound for the augmentation number of a hypergraph with respect to a local connectivity demand function.

**Theorem 3.3.4** Let $G = (V + s, E)$ be a hypergraph where $s$ is only incident with edges of size two, no cut edge is incident with $s$ and $d(s) \geq 4$. Then if there is no $\lambda$-split at $s$, $d(s) \leq |e_{\text{max}}|$, where $e_{\text{max}}$ is the largest edge in $E$.

**Proof:** Suppose the statement is false, and let $G$ be a counterexample with $|V| + |E|$ as small as possible. Then by Corollary 3.3.2 $d(s) = |N(s)|$.

**Claim 3.3.4.1** In $G$, every $\lambda$-tight set a singleton.

**Proof:** If there were a larger $\lambda$-tight set, $T \subset V$ say, consider the smaller hypergraph, $G' = G/T$. By Lemma 2.4.4, there would be no $\lambda$-split at $s$ and clearly $d_{G'}(s) = d_G(s)$. Also, the largest edge in $G'$ could be no larger than that in $G$. So $G'$ would be a counterexample, contradicting the minimality of $G$. □

Let $L_1$ be a fan centred on $x_1$. Lemma 2.3.2 implies that $|L_1| \geq 2$. Therefore Lemma 3.2.2 implies that $d_1(s, x_1) = 1$.

**Claim 3.3.4.2** $|L_1| \geq 3$

**Proof:** If $L_1 = \{X, Y\}$ then by Lemmas 3.2.2 and 3.2.3 we must have $d(s) = 3$. This contradicts our hypothesis that $d(s)$ is at least four. □

We know that $d_1(s, x_1) = 1$ and by hypothesis, the edge $sx_1$ is not a cut edge. Therefore $\delta_G(M_1)$ is non-empty. Lemma 3.2.4 implies that there is an edge, $e$ incident every $x_j \in N(s)$ with $j \geq 2$ and intersecting the heart of fan $L_1$, $M_1 = \bigcap_{X \in L_1} X$. Then we have $d(s) = |N(s)| = |L_1| + 1 \leq |e|$. Hence $d(s) \leq |e_{\text{max}}|$, contradicting our assumption that $d(s) > |e_{\text{max}}|$. □

**Remark:** The hypothesis that $d(s) \geq 4$ is necessary because there are many graphs with $d(s) = 3$ and $|e| = 2$ for all $e \in E$ with no $\lambda$-split at $s$. For instance $K_4$, with any of the vertices acting as $s$. Also, to see that the hypothesis that there is no cut edge incident with $s$ is necessary, consider the graph $G = (V + s, E)$ where $V = \{x_1, x_2, x_3, x_4\}$ and $E = \{x_3x_4, sx_1, sx_2, sx_3, sx_4\}$. It is not difficult to see that $G$ has no $\lambda$-split, and $d_G(s) = 4 > 2 = |e_{\text{max}}|$. 

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3.4 Characterisation of No Split Graphs

The next result gives a characterisation of hypergraphs of the form $G = (V + s, E)$ for which there is no $\lambda$-split at $s$. Recall that a fan centred on $x_i$ is a smallest possible family of $\lambda$-dangerous sets, all containing $x_i$, and covering $N_i^1(s)$. If a hypergraph $G = (V + s, E)$ has no $\lambda$-split, then we must be able to find a fan centred on each $x_i$ with $i \leq t - 1$. We will use such a collection of fans to characterise hypergraphs with no $\lambda$-split at $s$.

Up until now, we have been choosing the labelling of $s$-neighbours in order of degree. However, in the definition that follows, this ordering does not apply. In fact, it is given by the degree of tight sets containing the $s$-neighbours.

In a hypergraph $G = (V + s, E)$, with $N_G(s) = \{x_1, x_2, \ldots, x_t\}$, we define a break as a pair of sets $(\mathcal{M}, \mathcal{X})$ where $\mathcal{M} = \{M_1, M_2, \ldots, M_{t-2}\}$ and $\mathcal{X} = \{X_1, X_2, \ldots, X_t\}$ with the following properties.

(a) $M_1, M_2, \ldots, M_{t-2}, X_{t-1}, X_t$ is a partition of $V$.
(b) $d(X_1) \leq d(X_2) \leq \ldots \leq d(X_t)$.
(c) $X_i$ is $\lambda$-tight for each $i$.
(d) $x_i \in X_i$ for each $i \leq t$ and $X_i \subseteq M_i$ for all $i \leq t - 2$.
(e) For each $i \leq t - 3$, each edge in $\delta(M_i)$ intersects every $X_j$ with $j > i$.
(f) For $W = M_{i_1} \cup M_{i_2} \cup \ldots \cup M_{i_{k-1}} \cup X_{i_k}$, where $i_1 < i_2 < \ldots < i_k < t$, $d(W) \leq \lambda(W) + (k - 1)$.

The fact that in the general case we are allowed tight sets with more than one element, indicates why $N(s)$ is not ordered by degree. It is possible, for instance, to imagine a hypergraph in which $x_1$ is incident with, the edge $sx_1$ and an enormous number of size-two edges to a vertex $u$. In turn, $u$ is incident with just one additional (hyper)edge. In this hypergraph, we could have $X_1 = \{x_1, u\}$, so that $d(X_1) = 2$ despite $d(x_1)$ being very large. However, in the proof of the theorem below, we work with a minimum counterexample, in which every tight set is a singleton. Hence, in this hypergraph, we revert to choosing the labelling of $N(s)$ by degree.

Unfortunately, because of the complicated structure, we have not been able to use the following theorem to derive a characterisation of the hypergraphs for which there is no complete $\lambda$-split. We discuss this some more in the next chapter, which uses the results in Section 3.3 to provide a near-optimal augmentation result and an extension to a result of Szigeti about augmenting with hyperedges.
In previous works, the structure of the (hyper)graphs with no \( \lambda \)-split had been much simpler. There are two main points of interest here. The fact that we need two sets, \( M_i \) and \( X_i \) for each \( s \)-neighbour, and the “nested” nature of the underlying edge set of the break. As a further example of the difficulties of finding \( \lambda \)-splits, we point out that although it would have been nice to say that (e) holds for edges in \( \delta(M_{t-2}) \), there are examples of hypergraphs that have no \( \lambda \)-split, and have an edge from \( M_{t-2} \) to \( X_{t-1} \) and not to \( X_t \). For instance, consider \( G = (V + s, E) \), where \( V = \{x_1, x_2, x_3, x_4\} \) and \( E = \{x_2x_3, x_3x_4, x_2x_4, sx_1, sx_2, sx_3, sx_4\} \). Setting \( M_i = x_i \) for \( i = 1, 2 \) and \( X_i = x_i \) for \( i = 1, 2, 3, 4 \), it is not difficult to check that \( (M, X) \) is a break in \( G \), and that \( G \) has no \( \lambda \)-split. Then the edge \( x_2x_3 \), is in \( \delta(M_2) \), and is not incident with \( X_4 \) as required.

**Theorem 3.4.1** Let \( G = (V + s, E) \) be a hypergraph where \( s \) is only incident with edges of size two and \( d(s) = t \geq 4 \). There is no \( \lambda \)-split at \( s \) if and only if there is a break in \( G \).

**Proof:** Firstly, suppose there exists a break in \( G \). To see there is no \( \lambda \)-split at \( s \) we find, for every pair of vertices in \( N(s) \), a \( \lambda \)-dangerous set. Let \( x_i, x_j \in N(s) \) with \( i < j \). If \( j \leq t - 1 \) we can let \( W = M_i \cup X_j \) and then \( d(W) \leq \lambda(W) + 1 \) by (f). That is, \( W \) is \( \lambda \)-dangerous containing \( x_i \) and \( x_j \). Now suppose \( j = t \). If \( i \leq t - 2 \) let \( W = M_1 \cup M_2 \cup \ldots \cup M_{t-2} \cup X_{t-1} - M_i \) and let \( Y = V - W = M_i \cup X_t \). Then \( Y \) contains \( x_i \) and \( x_t \) and \( d(Y) = d(W) - (t - 2) + 2 = d(W) - t + 4 \leq \lambda(W) + (t - 2 - 1) - t + 4 = \lambda(W) + 1 = \lambda(Y) + 1 \). That is, \( Y \) is \( \lambda \)-dangerous. Finally, to find a \( \lambda \)-dangerous set containing \( x_{t-1} \) and \( x_t \), let \( W = M_1 \cup \ldots \cup M_{t-3} \cup X_{t-2} \) and again let \( Y = V - W = X_{t-1} \cup X_t \). Then by the same argument \( d(Y) \leq \lambda(Y) + 1 \). Therefore there is no \( \lambda \)-split at \( s \).

We now need to show that if there is no \( \lambda \)-split, there does exist a break. We argue by contradiction using a minimum counterexample. So, we assume that we can find hypergraphs with no \( \lambda \)-split and no break and let \( G \) be such a counterexample, which has \( |V| + |E| \) as small as possible. Because there is no \( \lambda \)-split at \( s \), Corollary 3.3.2 implies that \( d(s) = |N(s)| = t \). We write \( N(s) = \{x_1, x_2, \ldots, x_t\} \) as usual, and suppose that the labelling has been chosen such that \( d(x_1) \leq d(x_2) \leq \ldots \leq d(x_t) \).

In order that we can use our previous results, our first task is to show the following.

**Claim 3.4.1.1** In \( G \) every \( \lambda \)-tight set is a singleton.

**Proof:** Suppose not and let \( Z \) be a \( \lambda \)-tight subset of \( V \) with \( |Z| \geq 2 \). Let \( G' = G/Z \) and \( z^* \) be the new vertex. Then by Theorem 3.3.1 and Lemma
2.4.4, \( Z \) cannot contain more than one member of \( N_G(s) \). Now, \( G' \) is smaller than \( G \) and by Lemma 2.4.4, has no \( \lambda \)-split. So we can find a break, \((M', X')\), \\
\( \mathcal{M}' = \{M'_1, \ldots, M'_{|\mathcal{L}|} \} \) and \( \mathcal{X}' = \{X'_1, \ldots, X'_{|\mathcal{L}|} \} \) in \( G' \). We then inflate these sets to form \( \mathcal{M} = \{ M_i : M'_i \in \mathcal{M}' \} \) is the reduction of \( M_i \) and \( \mathcal{X} = \{ X_i : X'_i \in \mathcal{X}' \} \) is the reduction of \( X_i \). We show that \((\mathcal{M}, \mathcal{X})\) is a break in \( G \).

The definition of inflation means every \( M_i \) and \( X_i \) is either contained by \( V - Z \) or contain \( Z \) completely. Therefore \((a)\) holds for \((\mathcal{M}, \mathcal{X})\), Lemma 2.4.1 gives \((b)\) and Lemma 2.4.3 implies that \((c)\) holds. If \( x'_i \neq z^* \) and we consider \( x'_i \) as a singleton set contained in \( X'_i \), Proposition 2.4.5 implies the first half of \((d)\). If \( z^* \) is an \( s \)-neighbours in \( G' \), say \( z^* = x'_i \), then in \( G \), \( x_i \in Z \subseteq X_i \), so again the first part of \((d)\) holds. Proposition 2.4.5 implies the second half in both cases. If we identify an edge with its associated vertex set, Proposition 2.4.6 implies that \((e)\) holds. Finally Lemma 2.4.1 implies that \( d_G(W) = d_{G'}(W') \leq \lambda_G(W') + (k - 1) = \lambda_G(W) + (k - 1) \) and so \((f)\) holds. That is \((\mathcal{M}, \mathcal{X})\) is a break in \( G \) and therefore \( G \) is not a counterexample. This contradiction shows that our minimal counterexample has every \( \lambda \)-tight set a singleton.

We next find \( \mathcal{M} \) and \( \mathcal{X} \) that form a break in \( G \). For each \( i \leq t - 2 \) let \( L_i \) be the fan centred on \( x_i \), Lemma 3.2.3 says that if \( |L_i| \geq 2 \), we can find a set \( M_i \) with \( N(s) \cap M_i \) for some \( M_i \) such that \( M_i \cup x_i \) is \( \lambda \)-dangerous. Eventually we will have \( \mathcal{M} \) as the set of these \( M_i \). First, though, we must show that we do indeed have \( |L_i| \geq 2 \) for all \( 1 \leq i \leq t - 2 \).

To begin, notice that if \( |L_i| = 1 \) for some \( i \), then there is a set \( X \), \( \lambda \)-dangerous in \( G \), covering \( N^i(s) = \{ x_i, x_{i+1}, \ldots, x_1 \} \). This means \( X \) will have \( d_1(s, X) \geq |N^i(s)| = t - i + 1 \). We work case by case, considering all possible values of \( i \).

**Claim 3.4.1.2** If \( i \leq \frac{t}{2} \), \( |L_i| \geq 2 \).

**Proof:** Let \( i \leq \frac{t}{2} \). Let \( X \subseteq V \) be a set with \( N^i(s) \subseteq X \). Then \( d_1(s, X) \geq |N^i(s)| \geq t - \frac{t}{2} + 1 = \frac{t}{2} + 1 \). Therefore \( d_1(s, V - X) \leq t - |N^i(s)| \leq \frac{t}{2} - 1 \). So \( d_1(s, X) \geq d_1(s, V - X) + 2 \) and by Lemma 2.3.2, \( X \) is not \( \lambda \)-dangerous. Thus \( |L_i| \geq 2 \).

From this point, we need only deal with \( t \geq 5 \). If \( t = 4 \) the previous Claim gives us all we need, because \( \frac{t}{2} = t - 2 \).

**Claim 3.4.1.3** If \( t \) is odd, and \( i = \left\lceil \frac{t}{2} \right\rceil \) then \( |L_i| \geq 2 \).

**Proof:** Suppose \( t = a + (a + 1) \) and \( L_i = \{X\} \). Then \( d_1(s, X) \geq a + 1 \), because \( N^{a+1}(s) \subseteq X \), and in fact, equality must hold. Otherwise \( d_1(s, X) \geq a + 2 > a + 1 \geq d_1(s, V - X) + 1 \) which is contrary to Lemma 2.3.2. So \( d_1(s, X) = a + 1 \).
and \( d_1(s, V - X) = a \). Therefore \( d(V - X) = d(X) - 1 \leq \lambda(X) = \lambda(V - X) \) and so \( V_X \) is \( \lambda \)-tight. This is a contradiction, because every \( \lambda \)-tight set is a singleton in \( G \), and \( |V - X| \geq a \). So \( |L_i| \geq 2 \). \( \square \)

We can now assume \( t \geq 6 \), because if \( t = 5 \) then \( \lceil \frac{1}{2} \rceil = 3 = t - 2 \) and we have all we need.

The previous two Claims tell us that, in our counterexample, when \( i \leq \lceil \frac{1}{2} \rceil \), \( L_i \) is in the form of Lemma 3.2.3. To show that this is also true for larger values of \( i \), we remember that a fan is the smallest collection of \( \lambda \)-dangerous sets required to cover \( N^j(s) \). We then show that if there is some \( i \) with \( |L_i| = 1 \), we can find for some \( j \leq \lceil \frac{1}{2} \rceil \) a collection of \( \lambda \)-dangerous sets, covering \( N^j(s) \) which is smaller than \( L_j \) as given by Lemma 3.2.3.

So suppose that we can find \( \lceil \frac{1}{2} \rceil < k \leq t - 2 \) with \( |L_i| \geq 2 \) for all \( i < k \) and \( L_k = \{ X \} \), where \( X \subseteq V \) is \( \lambda \)-dangerous. Let \( p := t - k + 1 = d_1(s, X) \). Then \( p \geq 3 \). Let \( Y := V - X \).

By Lemma 3.2.3, for each \( i < k \) we can find \( M_i \) such that \( L_i = \{ X_{ij} = M_i + x_j : i < j \leq t \} \) and \( |L_i| = t - i \). Because \( i < k \leq t - 2 \), we have \( i \leq t - 3 \) which implies \( |L_i| \geq 3 \) for all \( i < k \).

**Claim 3.4.1.4** For every \( i < k \), \( M_i \cap X = \emptyset \).

**Proof:** For \( i < k \) consider \( X_{ik} = M_i + x_k \in L_i \). By Proposition 2.2.4 we must have either 2.2.4(a) or 2.2.4(b) for sets \( X \) and \( X_{ik} \). If 2.2.4 (b) holds we have

\[
1 + 1 \geq s(X) + s(X_{ik}) \geq s(X - X_{ik}) + s(X_{ik} - X) + 2d_{3}(X, X_{ik}) + d_{4}(X, X_{ik}) \geq 0 + 0 + 2 + 0
\]

since there is an edge \( sx_k \) and \( x_k \in X \cap X_{ik} \). Hence equality holds throughout, implying that \( X - X_{ik} \) and \( X_{ik} - X \) are \( \lambda \)-tight and therefore singletons. Since \( d_1(s, X) \geq 3 \), this means that \( X \cap X_{ik} \) contains at least two \( s \)-neighbours. This contradicts the fact that \( X_{ik} \) only contains two members of \( N(s) \), namely \( x_i \) and \( x_k \), by Lemma 3.2.3. Therefore Proposition 2.2.4 (b) does not hold for \( X \) and \( X_{ik} \).

So Proposition 2.2.4 (a) must hold. Recalling the minimality of \( L_i \) we see that \( X \cup X_{ik} \) cannot be \( \lambda \)-dangerous. So we have

\[
1 + 1 \geq s(X) + s(X_{ik}) \geq s(X \cap X_{ik}) + s(X \cup X_{ik}) \geq 0 + 2.
\]

Hence we have equality throughout and \( X \cap X_{ik} \) is \( \lambda \)-tight and a singleton. So \( X \cap X_{ik} = \{ x_k \} \) and therefore \( M_i \cap X = \emptyset \). \( \square \)
Therefore, using Lemma 3.2.3 we can find $M_y$-dangerous. (Note that if $t$ is even and $k = \frac{t}{2} + 1$ then $p = \frac{t}{2}$, $M^*$ is empty and thus, $A = Y$.)

Recall that for any $Z \subseteq V$, $\delta(Z)$ denotes the set of edges on the border of $Z$, (which does not include any edges to $s$). Then, $\delta(X) = \delta(Y)$. Also, because for $i < k$ we have $|L_i| \geq 3$, Lemma 3.2.4 applies and any edge on the border of $M^*$ intersects every $x_i$ with $k - p \leq i \leq 2m$.

**Claim 3.4.1.5** $\delta(A) = \delta(Y)$.

**Proof:** Let $e \in \delta(A)$ and suppose that $e \notin \delta(Y)$. Then $e \subseteq Y$. So $e$ intersects both $A$ and $Y - A = M^*$. That is, $e$ is on the border of $M^*$ and so must be incident with $x_k$ by Lemma 3.2.4. Thus $e \in \delta(X) = \delta(Y)$ which is a contradiction. So, if $e \in \delta(A)$ then $e \in \delta(Y)$. That is $\delta(A) \subseteq \delta(Y)$.

Now suppose $e \in \delta(Y)$ but $e \notin \delta(A)$. Then $e \in \delta(M^*)$ and hence, by Lemma 3.2.4, $e$ intersects $x_{k-1}$. But then $e \in \delta(A)$. So, every edge in $\delta(Y)$ is in $\delta(A)$.

So we have $\delta(A) = \delta(X)$ because $\delta(Y) = \delta(X)$. We chose $A$ to have $d_1(s, A) = d_1(s, X)$. So $d(X) = d(A)$.

Now, $X$ is $\lambda$-dangerous. So $d(X) \leq \lambda(X) + 1$. Let $\lambda(X) = \lambda(x, y)$ where $x \in X$ and $y \in Y = V - X$. Suppose $y \notin A$. Then $y \in M_i$ for some $i \leq k - 1 - p$. Also, $d_1(s, X) \geq 3$, $d_1(s, M_i) = 1$ by Lemma 3.2.3 and $\delta(M_i) \subseteq \delta(X)$ by Lemma 3.2.4. This implies that $d(M_i) \leq d(X) - 2$ and so we have

$$\lambda(x, y) \leq d(M_i) \leq d(X) - 2 \leq \lambda(x, y) - 1.$$ 

This contradiction tells us $y \in A$ and so $\lambda(A) \geq \lambda(y, x) = \lambda(X) \geq d(X) - 1 = d(A) - 1$. That is $A$ is $\lambda$-dangerous.

Now consider $L_{k-p}$, the smallest family of dangerous sets, all containing $x_{k-p}$ and covering $N^{k-p}(s)$. Then by Lemma 3.2.3, $L_{k-p} = \{ M_{k-p} + x_j : k - p + 1 \leq j \leq t \}$, because by our supposition, $|L_{k-p}| \geq 2$. Now, consider the set $L_{k-p}^*$ made up of $A$ and each $M_{k-p} + x_j$ with $j \geq k$. Each set, $M_{k-p} + x_j$ in $L_{k-p}^*$ is dangerous, (because they are all members of $L_{k-p}$, and $A$ is dangerous, as shown above). Then, because $A$ contains at least three $s$-neighbours, $L_{k-p}^*$ must be smaller than $L_{k-p}$, which is a contradiction. So our supposition that we can find $k$ with $|L_k| = 1$ was incorrect. Hence, for all $i \leq t - 2$, $|L_i| \geq 2$.

Therefore, using Lemma 3.2.3 we can find $M_i$ such that each $X_{ij}$ in $L_i$ is $M_i + x_j$, for every $i \leq t - 2$. Let $M = \{ M_1, \ldots, M_{t-2} \}$, each $M_i$ being that

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found above, and \( \mathcal{X} = \{X_1, X_2, \ldots, X_{t-1}, X_t\} \), where \( X_i = x_i \) for \( i \leq t - 1 \) and \( X_t = V - (M_1 \cup M_2 \cup \ldots \cup M_{t-2} \cup X_{t-1}) \). Note that, because we have every tight set a singleton, we order the \( s \)-neighbours by degree. We show that \((\mathcal{M}, \mathcal{X})\) is a break by considering each of the properties given in the definition.

The ordering of \( s \)-neighbours implies \( d(X_1) \leq d(X_2) \leq \ldots \leq d(X_{t-1}) \). Also, because every \( \lambda \)-tight set is a singleton in \( G \), using Proposition 3.2.1 we have \( \lambda(x_{t-1}, x_t) = \min\{d(x_{t-1}), d(x_t)\} = d(x_{t-1}) \). Therefore \( d(X_{t-1}) = d(x_{t-1}) \leq \lambda(X_t) \leq d(X_t) \) and so \( (b) \) holds. Again, every \( \lambda \)-tight set is single implies the tightness of each \( X_i \) for \( i \leq t - 1 \). We complete the proof that \( (c) \) holds (by proving the tightness of \( X_t \)) after we have shown that \( (f) \) holds. Clearly \( x_i \in \{x_i\} = X_i \) for \( i \leq t - 1 \) and \( x_t \in X_t \). Also, for \( i \leq t - 2 \) the choice of \( M_i \) implies that \( X_i \subseteq M_i \), and so \( (d) \) holds. For each \( i \leq t - 3 \) Lemma 3.2.3 implies that \( |L_i| \geq 3 \) and so we can use Lemma 3.2.4 to see that \( (e) \) holds.

Properties \( (a) \) and \( (f) \) are somewhat more complicated. We begin with \( (a) \) and point out that the choice of \( X_t \) implies that \( V \) is covered by \( M_1, M_2, \ldots, M_{t-2}, X_{t-1}, X_t \). Also, \( X_t \) is disjoint from \( M_1 \cup M_2 \cup \ldots \cup M_{t-2} \cup X_{t-1} \) by definition, and because \( X_{t-1} \) is the singleton set \( \{x_{t-1}\} \), \( X_{t-1} \) is disjoint from all the \( M_i \)'s.

To complete the proof that \( (a) \) holds, we show that the \( M_i \)'s are disjoint.

For \( 1 \leq i < j \leq t - 2 \), let \( X = M_i + x_t \) and \( Y = M_j + x_t \). Then \( X, Y \) are both \( \lambda \)-dangerous by the choice of \( M_i \). (Notice that we are using the vertex \( x_t \) not the set \( X_t \).) We use Proposition 2.2.4 and show that for every choice of \( i, j \), whichever of 2.2.4(a) and 2.2.4(b) hold for \( X \) and \( Y \), we have \( M_i \cap M_j = \emptyset \).

If 2.2.4(a) holds for \( X \) and \( Y \) we have

\[
1 + 1 \geq s(X) + s(Y) \geq s(X \cap Y) + s(X \cup Y) \geq 0 + 2
\]

because \( X \cup Y \) cannot be \( \lambda \)-dangerous. Equality must hold throughout implying that \( X \cap Y \) is \( \lambda \)-tight. Therefore \( X \cap Y = \{x_i\} \) and so \( M_i \cap M_j = \emptyset \).

If 2.2.4(b) holds we have

\[
1 + 1 \geq s(X) + s(Y) \geq s(X - Y) + s(Y - X) + 2d_3(X,Y) + d_4(X,Y) \\
\geq 0 + 0 + 2 + 0.
\]

So we have equality throughout, which implies that both \( X - Y \) and \( Y - X \) are \( \lambda \)-tight. Therefore \( X - Y = \{x_i\} \) and \( Y - X = \{x_j\} \). This implies that \( X \cap Y = P + x_t \) where \( P := M_i - x_i = M_j - x_j \).

**Claim 3.4.1.6** \( \delta(M_j) \subseteq \delta(M_i + x_j) \).

**Proof:** Suppose there is an edge \( e \in \delta(M_j) \) which is not in \( \delta(M_i + x_j) \). Then \( e \) must only intersect \( x_i \) and \( M_j \). Further, \( e \) is not incident with \( x_j \), for otherwise
it would be in $\delta(M_i)$. But, because $i \leq t - 3$, Lemma 3.2.4 would imply that $e$ is incident with $x_i$ and hence is in $\delta(M_i + x_j)$. Therefore, $e$ only intersects $P$ and $x_i$. That is $d_1(x_i, P) \geq 1$.

Now define $Q := \{x_j, x_t\}$ so that $Y = P \cup Q$. Any edge intersecting $P$ and $V - (Y + x_t)$ is in $\delta(M_i)$ and, by Lemma 3.2.4 is incident with $x_j$, and so is in $\delta(Q)$. Therefore $d(Y) = d(Q) + d_1(x_t, P) \geq d(Q) + 1$. Because $Q$ is not a singleton, it is not $\lambda$-tight. This means that $d(Q) \geq \lambda(Q) + 1$ and so, $\lambda(Y) + 1 \geq d(Y) \geq d(Q) + 1 \geq \lambda(Q) + 2$. Therefore $\lambda(Y) > \lambda(Q)$. But $\lambda(Y) = \lambda(y, v)$ for some $y \in Y$ and $v \in V - Y$. So $y$ cannot be in $Q$ and must be in $P$, meaning $\lambda(Y) \leq \lambda(P)$.

We now show that there is no edge intersecting both $P$ and $Q$ and contained in $P \cup Q = Y$. Any edge intersecting both $P$ and $Q$ is in $\delta(M_i)$. Therefore by Lemma 3.2.4, any such edge must intersect every $x_n$ with $n \geq i$. Because $i \leq t - 3$, there is some such $n$ with $n \neq j$ and $n \neq t$. Thus, any such edge is incident with some $x_n \notin P \cup Q$. Therefore $\delta(P) \subseteq \delta(Y)$. This means that $d(Y) \geq d(P) + 2$, because $Q = Y - P$ contains two distinct s-neighbours.

Therefore, we have

$$\lambda(Y) + 1 \geq d(Y) \geq d(P) + 2 \geq \lambda(P) + 2 \geq \lambda(Y) + 2.$$ 

This is contradiction and so $\delta(M_j) \subseteq \delta(M_i + x_j)$ as required. \hfill \qed

If we recall that $M_j$ has one edge to $s$ and $M_i + x_j$ has two, we see that Claim 3.4.1.6 implies that $d(M_j) \leq d(M_i + x_j) - 1$. We next show that $\lambda(M_i + x_j) \leq \lambda(M_j)$. To see this note that because $i \leq t - 3$, $|L_i| \geq 3$ and so by Lemma 3.2.4, we have $\delta(M_i) \subseteq \delta(x_j)$. Therefore, $d(M_i) \leq d(x_j)$, because $d_1(s, M_i) = d_1(s, x_j) = 1$. Also, if $x \in M_i$, $y \in V - M_i$ we have $\lambda(x, y) \leq d(M_i) \leq d(x_j) = \lambda(x_j, x_t)$ by Proposition 3.2.1. Therefore $\lambda(M_i + x_j) = \lambda(x_j, x_t) \leq \lambda(M_j)$ as required.

So we have that $d(M_j) \leq d(M_i + x_j) - 1 \leq \lambda(M_i + x_j) \leq \lambda(M_j)$ and thus, $M_j$ is $\lambda$-tight. Therefore $M_j = x_j$ and so $P = M_j - x_j = M_i \cap M_j = \emptyset$. Thus, the $M_i$’s are disjoint and (a) holds.

We now deal with (f). Consider the set $W = M_{i_1} \cup M_{i_2} \cup \ldots \cup M_{i_{k-1}} \cup \{x_{i_k}\}$ where $i_1 < i_2 < \ldots < i_k < t$. We know that the set $A = M_{i_{k-1}} \cup \{x_{i_k}\}$ is $\lambda$-dangerous. Therefore $d(A) \leq \lambda(A) + 1$, and in fact, equality must hold, because $|A| \geq 2$ which means that $A$ is not $\lambda$-tight.

**Claim 3.4.1.7** For $j \leq k - 2$, $\delta(M_{i_j}) \subseteq \delta(A)$, and thus, $d(M_{i_j}) < d(A)$.

**Proof:** Because $i_k < t$, when $j \leq k - 2$ we must have $i_j \leq t - 3$. Therefore $|L_{i_j}| \geq 3$, and so every edge in $\delta(M_{i_j})$ is incident with all $x_n$ with $n \geq i_j + 1$. 49
Thus, any edge in $\delta(M_{i_j})$ is in $\delta(A)$, as required. Now, because $M_{i_j}$ has only one edge to $s$, and $A$ has two, we have $d(M_{i_j}) < d(A)$, as required. \qed

This means that $\lambda(M_{i_j}) \leq d(M_{i_j}) < d(A) = \lambda(A) + 1$ and hence $\lambda(M_{i_j}) \leq \lambda(A)$.

**Claim 3.4.1.8** $d(W) = d(A) + k - 2$.

**Proof:** Firstly we show that $\delta(W) = \delta(A)$. Suppose there is an edge $e \in \delta(W)$ which is not in $\delta(A)$. Then $e$ must be in some $\delta(M_{i_j})$. But then the previous Claim implies that $e$ must be in $\delta(A)$. Now suppose that some edge $e \in \delta(A)$ is not in $\delta(W)$. Then $e$ must be incident with a vertex $v \in W$ which is not in $A$. Therefore $v \in M_{i_j}$ for some $j \leq k - 2$, and so $e$ is in $\delta(M_{i_j})$. But then $e$ must be incident with $x_t$ and so is in $\delta(W)$. These two contradictions imply that $\delta(W) = \delta(A)$. Therefore $d(W) = d(A) - d_1(s, A) + d_1(s, W) = d(A) + k - 2$. \qed

**Claim 3.4.1.9** $\lambda(W) = \lambda(A)$

**Proof:** Firstly we show that $\lambda(A) \leq \lambda(W)$. This is clearly the case if $\lambda(A) = \lambda(a, x)$ and $x \in V - W$. So suppose $x \in W - A \subseteq V - A$. Then $x \in M_{i_j}$ for some $j \leq k - 2$. Then $\lambda(A) = \lambda(a, x) \leq d(M_{i_j})$. But every edge in $\delta(M_{i_j})$ intersects both $x_{i_k}$ and $x_t \in V - W$ and each of $M_{i_j}, x_{i_k}$ and $x_t$ have exactly one edge to $s$. So $d(M_{i_j}) \leq d(x_{i_k}) = \lambda(x_{i_k}, x_t) \leq \lambda(A)$. So in fact, $\lambda(A) \leq \lambda(x_{i_k}, x_t) \leq \lambda(W)$.

Now suppose $\lambda(W) = \lambda(w, x)$ for some $w \in W$ and $x \in V - W$. We need to show that $\lambda(W) \leq \lambda(A)$. This is clear if $w \in A$. If $w \notin A$ it must lie in some $M_{i_j}$ where $j \leq k - 2$. Then using Claim 3.4.1.7 we have that $\lambda(W) = \lambda(w, x) \leq \lambda(M_{i_j}) \leq d(M_{i_j}) \leq d(A) - 1 \leq \lambda(A)$.

So we have both directions and thus, $\lambda(W) = \lambda(A)$. \qed

So we have $d(W) = d(A) + k - 2 = \lambda(A) + k - 1 = \lambda(W) + (k - 1)$. Therefore (f) holds.

All that remains, is to show that $X_t$ is $\lambda$-tight. Let $W_T = M_1 \cup M_2 \cup \ldots \cup M_{t-2} \cup X_{t-1}$, that is $W_T = V - X_t$. Then we have

$$
\lambda(X_t) \leq d(X_t) = d(W_T) - d_1(s, W_T) + 1 \\
\leq \lambda(W_T) + (t - 1 - 1) - (t - 1) + 1 \\
= \lambda(W_T) \\
= \lambda(X_T).
$$

So we have equality throughout and $X_t$ is $\lambda$-tight and thus, (c) holds.
So, in our counterexample $G$ (a hypergraph with no split and no break), we have found a break $(\mathcal{M}, \mathcal{X})$. This contradiction proves the theorem.

**Remark - What next?**

Given a hypergraph $G = (V + s, E)$, we would like to be able to determine the maximum number of $\lambda$-splits we can perform at $s$. If there is a break $(\mathcal{M}, \mathcal{X})$ in $G$ there is no $\lambda$-split. We can think of the size of a break, as the number of sets in its implied partition of $V$, $\{M_1, M_2, \ldots, M_t, X_t\}$. Then in a hypergraph with a break with size equal to $d(s)$, the maximum number of $\lambda$-splits we can perform is $0 = d(s) - d(s)$. From this idea, we generalise the idea of a break, in order to make conjectures about the maximum number of $\lambda$-splits available in a hypergraph $G = (V + s, E)$, and the structure that may preclude the existence of a complete $\lambda$-split.

A possible generalisation of a break could be as follows.

A **generalised break of size** $t$ in a hypergraph $G = (V + s, E)$ is a pair of sets $B = (\mathcal{M}(B), \mathcal{X}(B))$ where $\mathcal{M}(B) := \{M_1, M_2, \ldots, M_t\}$ and $\mathcal{X}(B) := \{X_1, X_2, \ldots, X_t\}$ with the following properties.

(a) $M_1, M_2, \ldots, M_{t-2}, X_{t-1}, X_t$ is a partition of $V$.
(b) $N(s) \subseteq X_1 \cup X_2 \cup \ldots \cup X_t$.
(c) $X_i \subseteq M_i$ for all $i \leq t - 2$.
(d) For each $i \leq t - 3$, each edge in $\delta(M_i)$ intersects every $X_j$ with $j > i$.
(e) For $W = M_{i_1} \cup M_{i_2} \cup \ldots \cup M_{i_k} \cup X_{i_k}$, where $i_1 < i_2 < \ldots < i_k < t$, $d(W) \leq \lambda(W) + d_1(s, W) - 1$.

With this we suggest the following conjectures.

**Conjecture 3.4.2** Let $G = (V + s, E)$ have $d(s) = 2m \geq 4$ and no complete $\lambda$-split at $s$. Then there exists a generalised break in $G$ with size at least $m + 2$.

**Conjecture 3.4.3** Let $G = (V + s, E)$ have $d(s) = 2m$ and no complete $\lambda$-split. Let $\alpha$ be the maximum number of $\lambda$-splits available at $s$ and let $\beta$ be the size of the largest generalised break in $G$. Then $\alpha = 2m - \beta$.

Results of this type could lead to improvements in the upper bound given in Chapter 4 for the local-edge-connectivity augmentation problem.
3.5 Splitting to Preserve $\lambda_k$

This work was motivated by the special case of augmenting a $(k-1)$-edge-connected hypergraph, to ensure at least $k$ paths between some pairs of vertices. We consider a hypergraph $G = (V + s, E)$ with $s$ special, and such that $G - s$ is $(k-1)$-edge-connected. We wish to split edges from $s$ and in such a way as to ensure that if $\lambda(x, y) \geq k$, we maintain at least $k$-edge-disjoint paths in the result, and if $\lambda(x, y) < k$ we do not make their situation worse! That is we wish to find $\lambda_k$-splits, where $\lambda_k(x, y) := \min\{\lambda(x, y), k\}$.

The $\lambda_k$ function is clearly symmetric on $V^2$ and so we extend it to a set function in the usual way. It is also a routing function on $G$, because $\lambda_k(x, y) \leq \lambda(x, y)$ for all $x, y \in V$ and hence $d(X) \geq \lambda(X) \geq \lambda_k(X)$ for all $X \subseteq V$. So we can apply the results in Chapter 2.

The main problem with splitting in hypergraphs is the existence of a set of (a small number of) edges that disconnects the hypergraph leaving more than two components. These are the sets we need to deal with. In a hypergraph $H = (U, E)$, an $n$-plate is a set, $A$, of $n$ edges such that $H - A$ has at least 3 components, and each edge in $A$ intersects each component of $H - A$.

**Theorem 3.5.1** Let $G = (V + s, E)$ have $s$ only incident with size two edges and $d(s) = t \geq 4$. Let $G - s$ be $(k-1)$-edge-connected. Suppose there is no $\lambda_k$-split at $s$. Then there is a set of edges $A$ such that

(a) $A$ is a $(k-1)$-plate,

(b) $G - s - A$ has at least $t$ components and,

(c) $s$ has exactly one neighbour in each of $t$ of them.

**Proof:** Suppose there is no $\lambda_k$-split. Then there can be no $\lambda$-split and we can find a break $(M, X)$. We will assume that each $X_i$ is the singleton set containing $x_i$. (If this were not the case we could contract each $X_i$ and use the contraction results in Chapter 2, and then re-inflate at the end.)

Let $A := \delta(M_1)$. Then by the definition of a break, every edge in $A$ is incident with every $x_n$ for $2 \leq n \leq t$. Also, because $G - s$ is $(k-1)$-edge-connected, $|A| \geq k - 1$. Because the $M_i$'s are disjoint, $x_1$ is in a different component of $G - s - A$ to all other neighbours of $s$.

By Lemma 2.3.1 every pair of neighbours of $s$ must lie in some $\lambda_k$-dangerous set, $X$, say. Then by the $(k-1)$-connectedness of $G - s$ we have for any such $\lambda_k$-dangerous set, $k + 1 \leq d(X) \leq \lambda_k(X) + 1 \leq k + 1$. So we have equality
throughout implying $d(X) = k + 1$, and hence $|\delta(X)| = k - 1$. Also no $\lambda_k$-dangerous set can contain more than two neighbours of $s$, otherwise it will have degree at least $k + 2$.

Let $x_i, x_j$ and $x_k$ be three neighbours of $s$ with $2 \leq i < j < k \leq t$, and $X_{ij}$ and $X_{jk}$ be the $\lambda_k$-dangerous sets containing the pairs $\{x_i, x_j\}$ and $\{x_j, x_k\}$ respectively. Then $x_k \notin X_{ij}$ and $x_i \notin X_{jk}$.

Every edge in $A$ is incident with both $x_j \in X_{ij}$ and $x_k \in X_{jk}$ (because $j \geq 2$) and hence $A \subseteq \delta(X_{ij})$. Therefore $k - 1 \leq |A| \leq |\delta(X_{ij})| = k - 1$. So $|A| = k - 1$ and we have $A = \delta(X_{ij})$. This means that in $G - s - A$, $x_k$ must lie in a component that does not contain $x_i$ or $x_j$.

Similarly, every edge in $A$ is incident with both $x_k \in X_{jk}$ and $x_i \notin X_{jk}$. So $x_i$ is in a component of $G - s - A$ that contains neither $x_j$ nor $x_k$. That is $x_i, x_j$ and $x_k$ all lie in separate components.

We can use this argument for any triple of neighbours of $s$ and hence each neighbour of $s$ must lie in a separate component of $G - s - A$. Therefore $G - s - A$ has at least four components. We have seen that $|A| = k - 1$, and every edge in $A$ must intersect every component - or else there would be some set in $G - s$ with degree less than $k - 1$. Therefore $A$ is a $(k - 1)$-plate, and satisfies the conditions of the theorem. $\Box$ $\Box$ $\Box$

**Corollary 3.5.2** Let $G = (V + s, E)$ be a hypergraph with $s$ only incident with edges of size two, $t = d(s) \geq 4$, and suppose $G - s$ is $(k - 1)$-edge-connected. If $G$ has no $(k - 1)$-plates, there is a $\lambda_k$-split at $s$. $\Box$
Chapter 4

Applications of $\lambda$-Splitting

This chapter falls into two areas. Sections 4.1, 4.2 and 4.3, deal with $r$-Augmentation and provide an upper bound for the local problem. Sections 4.4, 4.5 and 4.6 look at augmenting with hyperedges and provide a refinement of a theorem due to Szigeti.

4.1 Augmenting with Size-two Edges - Introduction

The results in the previous chapter were motivated by the local-edge-connectivity augmentation problem. We are given a hypergraph $H = (V, E)$ and a (symmetric) demand function $r$ defined for every pair of vertices. We wish to augment $H$ with a set of edges, $F$, so that in the resulting hypergraph $H^+ = (V, E \cup F)$ we have $\lambda_{H^+}(x, y) \geq r(x, y)$ for all $x, y \in V$. We call such a set of edges an $r$-augmenting set for $H$.

Our approach is to add a new vertex $s$, enough edges from $s$ into $V$ to satisfy $r$, and then to split these edges from $s$ in such a way as to preserve $r$. A complete split generates an $r$-augmenting set. In the graph versions of the problem, the theorems of Mader and Lovasz ([53], [51, 52]) imply that in most graphs $G = (V + s, E)$, there is a split available, preserving the newly acquired connectivity in $V$. The graphs for which such splits are not available are relatively small, and easy to handle in other ways. This means that we can almost always find a “complete” split from $s$. However, when dealing with hypergraphs this is not always the case - as we have already shown.

The difficulty is that there are hypergraphs with no “good” split available, that are by no means small or easy to handle. So, we can add $s$, lots of edges, and start splitting only to get stuck with $d(s)$ still quite large. The question then is - How do you know that you have performed the most sensible sequence of
splits? To explain this, suppose that $G = G_0, G_1, G_2, \ldots, G_t$ is a sequence of hypergraphs formed by performing good splits from $s$. (Recall - a split is good when it preserves the connectivity requirement in $V$.) It is possible that the split that formed $G_7$ from $G_6$, say, forced us into a position where we could not avoid getting stuck at $G_t$, whereas, had we performed another split, creating $G'_t$ say, we could have finished a complete splitting of $s$.

In [1], Bang-Jensen and Jackson devised a method for dealing with this, when looking for $k$-splits. When a sequence of splits gets stuck, they backtrack through those already done, hoping to find the “foolish” split. (Foolish is well defined in their paper!) If such a split is present, they go on to describe the “better” splits. In this way, they are able to characterise those hypergraphs which have a complete $k$-split, and then provide a minimax result for the smallest $k$-augmenting set.

However, thanks to the complexity of the structure preventing a single $\lambda$-split, we have not been able to find a similar technique for dealing with $\lambda$-splits. Moreover, there is another difficulty to consider. When we set up $G = (V + s, E)$ that has $\lambda_G(x, y) \geq r(x, y)$, it is sensible to look for $\lambda$-splits, because a split that preserves $\lambda$ ensures that $\lambda$ is still greater than or equal to $r$ across the whole of $V$. However, it is also possible to have $\lambda_G(x, y) > r(x, y)$ for a pair of vertices and a split $su, sv$ for which $\lambda_G(uv)(x, y) < \lambda_G(x, y)$ but preserves $r$ in that $\lambda_G(uv)(x, y) \geq r(x, y)$.

For instance, consider an initial hypergraph with four vertices, $V = \{v_1, v_2, v_3, v_4\}$ and just one edge incident with every $v_i$. In this hypergraph $\lambda(v_i, v_j) = 1$ for all $i, j$ and we specify the demand function $r(v_1, v_2) = r(v_3, v_4) = 2$ and $r(x, y) = 1$ otherwise. We would begin by adding $s$ and an edge from $s$ to every vertex in $V$, forming $G$, in which $\lambda(v_i, v_j) = 2$. It is not difficult to see (by inspection or, indeed, by the results in the last chapter) that there is no $\lambda$-split from $s$. However, there are two splits preserving $r$. Namely $v_1v_2$ and $v_3v_4$. This means that we must be careful before drawing conclusions from finding or not finding $\lambda$-splits.

As it turns out - and we see this in the next chapter - the local-edge-connectivity augmentation problem is NP-complete. However, using the results on $\lambda$-splitting, we provide here a (sharp) upper-bound on the size of a minimum augmentation and describe an algorithm to produce an augmentation that stays within this upper-bound.
4.2 A Lower Bound, Marginal Components and Tight Sets

We have a hypergraph $H = (V, E)$ and symmetric requirement function $r : V^2 \rightarrow \mathbb{N}$. We use $\theta_r(H)$ to denote the smallest number of size-two edges we must add to $H$ to ensure that in the resulting hypergraph $H^+$, we have $\lambda_{H^+}(x, y) \geq r(x, y)$. By the hypergraph version of Menger's Theorem, (see [63]), this is equivalent to saying that in the resulting hypergraph, we require $d_{H^+}(X) \geq r(X)$ for all subsets $\emptyset \subset X \subset V$. Also, it is useful (and reasonable) to assume the following about our demand function, $r$.

(a) $r(x, y) \geq \lambda_H(x, y)$, and
(b) If $r(x, u) \geq t$ and $r(u, y) \geq t$ for some integer $t$, then $r(x, y) \geq t$.

We define the following.

\[ q_H(X) := r(X) - d_H(X) \text{ for } X \subset V; \]
\[ \beta(H) := \max\{\sum_{X_i \in U} q_H(X_i) : U \text{ is a subpartition of } V\}; \]
\[ m(H) := \lceil \frac{\beta(H)}{2} \rceil. \]

By considering that every new edge can contribute to the deficiencies of at most two subsets of $V$, the following result is straightforward.

**Proposition 4.2.1** For a hypergraph $H = (V, E)$ and requirement function $r$ as above, $\theta_r(H) \geq m(H)$. \hfill \Box

As with the graph version of this case, before attempting an augmentation, we must deal with so-called marginal components - as defined for graphs by Frank in [22]. Let $C \subset V$ be the vertex set of a component of $H$. We say that the component is marginal (with respect to the requirement function $r$) when $q_H(C) \leq 1$ and $q_H(X) = 0$ for every proper subset $X \subset C$. (Note that if $q_H(C) \leq 1$, then we have $r(C) \leq 1$.) The following result is simply a restatement of Theorem 1.2.3 for hypergraphs. The proof given is exactly that used for graphs by Frank in [22]. It follows through directly for hypergraphs and is reproduced here for completeness. We use the following notation. If $C$ is a marginal component of $H$, we use $H_1 = H - C$, $r_1$ for the restriction of $r$ to the vertices in $H_1$ and $\theta_{r_1}(H_1)$ for the size of a minimum augmentation of $H_1$. 

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Theorem 4.2.2 Let $H = (V, E)$ be a hypergraph and $r : V^2 \to \mathbb{N}$ be a symmetric requirement function. If $C$ is a marginal component of $H$, then $\theta_r(H) = \theta_r(H_1) + q_H(C)$.

Proof: To see that $\theta_r(H) \leq \theta_r(H_1) + q(C)$, let $F_1$ be a minimum $r_1$-augmenting set for $H_1$. If $q_H(C) = 0$, then clearly $F_1$ is a minimum $r$-augmentation set for $H$. If $q_H(C) = 1$ then there is a pair of vertices $a \in C$, $b \in V - C$ with $r(a, b) = 1$. We show that $F_1 + ab$ is an $r$-augmenting set for $H$. To see this, we suppose the contrary, which implies that there are vertices $x, y \in V$ such that $\lambda_H(x, y) < r(x, y)$ where $H^* = H + F_1 + ab$. Then because $F_1$ is $r_1$-augmenting set for $H_1$, and because $q_H(X) \geq 0$ for all $X \subset C$, we have (wlog) $x \in C, y \in V - C$. Because $C$ is marginal, we have $r(x, y) = 1$ and $\lambda_H(x, y) = 0$. Now, $C$ is a component of $H$, so there is a path joining $x$ and $a$ in $H^*$, and clearly there is a path joining $a$ and $b$. Therefore $0 = \lambda_H(b, y) \geq \lambda_H(b, y)$. So $\lambda_H(b, y) = 0$ and hence $r(b, y) = r_1(b, y) = 0$. Further, by our assumptions about $r$, $r(x, a) \geq \lambda_H(x, a) \geq 1$ and because $r(a, b) = 1$ we have $r(x, b) \geq 1$. Also, $r(y, x) = r(x, y) = 1$, so by our assumption, $r(y, b) = 1$ which contradicts the symmetry of $r$. So $F_1 + ab$ is an $r$-augmenting set for $H$ and $\theta_r(H) \leq \theta_r(H_1) + q_H(C)$, as required.

To see that $\theta_r(H) \geq \theta_r(H_1) + q(C)$, we consider $H_2 := H - C + v_C$, with $r_2(x, y) = r(x, y)$ for $x, y \in V - C$ and $r_2(v_C, y) = q_H(C)$, for all $y \in V - C$. Then clearly, $\theta_r(H_2) \leq \theta_r(H)$.

Suppose $F_2$ is a minimal $r_2$-augmenting set, such that $F_2(v_C)$, the set of edges in $F_2$ that are incident with $v_C$, is as small as possible. We shall show that $|F_2(v_C)| \leq 1$. To see this, suppose that $v_Cu_1, v_Cu_2 \in F_2(v_C)$ and let $F'_2 = F_2 - v_Cu_2 + v_Cu_2$. Then $F'_2$ is also an $r_2$-augmenting set for $H_2$, with fewer edges to $v_C$ than $F_2$ - which is a contradiction.

Now, if $|F_2(v_C)| = 1$, say $F_2(v_C) = \{ f \}$, then $F_2 - f$ is clearly a good augmentation of $H - C$. Hence $\theta_r(H_1) \leq |F_2| - 1 = \theta_r(H_2) - 1 \leq \theta_r(H) - q_H(C)$ as required. Also, if $|F_2(v_C)| = 0$, then $q_H(C) = 0$ and all the edges in $F_2$ are contained in $V - C$. Therefore, we have $\theta_r(H_1) \leq |F_2| = \theta_r(H_2) \leq \theta_r(H) = \theta_r(H) - q_H(C)$, as required. $\square$

This means that when we are given a hypergraph to augment, we first check if there are any marginal components. If there are, we remove them and augment what is left. We then put them back, with one extra edge for each component $C$ with $q_H(C) = 1$. Because this process is straightforward, we restrict our augmentation theorem to hypergraphs that have no marginal components.

Before the main result, we need a Lemma about tight sets. In order to do the augmentation, we add a new vertex $s$ and edges from $s$ into $V$, forming $G = (V + s, E)$. We add enough edges to ensure that $\lambda_G(x, y) \geq r(x, y)$ for all $x, y \in V$, and hence we have $d(X) \geq \lambda(X) \geq r(X)$ for all $\emptyset \subset X \subset V$, that
is, \( r \) is a routing function for \( G \). In this version of the problem, we say a set \( \emptyset \subset X \subset V \) is \textbf{tight} when \( d_G(X) = r(X) \). With help from Chapter 2, we get the following.

**Lemma 4.2.3** In a hypergraph \( G = (V + s, E) \), if \( X, Y \subset V \) are both tight, then at least one of the following holds.

(a) Both \( X \cap Y \) and \( X \cup Y \) are tight, or

(b) both \( X - Y \) and \( Y - X \) are tight and \( d_3(X,Y) = 0 \).

**Proof:** As in Chapter 2, we define \( s(X) := d_G(X) - r(X) \). Then for a tight set \( X \) we have \( s(X) = 0 \). We use Proposition 2.2.4. If 2.2.4(a) holds, we have \( 0 + 0 \geq s(X) + s(Y) \geq s(X \cap Y) + s(X \cup Y) \geq 0 + 0 \), so there is equality throughout, and (a) (in this Lemma) holds. Similarly, if 2.2.4(b) holds, we have \( 0 + 0 \geq s(X) + s(Y) \geq s(X - Y) + s(Y - X) + 2d_3(X,Y) + d_4(X,Y) \geq 0 + 0 + 0 + 0 \) and there is equality throughout and (b) (this Lemma) holds.

**4.3 The Augmentation Result**

We now present our augmentation result, for a hypergraph \( H = (V, E) \). It combines Proposition 4.2.1 with an upper bound, based on, \( |e_{\text{max}}| \), the size of the largest edge in \( E \). The idea is that when we get stuck in a sequence of splits, we have \( d(s) = |N(s)| \leq |e_{\text{max}}| \). So if we do get stuck, we add extra edges to \( s \) to split. This leads us to the main result. Before the theorem, however, we need two lemmas. The first of which describes how we initialise the hypergraph and we use it again in Section 4.6. In a hypergraph \( G = (V + s, E) \) that has \( d(X) \geq r(X) \) for all \( \emptyset \subset X \subset V \) we say an edge \( sx \) is \textbf{critical with respect to} \( r \) when \( x \) lies in some set with \( d_G(X) = r(X) \). If all edges incident with \( s \) are critical, we cannot remove any of them without “violating” \( r \).

**Lemma 4.3.1** Let \( H = (V, E) \) be a hypergraph and \( r : V^2 \to \mathbb{N} \) be a symmetric requirement function. Then we can form a hypergraph \( G_2 = (V + s, E \cup E_2(s)) \) such that

(a) \( d_{G_2}(X) \geq r(X) \) for all \( X \subseteq V \),

(b) \( d_{G_2}(s) = \beta(H) \), and

(c) every edge incident with \( s \) is critical with respect to \( r \).
Proof: Let $G_1 = (V + s, E \cup E_1(s))$ be the hypergraph created from $H$, by adding a new vertex, $s$, and enough edges from $s$ into $V$ to ensure that $d(X) \geq r(X)$ for every $\emptyset \subset X \subset V$. (We can do this by adding, say, $\max\{r(x, y) : x, y \in V\}$ edges from $s$ to every vertex in $V$.) Then form $G_2 = (V + s, E \cup E_2(s))$ by deleting the edges in $E_1(s)$ until every edge incident with $s$ is critical. We now show that $d_{G_2}(s) \leq \beta(H)$.

Because each edge incident with $s$ is critical, every $u \in N_{G_2}(s)$ lies in some tight set. Let $X(s) = \{X_1, X_2, \ldots, X_t\}$ be a collection of tight sets, that covers $N_{G_2}(s)$ so that $t$ is as small as possible, and subject to this $\sum |X_i|$ is as small as possible. We begin, by showing that the $X_i$'s are disjoint, and thus, that $X(s)$ is a subpartition of $V$.

We use Lemma 4.2.3 on distinct sets, $X_i, X_j \in X(s)$. If 4.2.3 (a) holds, then $X_i \cup X_j$ is tight, contradicting the minimality of $t$. So 4.2.3 (b) must hold. Hence, both $X_i - X_j$ and $X_j - X_i$ are tight, and $d_3(X_i, X_j) = 0$. Therefore there are no $s$-neighbours in $X_i \cap X_j$. Hence, $X(s) - \{X_i, X_j\} + \{X_i - X_j, X_j - X_i\}$ covers $N_{G_2}(s)$. Hence, by the minimality of $\sum |X_i|$, we have $|X_i| = |X_i - X_j|$ and $|X_j| = |X_j - X_i|$. That is, $X_i$ and $X_j$ are disjoint, and $X(s)$ is a subpartition of $V$.

We now consider $d_{G_2}(s)$. Because $X(s)$ contains all the $s$-neighbours in $G_2$, we have that

$$d_{G_2}(s) = \sum_{X_i \in X(s)} (d_{G_2}(X_i) - d_H(X_i))$$

$$= \sum_{X_i \in X(s)} (r(X_i) - d_H(X_i))$$

$$= \sum_{X_i \in X(s)} q_H(X_i)$$

$$\leq \beta(H).$$

To complete the proof, we show that $d_{G_2}(s) \geq \beta(H)$. Let $Y(s) = \{Y_1, \ldots, Y_t\}$ be the subpartition that defines $\beta(H)$. Then

$$d_{G_2}(s) \geq \sum_{Y_i \in Y(s)} (d_{G_2}(Y_i) - d_H(Y_i))$$

$$= \sum_{Y_i \in Y(s)} (r(Y_i) - d_H(Y_i))$$

$$= \beta(H).$$

Therefore, $d_{G_2}(s) = \beta(H)$ as required. \qed
We also have the following result used at the last stage of finding an \( r \)-augmenting set.

**Lemma 4.3.2** Let \( G = (V + s, E) \) be a hypergraph with \( s \) special, \( d_G(s) = 2t \) and such that for some \( x \in N_G(s), d_1(s, x) = t \) and \( d_1(s, y) = 1 \) for all \( y \in N_G(s) - x \). Then there is a complete \( \lambda \)-split from \( s \).

**Proof:** We begin with finding the first split. The way we choose this split leaves us with a hypergraph satisfying the same hypotheses as the hypergraph we started with. Because of this, repeated application of the following claim implies the Lemma.

**Claim 4.3.2.1** Edges \( sx, sy \) form a \( \lambda \)-split, for every \( y \in N_G(s) - x \).

**Proof:** To see this, suppose that they do not and let \( X \) be a dangerous set containing both \( x \) and \( y \). Then \( d_1(s, X) \geq d_G(s)/2 + 1 > d_1(s, V - X) \) and contradicts Lemma 2.3.2.

We can now give the augmentation result. The proof is constructive, describing how to produce an \( r \)-augmentation with the desired number of edges.

**Theorem 4.3.3** Let \( H = (V, E) \) be a hypergraph, \( r : V^2 \to N \) be a symmetric requirement function and suppose that no component of \( H \) is marginal with respect to \( r \). Then \( m(H) \leq \theta_r(H) \leq m(H) + \frac{|e_{\text{max}}|}{2} - 1 \).

**Proof:** The first inequality is clear - and was seen already in Proposition 4.2.1. To see the second inequality we show that there is an \( r \)-augmentation of \( H \) with at most \( m(H) + \frac{|e_{\text{max}}|}{2} - 1 \) edges. By Lemma 4.3.1 we can form \( G_2 = (V + s, E \cup E_2(s)) \) in which \( d_{G_2}(X) \geq r(X) \) for all \( X \subseteq V \) and \( d_{G_2}(s) = \beta(H) \).

Now, if necessary, we add an extra edge, parallel to some existing edge \( sx \), forming \( G_3 \) with \( d_{G_3}(s) = 2m(H) \). (If \( d_{G_2}(s) \) is odd, we must add one edge.) There is no cut edge incident with \( s \). If there were such an edge, \( e \) say, then the component of \( H - e \) that doesn’t include \( s \), would be a marginal component of \( H \), which is contrary to our assumption.

We now perform as many \( \lambda \)-splits as possible. Let \( \alpha \) be the number of splits, let \( G_4 \) be the hypergraph formed and let \( \gamma = d_{G_4}(s) = 2(m - \alpha) \). If \( \alpha = m(H) \) we have found a complete \( \lambda \)-split from \( s \), generating an \( r \)-augmentation with
For a hypergraph \( H \) edges. If this is not the case, there is no \( \lambda \)-split at \( s \) in \( G_4 \) and we have \( \alpha \leq m(H) - 2 \), (because performing \( m(H) - 1 \) splits leaves a hypergraph with \( d(s) = 2 \) in which (trivially) there is a \( \lambda \)-split). We point out here that \( \alpha + \gamma / 2 = m(H) \). Moreover, by Theorem 3.3.1, \( \gamma = |N_{G_4}(s)| \geq 4 \) and by Theorem 3.3.4, \( \gamma \leq |e_{\max}| \).

Now, form \( G_5 \) by adding \( \gamma - 2 \) edges between \( s \) and some \( x \in N_{G_4}(s) \). Then \( d_{G_5}(s) = 2(\gamma - 1) \), with \( d_1(s, x) = \gamma - 1 \), \( d_1(s, y) = 1 \) for all \( y \in N_{G_5}(s) - x \). So by Lemma 4.3.2 there is a complete \( \lambda \)-split from \( s \) in \( G_5 \). Perform this split and delete \( s \) to form \( H^+ \). By splitting, we have added \( \alpha + \gamma - 1 = m - \gamma / 2 + \gamma - 1 = m(H) + \gamma / 2 - 1 \leq m(H) + \frac{|e_{\max}|}{2} - 1 \) new edges and because every split was \( \lambda \)-good, these new edges form an \( r \)-augmenting set. That is, \( \theta_r(H) \leq m(H) + \frac{|e_{\max}|}{2} - 1 \), as required. \( \square \)

The upper bound is sharp in the sense that there are hypergraphs which require this many edges. For instance, consider a hypergraph \( H = (V, E) \) with \( |V| = 2t \), and just one edge that is incident to all the vertices. Then \( |e_{\max}| = 2t \). Set \( r(x, y) = 2 \) for all \( x, y \in V \), then \( \beta(H) = 2t \) and \( m(H) = t \). It is not difficult to see that we must add \( 2t - 1 = t + 2t - 1 = m(H) + \frac{|e_{\max}|}{2} - 1 \) edges to successfully augment \( H \).

Also, for an example with a non-uniform \( r \), consider a hypergraph \( H' = (V', E') \) with \( V' = \{x_1, x_2, x_3, x_4\} \) and \( E' = \{e_1, e_2\} \) with \( e_1 = \{x_1, x_2, x_3, x_4\} \), \( e_2 = \{x_2, x_3, x_4\} \). Set \( r'(x_1, x_i) = 2 \) for \( i = 1, 2, 3 \) and \( r'(x_j, x_k) = 3 \) for \( j, k = 2, 3, 4 \). Again \( m(H') = 2 \) and we must add \( 3 \) edges to augment \( H' \) and satisfy \( r' \). Furthermore, by adding a vertex \( x_5 \), including it in \( e_1 \), and setting \( r'(x_5, x_i) = 0 \) for all other \( i \), we see that there are hypergraphs for which \( \theta_r(H) \) falls between the upper and lower bound.

### 4.4 Augmenting with Hyperedges - Introduction

As mentioned in Chapter 1, we could consider the problem of augmenting a hypergraph to meet a general requirement function, but using hyperedges instead of just size-two edges. In [58], Szegti considered the problem of finding an edge set of minimum value. The value of a set of edges, \( F \), is defined as \( \text{val}(F) := \sum_{e \in F} |e| \). For a hypergraph \( H = (V, E) \) and demand function, \( r \), we define \( q_H(X) \) and \( \beta(H) \) as in Section 4.2 and use \( \nu_r(H) \) to denote the minimum value of an \( r \)-augmenting set of (hyper)edges. The definition of \( \beta \) implies that any \( r \)-augmenting set must have value at least as large as \( \beta(H) \). Szegti showed that this bound is achievable.

**Theorem 4.4.1 (Szegti)** For a hypergraph \( H = (V, E) \) and a symmetric requirement function, \( r : V^2 \to \mathbb{N} \), \( \nu_r(H) = \beta(H) \) where the maximum is taken...
over all subpartitions $U$ of $V$. 

In what follows, we show that there exists an $r$-augmenting set, with value $\beta(H)$, that contains at most one edge with size greater than two. To do this, we take the same kind of approach as in Theorem 4.3.3. We initialise as before, and then perform $\lambda$-splits until either we have a complete split, or we are left with no further $\lambda$-split. If the former case, we have an augmenting set of size-two edges, that has minimum value. In the latter, we add a single hyperedge, which is incident with exactly the $s$-neighbours of $G$. In the next section we provide the results needed to show that this approach does indeed produce an $r$-augmentation.

4.5 Adding a Single Hyperedge Does Work

Suppose $G = (V + s, E)$ has no $\lambda$-split. We wish to show that $G^* = G - s + e^*$, where $e^* = \{x : x \in N_G(s)\}$, satisfies $\lambda_{G^*}(x, y) \geq \lambda_G(x, y)$ for all $x, y \in V$. We do this with the following three lemmas.

Lemma 4.5.1 Let $G = (V + s, E)$ have $s$ only incident with size-two edges, no $\lambda$-split and let $x, y \in N_G(s)$. Let $G^+ = G + sx$. Then $x, y$ form a $\lambda_{G^+}$-split in $G^+$.

Proof: We begin by noting that $\lambda_G$ is a routing function in $G^+$ because we have $d_{G^+}(X) \geq \lambda_{G^+}(X) \geq \lambda_G(X)$ for all $X \subseteq V$. Now suppose that $Y \subseteq V$ contains both $x, y$. We claim that $d_G(Y) \geq \lambda_G(Y) + 1$. To see this, recall that $G$ has no $\lambda_G$-split. Hence, $Y$ cannot be $\lambda_G$-tight, for otherwise, we could contract it, forming a hypergraph, $G'$. In this we have $d_{G'}(s) > |N_{G'}(s)|$ and hence by Theorem 3.3.1, $G'$ has a $\lambda_{G'}$-split. But then Lemma 2.4.4 implies that $G$ has a $\lambda_G$-split, which is contrary to our hypothesis. So $d_G(Y) \geq \lambda_G(Y) + 1$ as required.

Now consider the degree of $Y$ in $G^+$. We have $d_{G^+}(Y) = d_G(Y)+1 \geq \lambda_G(Y)+2$. That is, there is no $\lambda_G$-dangerous set in $G^+$, containing both $x$ and $y$, and hence, by Lemma 2.3.1, $x, y$ form a $\lambda_G$-split in $G^+$ as required. 

Lemma 4.3.2 showed that in a hypergraph with no $\lambda$-split, we can add a (specific) tree to the $s$-neighbours and maintain local edge-connectivity in $V$. The next result refines this, implying that any tree will do!

Lemma 4.5.2 Let $G = (V + s, E)$ have $s$ only incident with size-two edges and no $\lambda$-split. Let $T = (N_G(s), F_T)$ be a tree on the vertices in $N_G(s)$ and
\[ H_1 = G - s + F_T. \] Then we have \( \lambda_{H_1}(u,v) \geq \lambda_G(u,v) \) for all \( u,v \in V \) or equivalently, we have \( d_{H_1}(X) \geq \lambda_G(X) \) for all \( X \subseteq V \).

**Proof:** Every tree on \( N_G(s) \) can be formed by \(|N_G(s)| - 1\) repeats of the operation of Lemma 4.5.1. Because each split in this process is a \( \lambda_G \)-split, adding a tree in this way does preserve the local-connectivity in \( V \) that was present in the starting hypergraph. \( \square \)

We can now prove the result we need.

**Lemma 4.5.3** Let \( G = (V + s, E) \) have \( s \) only incident with size-two edges and no \( \lambda \)-split. Let \( e^* \) be a new edge incident with every vertex in \( N_G(s) \) and no others and let \( H_2 = G - s + e^* \). Then \( \lambda_{H_2}(u,v) \geq \lambda_G(u,v) \) for all \( u,v \in V \).

**Proof:** We show that \( d_{H_2}(X) \geq \lambda_G(X) \) for all \( X \subseteq V \). Indeed, if \( X \cap N_G(s) = 0 \) then \( d_{H_2}(X) = d_G(X) \geq \lambda_G(X) \). If \( X \cap N_G(s) = N_G(s) \) then \( d_{H_2}(X) = d_{H_1}(V - X) = \lambda_G(X) \).

We are left with the case when \( X \) contains at least one, but not all, of the \( s \)-neighbours in \( G \). This means we have \( d_{H_2}(X) = d_{G - s}(X) + 1 \). Also for all such \( X \), there is a tree, \( T = (N_G(s), F_T) \), such that \( d_T(N_G(s) \cap X) = 1 \). If we let \( H_1 = G - s + F_T \), then \( d_{H_1}(X) = d_{G - s}(X) + 1 \) and by Lemma 4.5.2, \( d_{H_1}(X) \geq \lambda_G(X) \). Thus, we have \( d_{H_2}(X) = d_{H_1}(X) \geq \lambda_G(X) \), as required. \( \square \)

### 4.6 Minimum Value Augmentation

By beginning with the method of Theorem 4.3.3 and then (if necessary) adding a single hyperedge, as above, we now prove the following refinement of Szigeti’s Theorem.

**Theorem 4.6.1** Let \( H = (V, E) \) be a hypergraph and \( r : V^2 \to \mathbb{N} \) be a symmetric requirement function. Then there is an \( r \)-augmenting set, \( F^* \) such that \( \text{val}(F^*) = \nu_r(H) = \beta(H) \) and at most one edge in \( F^* \) has size greater than two.

**Proof:** By Lemma 4.3.1 we can form a hypergraph \( G_2 = (V + s, E \cup E_2(s)) \) by adding \( s \) and size-two edges from \( s \) into \( V \) such that \( d_{G_2}(X) \geq \lambda_{G_2}(X) \geq r(X) \) for all \( X \subseteq V \), every edge incident with \( s \) is critical and \( d_{G_2}(s) = \beta(H) \). Note that in this case we do not mind if \( d_{G_2}(s) \) is odd.

We now perform as many \( \lambda \)-splits as possible. Let \( \alpha \) be the number of splits, let \( G_4 = (V + s, E \cup E_4(s) \cup F) \) (where \( E_4(s) \) is the set of any edges remaining
that are incident with $s$, and $F$ is the set of size-two edges added into $V$ by splitting) be the hypergraph formed. Then $d_{G_4}(s) = \beta - 2\alpha$. If $d_{G_4}(s) = 0$ we have found a complete $\lambda$-split from $s$. Thus, $E_4(s)$ is empty and the edges in $F$ are an $r$-augmenting set for $H$. In this case, $|F| = \beta/2$ and each edge has size-two, so $F$ has value $\beta$ as required. We now deal with the case when $d_{G_4}(s) > 0$.

Claim 4.6.1.1 $d_{G_4}(s) \neq 1$ or 2.

Proof: If $d_{G_4}(s) = 2$ there is, trivially, a $\lambda$-split, contradicting the fact that $G_4$ is formed by performing as many $\lambda$-splits as possible. So now suppose that $d_{G_4}(s) = 1$, say $E_4(s) = \{sx\}$. We show now that in this case $d_{G_2-sx}(X) \geq r(X)$ for all $X \subseteq V$. Suppose there is some $X$ contrary to this. Then $d_{G_2-sx}(X) < r(X)$, for otherwise, $d_{G_2}(X) < r(X)$. So because we are only deleting one edge, we have $d_{G_2}(X) = r(X)$. Further, we have $d_{G_4}(X) = d_{G_2}(X)$ because to form $G_4$ we only perform $\lambda$-splits, which (by their very nature) preserve $r$ and cannot increase the degree of any subset of $V$. Therefore we have that

$$r(X) = d_{G_2}(X) = d_{G_4}(X) = d_{G_4}(V - X) + 1 \geq r(V - X) + 1 = r(X) + 1$$

which is clearly a contradiction.

So, $d_{G_2-sx}(X) \geq r(X)$ for all $X \subseteq V$. Therefore, every $X \subset V$ containing $x$ has $d_{G_2}(X) \geq r(X) + 1$, and so $sx$ is not critical. This is a contradiction because every edge incident with $s$ is critical in $G_2$. So $d_{G_4}(s) \neq 1$ as required.

So we have $d_{G_4}(s) \geq 3$ and no $\lambda$-split at $s$. Also, at this point, $val(F) = 2\alpha$, because $F$ contains just $\alpha$ size-two edges. Because all the splits we have done are $\lambda$-splits, we have $d_{G_4}(X) \geq r(X)$ for all $X$ with $\emptyset \subset X \subset V$. Now let $e^* = \{x : x \in N_{G_4}(s)\}$ be a new edge. Then by Lemma 4.5.3, the hypergraph $H^+ = G_4 - s + e^*$ is such that $\lambda_{H^+}(u,v) \geq \lambda_G(u,v) \geq r(u,v)$ for all $u,v \in V$. So we have $H^+ = H + F^*$ where $F^* = F + e^*$, is an $r$-augmenting set for $H$ and $val(F^*) = 2\alpha + d_{G_4}(s) = \beta = \nu_r(H)$ as required.

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Chapter 5

Local Connectivity Augmentation is NP-Complete

5.1 Introduction

The main result in this chapter, is the product of a workshop in Bonn, in 1999. Many connectivity augmentation problems were discussed, one of which being the hypergraph augmentation problem. The requirement function we were interested in satisfying was based on a subpartition of the vertex set of the starting hypergraph, \( G = (V, E) \). The starting point for this was the case where the subpartition has just one element.

For a set \( T \subset V \), we say that \( G \) is \( k \)-edge-connected in \( T \) when for all pairs of vertices \( x, y \in T \) we have \( \lambda(x, y) \geq k \). We say a set \( X \) separates \( T \) when both \( X \cap T \) and \( X - T \) are non-empty, and a subpartition \( \mathcal{P} \) separates \( T \) when every member of the subpartition separates \( T \). With this terminology an equivalent definition for \( k \)-edge-connected in \( T \) is having \( d(X) \geq k \) for every set \( \emptyset \subset X \subset V \) separating \( T \). In [6], Benczúr and Frank determined the minimum number of edges that must be added to a given hypergraph, to ensure that the result was \( k \)-edge-connected in a given set \( \emptyset \subset T \subset V \). To state their result we need one more definition. We use \( c_T(G) \) to denote the number of components of \( G \) that have non-empty intersection with \( T \).

**Theorem 5.1.1 (Benczúr and Frank)** A hypergraph \( G = (V, E) \) can be made \( k \)-edge-connected in a set \( T \), with \( \emptyset \subset T \subset V \), by adding at most \( \gamma \) new size-two edges if and only if...
(a) \[ \sum_{X \in \mathcal{P}} (k - d(X)) \leq 2\gamma \] for every subpartition \( \mathcal{P} \) of \( V \) separating \( T \), and

(b) \( c_T(G - A) - 1 \leq \gamma \) for every set of \( k - 1 \) edges \( A \subset E \).

\[ \Box \]

A natural extension to this problem is where there are several sets \( T_1, T_2, \ldots, T_n \) to be made \( k \)-edge-connected. That is, we try to satisfy a local demand function, \( r : V^2 \to \mathbb{N} \) where

\[ r(x, y) = \begin{cases} k & \text{if } x, y \in T_i \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases} \]

In this chapter, we solve this problem for the case when \( k = 1 \) and show that the associated decision problem is NP-complete for the general case.

**Acknowledgement:** The idea to use a transformation to the Bin-Packing problem came about during a discussion at the Bonn workshop between the present author and Zoltán Király. This is the first time the details have been presented.

### 5.2 Subpartitions and \( k = 1 \)

Our problem is the following. Given a hypergraph \( G=(V,E) \), an integer \( \gamma \) and a subpartition of \( V \), \( T = \{T_1, T_2, \ldots, T_n\} \), can we add \( \gamma \) size two edges to \( G \) to ensure that each \( T_i \) is connected? That is, can we make \( G \) \( 1 \)-edge-connected in \( T \)?

A set of size-two edges, \( F \), is a **good augmenting set with respect to \( T \)**, when each set \( T_i \in T \) is connected in \( G + F \). If \( |T| = 1 \), say \( T = \{T_1\} \) we will refer to a **good augmenting set with respect to \( T_1 \)**. We also continue to use the \( c_T \) notation, but here \( c_T(G) \) is the number of components that separate at least one \( T_i \in T \). In Benczúr and Frank’s result, it was necessary to consider components arising when \( k - 1 \) edges are removed from \( G \). Here though, we have \( k = 1 \), and so we look at the components of \( G \) (that we get when remove no edges).

Now, if a component contains some \( T_i \), then that set is already connected and so needs no augmentation. This is why we consider only those components that **separate** some element of \( T \). Also, if a component separates two or more elements of the subpartition, in \( G + F \) the union of these sets is connected. This is important in terms of finding a minimum augmenting set, because the fewer sets in the subpartition the better. The following result formalises this process.
Lemma 5.2.1 Let $T = \{T_1, T_2, \ldots, T_n\}$ and $C$ be a component of $G$ such that $C$ separates both $T_i$ and $T_j$. Then $F$ is a minimum augmenting set with respect to $T$ if and only if it is a minimum augmenting set with respect to the subpartition $T' := T - \{T_i, T_j\} \cup \{T_i \cup T_j\}$.

Proof: We first show that $F$ is a good augmenting set with respect to $T$ if and only if it is good with respect to $T'$. One direction is not difficult. If $F$ makes $T'$ connected, then clearly both $T_i$ and $T_j$ are connected in $G + F$. Hence, $F$ is good with respect to $T$. Now assume that $F$ is a good augmenting set with respect to $T$. Then in $G + F$ both $T_i$ and $T_j$ are connected. Also, because $C$ intersects both $T_i$ and $T_j$, there exist $t_i \in T_i$ and $t_j \in T_j$, such that there is a $(t_i, t_j)$-path in $G$. This path remains intact in $G + F$ and so $T_i \cup T_j$ is connected. That is, $F$ is a good augmenting set with respect to $T'$.

Now let $F$ be a minimum augmenting set with respect to $T$. By the above, it is good with respect to $T'$. Suppose, indirectly, that $F_1$ is a good augmenting set with respect to $T'$ and $|F_1| < |F|$. Then $F_1$ is also good with respect to $T$ (by the previous paragraph), contradicting the minimality of $F$. Therefore $F$ is a minimum augmenting set with respect to $T'$. The other direction holds with a similar argument.

From this point forward, we call the action of forming $T' = T - \{T_i, T_j\} \cup \{T_i \cup T_j\}$ a simple welding of $T$. Note that welding does not change either the vertex set or the edge set of the hypergraph. We simply change the way we look at the sets we wish to augment, by using Lemma 5.2.1 to reduce $T$ to a small (more useful) subpartition. We call a subpartition $X = \{X_1, \ldots, X_n\}$ open, if each component of $G$ separates at most one element of $X$.

Lemma 5.2.2 Let $G = (V, E)$ be a hypergraph and $T = \{T_1, \ldots, T_n\}$ be a subpartition of $V$. There exists an open subpartition $T^* = \{T_1^*, T_2^*, \ldots, T_m^*\}$ such that $F$ is a minimum augmenting set with respect to $T$ if and only if it is minimum with respect to $T^*$.

Proof: Let $V(G) = C_1 + C_2 + \ldots + C_q$, where each $C_i$ is the vertex set of a component of $G$. Consider each component in turn. For each $C_i$, weld any pair $T_i, T_j$, both separated by $C_i$ until no such pair exists. Repeated application of Lemma 5.2.1 implies that $F$ is minimum with respect to the resulting subpartition $T^*$ and it is open.

Lemma 5.2.2 means that we need only consider attempts to augment with respect to an open subpartition of $V$. The significance of open-ness is that we can augment the sets in the subpartition one at a time and still have a minimum augmenting set.
Theorem 5.2.3 Let $G = (V, E)$ be a hypergraph and $T = \{T_1, T_2, \ldots, T_n\}$ be an open subpartition of $V$. Then $G$ can be made 1-edge-connected in $T$ by adding at most $\gamma$ new size-two edges if and only if $c_T - n \leq \gamma$.

Proof: Firstly we note that, because each component separates at most one element of $T$, we must have at least $c_T - n$ edges in any good augmenting set. We now show that there is such a set.

For each $i$, let $C_{T_i}$ be the set of components of $G$ that separate the set $T_i$. Because $T$ is open, each component separates at most one member of $T$ and so $\sum |C_{T_i}| = c_T$. Then, we can form a set of $|C_{T_i}| - 1$ size-two edges, $F_i$, by choosing one vertex from each component in $C_{T_i}$ and joining them in a path. Then in $G + F_i$ the set $T_i$ is connected. Furthermore, any set with fewer than $|C_{T_i}| - 1$ size-two edges cannot connect $T_i$. If we let $F = F_1 \cup F_2 \cup \ldots \cup F_n$, it is certainly a good augmenting set with $c_T - n$ edges. \[\square\]

So given a hypergraph and a subpartition, we would first weld it until it was open, and then perform the augmentation one part at a time. In the case where we wish to make $T$ $k$-edge-connected with $k \geq 2$, though, there seems to be no equivalent to the welding operation and no reason that a minimum augmentation can be found part by part.

5.3 Bin-Packing

Before we can tackle the $k \geq 2$ case of the problem above, we need to know a little about the Bin-Packing Problem.

Suppose we are given a set of “weights” and are asked to fit them into bins, each of which can only carry a certain total weight. It is sensible to ask, “What is the smallest number of bins required?” The decision problem associated with this (optimization) question is known to be strongly NP-complete. (See Garey and Johnson, [35].)

PROBLEM: BIN-PACKING (BP)
Instance $I_{BP}$: A set of weights $W = \{w_1, w_2, \ldots, w_n\}$ with each $w_i \in \mathbb{N}$, a bin size $b^*$ and an integer $m$ such that $\sum_{w_i \in W} w_i \leq mb^*$.

Question: Is there a partition $W_1, \ldots, W_m$ of $W$ such that the sum of the weights in each $W_i$ is at most $b^*$?

Note that we only deal with instances where $\sum_{w_i \in W} w_i \leq mb^*$, as this is an obvious necessary condition for there to be a solution.
We now show that $I$ is that in what follows, we will be creating a set of vertices of size $w_i$ for each weight. If the weights were stored as binary strings, we would be transforming from an instance with size $\log w_i$ to one of size $w_i$ - which is not a polynomial transformation. However, by storing weights and bin-sizes as unary strings, we avoid this difficulty. For a more complete discussion of strong NP-completeness see [35].

For our purposes we require a slight refinement of this problem. Namely, one in which we can vary the size of the bins, but only in such a way that the sum of the weights equals the sum of the bins. We also include the condition that each weight and bin must have size at least three. We use this condition in our reduction for the hypergraph problem.

**PROBLEM: SPECIAL BIN-PACKING (SBP)**

**Instance** $I_{SBP}$: A set of weights $W = \{w_1, w_2, \ldots, w_n\}$ with each $w_i \in \mathbb{N}$, a set of bins $B = \{b_1, b_2, \ldots, b_m\}$, such that every $w_i$ and $b_i$ is at least 3 and $\sum_{w_i \in W} w_i = \sum_{b_i \in B} b_i$.

**Question:** Is there a partition $W_1, \ldots, W_m$ of $W$ such that $\sum_{w_i \in W_j} w_i = b_j$, for each $j = 1, \ldots, m$?

When there is a solution for a given instance of a problem $I_{\text{prob}}$, we say $I_{\text{prob}} \in Y_{\text{prob}}$ the set of instances for which the answer to the question is “yes”.

**Theorem 5.3.1** Special Bin-Packing is NP-complete.

**Proof:** SBP is in NP so we consider the following instance, $I_{BP}$, of BP. Weight set $W' = \{w'_1, \ldots, w'_n\}$, binsize $b^*$, and integer $m$. We construct an instance of SBP as follows. Firstly we let $B' = \{b'_1, b'_2, \ldots, b'_m\}$ where each $b'_i = b^*$. We then check whether or not $\sum w'_i = \sum b'_i$ and if necessary add $l := mb^* - \sum w'_i$ unit value weights, to give weight set $W'' = \{w'_1, w'_2, \ldots, w'_n, w'_{n+1}, \ldots, w'_{n+l}\}$. We now have $\sum w'_i = \sum b'_i$. Finally, we multiply every weight and bin by three. That is, let $W = \{w_1, \ldots, w_{n+l}\}$ and $B = \{b_1, \ldots, b_m\}$ where $w_i = 3w'_i$ for all $i = 1, \ldots, n+l$ and $b_i = 3b'_i$ for all $i = 1, 2, \ldots, m$. Then let $I_{SBP}$ be weight set $W$, bin set $B$, and let $\gamma = \sum w_i$.

We now show that $I_{BP} \in Y_{BP}$ if and only if $I_{SBP} \in Y_{SBP}$. Indeed, suppose $W'_1, W'_2, \ldots, W'_m$ is a solution partition for $I_{BP}$. Let $l_i := b^* - \sum_{w \in W'_i} w$. Then $l_1 + l_2 + \ldots + l_m = l$ and we can partition the set $\{w'_{n+1}, w'_{n+2}, \ldots, w'_{n+l}\}$ into $m$ sets $L'_i$, each having $l_i$ members. (Some of them may be empty.) Then each
set $W_i' \cup L_i'$ has “size” $b_i$. Therefore if we let $W_i = \{w_j : w_j' \in W_i' \cup L_i'\}$ for each $i = 1$ to $m$ we have a solution partition for $I_{SBP}$. That is if $I_{BP} \in Y_{BP}$ then $I_{SBP} \in Y_{SBP}$.

The other direction holds by the reverse argument. That is, if $W_1, W_2, \ldots, W_m$ is a solution partition for $I_{SBP}$ we set $W_i' = \{w_i' : w_i \in W_i$ and $i \leq n\}$. That is, we remove the weights $w_{n+1}, \ldots, w_{n+l}$ and then switch from $w_i$ to $w_i'$ to take care of the multiplying/dividing by three. Then $W_1', \ldots, W_m'$ is a solution partition for $I_{BP}$. Hence, SBP is NP-complete, as required.

5.4 Connectivity Augmentation

As in our earlier work, we are considering using size-two edges to augment a hypergraph and satisfy a connectivity requirement. Here our requirement is to ensure $k$-edge-connectivity within each element of a subpartition of the vertex set. We formulate the decision problem as follows.

**PROBLEM: SUBPARTITION CONNECTIVITY AUGMENTATION (SPCA)**

**Instance** $I_{SPCA}$: A hypergraph $G = (V,E)$, a subpartition $V_1, V_2, \ldots V_t$ of $V$, and an integer $\gamma$.

**Question:** Can we add $\gamma$ size-two edges to $G$ so that in the resulting hypergraph, whenever $x$ and $y$ are both members of the same $V_i$, $\lambda(x,y) \geq 2$?

SPCA is a special case of the general local edge-connectivity augmentation problem, which can be expressed as the following decision problem.

**PROBLEM: LOCAL EDGE-CONNECTIVITY AUGMENTATION (LECA)**

**Instance:** A hypergraph $G = (V,E)$, a function $r : V^2 \rightarrow \mathbb{N}$ defined for each pair of vertices and an integer $\gamma$.

**Question:** Can we add $\gamma$ size-two edges to $G$ so that in the resulting hypergraph $\lambda(x,y) \geq r(x,y)$ for all pairs $x, y \in V$?

We show this problem is NP-complete by showing SPCA is NP-complete. We use a transformation from SBP.

**Theorem 5.4.1** SPCA is NP-complete.

**Proof:** SPCA is in NP, so we start by describing how we would construct an
Before dealing with the other direction, we introduce some notation. Let $I_{SBP}$ be a weight set, $\{w_1, w_2, \ldots, w_n\}$ and a bin set $\{b_1, b_2, \ldots, b_m\}$ such that $\sum w_i = \sum b_i$ and all $w_i, b_i \geq 3$.

For each $w_i$, create a vertex set $X_i$ with $w_i$ vertices and for each $b_i$ let $Y_i$ be a set of $b_i + 1$ vertices. We form a hypergraph $G = (V, E)$ as follows. Let $V = X_1 \cup \ldots \cup X_n \cup Y_1 \cup \ldots \cup Y_m$. Let $E = \{e_0, e_1, e_2, \ldots, e_m\}$ where $e_0$ is incident with every vertex in $X_1 \cup X_2 \cup \ldots \cup X_n$ and exactly one vertex from each $Y_i$, calling this vertex $y_i$ in each case, and for $i = 1, \ldots, m$, the edge $e_i$ is incident with every vertex in $Y_i$ and no others.

Now, let $I_{SPCA}$ be the hypergraph $G = (V, E)$ above, let the subpartition be $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m$ (actually this is a partition of $V$, but this makes no difference), let $k = 2$ and $\gamma = \sum w_i$. Recall that, thanks to the strong NP-completeness of the bin-packing problem, we can consider the weights and bins to have been stored as unary strings. Hence, the size of $I_{SPCA}$ is bounded by a polynomial of the size of $I_{SBP}$ and thus we have indeed performed a polynomial transformation.

We now show that $I_{SPCA} \in Y_{SPCA}$ if and only if $I_{SBP} \in Y_{SBP}$, we must

**Claim 5.4.1.1** If $I_{SBP} \in Y_{SBP}$, then $I_{SPCA} \in Y_{SPCA}$.

**Proof:** Suppose that $I_{SBP} \in Y_{SBP}$ and let $W_1, W_2, \ldots, W_m$ be the solution partition. We can assume, by renumbering if necessary, that $W_1 = \{w_1, w_2, \ldots, w_s\}$. Then $\sum_{w_i \in W_1} = b_1$ and so $X_1 \cup X_2 \cup \ldots \cup X_s$ has $b_1 = |Y_1| - 1$ vertices. Thus we can form a bijection, $f$, between $X_1 \cup X_2 \cup \ldots \cup X_s$ and $Y_1 - y_1$ (where $y_1$ is the vertex incident with both $e_0$ and $e_1$), and let $F_1$ be the set of edges $\{xy : x \in (X_1 \cup X_2 \cup \ldots \cup X_s) \text{ and } y = f(x)\}$. Note that $|F_1| = b_1$.

Then we can quickly see that in $G + F_1$, if $Z$ separates any of $X_1, X_2, \ldots, X_s, Y_1$, it has degree at least 2, and hence $G$ is 2-edge-connected "inside" these sets. Repeating this process for each of the "bins" leads to a good augmenting set $F = F_1 \cup \ldots \cup F_m$ with $\sum b_i = \sum w_i = \gamma$ edges. Thus $I_{SPCA} \in Y_{SPCA}$. $\square$

Before dealing with the other direction, we introduce some notation. Let $X = X_1 \cup \ldots \cup X_n$, and $Y = (Y_1 - y_1) \cup \ldots \cup (Y_m - y_m)$. Then $|X \cup Y| = 2\gamma$.

**Claim 5.4.1.2** If $I_{SPCA} \in Y_{SPCA}$, then $I_{SBP} \in Y_{SBP}$.

**Proof:** Suppose that $I_{SPCA} \in Y_{SPCA}$, let $F$ be a good augmenting set, with $\sum w_i$ edges. We note that every singleton set containing a vertex from $X \cup Y$ separates some $X_i$ or $Y_i$, and has degree one in $G$. Thus, in $G + F$ each vertex in $X \cup Y$ is incident with exactly one edge from $F$ and further, given a vertex
in $X \cup Y$ it has exactly one “image” defined by $F$. That is, $F$ is a perfect matching.

We now show that if $uv \in F$ and $u \in X$, then $v \in Y$. To see this we suppose $uv \in F$ and that both $u$ and $v$ are in $X$. Then $u$ is a member of some $X_i$, and because $|X_i| \geq 3$ the set $Z = \{u, v\}$ separates $X_i$. But then in $G + F$, $d(Z) = 1$, because the only edges incident with $u$ and $v$ are $uv$ and $e_0$. That is $X_i$ is not 2-edge-connected in $G + F$, which contradicts the fact that $F$ is a good augmenting set for $G$.

We now wish to show that if $uv \in F$ with $u \in X_i$ and $v \in Y_j$, then every $xy \in F$ with $x \in X_i$ has $y \in Y_j$. So we suppose that this is false and let $Z = \{z \in X : \text{there exists } zy \in F \text{ such that } y \in Y_j\}$. Then our supposition implies that $X_i$ is not contained in $Z$ and moreover that $Z \cup Y_j$ separates $X_i$. But in $G + F$, $d(Z \cup Y_j) = 1$ and so $X_i$ is not 2-edge-connected in $G + F$. This is a contradiction.

Thus we have shown that every edge in $F$ is from $X$ into $Y$, and that for each $X_i$, there is a $Y_j$ such that every edge $uv \in F$ with one end in $X_i$ has the other end in $Y_j$. Therefore there is partition $U_1, \ldots, U_m$ of the set $\{X_1, X_2, \ldots, X_n\}$ such that

$$\sum_{X_i \in U_j} |X_i| = |Y_j| - 1.$$ 

Therefore, if we let $W_j = \{w_i : X_i \in U_j\}$ for each $j = 1$ to $m$, we have a solution partition of the weight set from $I_{SBP}$. That is $I_{SBP} \in Y_{SBP}$. \hfill $\Box$

So we have shown that we can polynomially create an instance of SPCA from one for SBP, and that there is a solution for SPCA if and only if there is one for SBP. That is, SPCA is NP-complete. \hfill $\square$

Because every instance of SPCA is an instance of LECA, we have the following Corollary.

**Corollary 5.4.2** *Local Edge-Connectivity Augmentation in Hypergraphs, is NP-complete.* \hfill $\square$

**Remark - What next?**

Recently, András Frank has asked whether progress might still be possible with the following special case.

*Given a $k$-edge-connected hypergraph $H = (V, E)$, and a collection of sets, $T = \{T_1, \ldots, T_n\}$ such that $T_n \subseteq T_{n-1} \subseteq \ldots \subseteq T_1 \subseteq V$, what is the smallest number*
of size two edges we must add to satisfy the requirement function \( r : V^2 \rightarrow \mathbb{N} \) where \( r(x, y) = k + i \) when \( x, y \in T_i \) and \( r \equiv k \) otherwise?

This is related to the case considered throughout this chapter, but the nesting of the sets may make progress possible. For instance, it may be possible to augment a nested subpartition one-element-at-a-time. Alternatively, the nesting property may simplify the structure of a “break” in hypergraph \( G = (V + s, E') \) formed by adding \( s \) to \( H \), (as described in Chapter 3), which may make determining the maximum number of \( \lambda \)-splits easier.
Chapter 6

Bipartition Constrained $k$-Splitting

6.1 Introduction

In this chapter, we are again splitting edges from a special vertex $s$ in a hypergraph $G = (V + s, E)$, and as always we have a requirement that must be satisfied for a split to be “good.” Previously, our requirements have all been in terms of preserving connectivity in $V$. For instance, we might look for $k$-splits - those that preserve $k$-edge-connectivity in $V$. Now though, once we have determined our “connectivity requirement” we also have a further “locational” constraint that must also be satisfied. That is, we will look for $PC$-splits, where a $C$-split satisfies the connectivity requirement, a $P$-split satisfies the locational requirement and a $PC$-split satisfies both.

To analyse the availability of $PC$-splits, we follow the approach introduced by Jordán, in [42]. Given any $G = (V + s, E)$ in which we wish to find a $PC$-split, we form the following graphs on copies of $N(s)$.

$B_G(s)$: where vertices $x, y$ are joined if and only if they do not form a $C$-split.
(This is equivalent to Jordán’s “non-admissibility” graph).

$D_G(s)$: where vertices $x, y$ are joined if and only if they do form a $P$-split.

If it is clear which hypergraph we are dealing with, we drop the subscripts. We compare $B(s)$ and $D(s)$ and can determine whether or not there is a $PC$-split by using the following idea.

There is no $PC$-split if and only if $D(s)$ is a subgraph of $B(s)$.
By analysing and comparing the structures of $B(s)$ and $D(s)$, we can determine the structures that prevent us from performing $\mathcal{PC}$-splits, and those that prevent us forming complete $\mathcal{PC}$-splits.

6.2 Notation and Some Definitions

Throughout this chapter, we consider hypergraphs in the form $G = (V + s, E)$ where $s$ is special - that is, has even degree of at least 4 and is only incident with size-two edges - and $d(X) \geq k$ for all $X$ with $\emptyset \subset X \subset V$. $P = \{P_1, P_2\}$ is a bipartition of $V$ and we think of $P_1$ as red and $P_2$ as blue. Our good split is a $k$-split that has endpoints in both $P_1$ and $P_2$. We call such a split a $P_k$-split.

Thus we form $B(s)$ and $D(s)$ on copies of $N(s)$ by joining vertices $x, y$ in $B(s)$ if and only if they are not a $k$-split, and in $D(s)$ if and only if $x \in P_1$ and $y \in P_2$ or vice versa.

Now that $B(s)$ and $D(s)$ are properly defined, we state the following proposition for later use.

**Proposition 6.2.1** There is no $P_k$-split at $s$ if and only if $D(s)$ is a subgraph of $B(s)$. □

We also note here, that in this case $D(s)$ is the complete bipartite graph on $N(s) \cap P_1$ and $N(s) \cap P_2$. Analysing the structure of $B(s)$ takes a little more work.

6.3 The Structure of $B(s)$

We form $B(s)$ by considering whether vertices $x, y$ form $k$-splits. To analyse $B(s)$ we use the “tight/dangerous” system for $k$-splits. That is, throughout this chapter, a set $X$ with $\emptyset \subset X \subset V$ is tight when $d(X) = k$ and dangerous when $d(X) \leq k + 1$ and $d_1(s, X) \geq 2$. When referring to maximal dangerous sets, we mean with respect to inclusion.

**Lemma 6.3.1** If $X$ is dangerous, then $d_1(s, X) \leq d_1(s, V - X)$.

**Proof:** This follows immediately from Lemma 2.3.2, considering the simple routing function $r \equiv k$. □

**Lemma 6.3.2** A maximal dangerous set does not cross any tight set.
Proof: We apply Proposition 2.1.1, considering a tight set $X$ and a maximal dangerous set $Y$. Then

$$k + k + 1 \geq d(X) + d(Y) \geq d(X \cap Y) + d(X \cup Y) \geq k + k + 2$$

as long as $\emptyset \neq X \cap Y \subset V$ and $\emptyset \neq X \cup Y \subset V$. This is a contradiction, so the two sets cannot cross. □

In order to prove our results on the structure of $B(s)$, we must categorise the $s$-neighbours according to the number of maximal dangerous sets in which they lie.

Suppose $t \in N(s)$. If there is no dangerous set containing $t$ (that is, an edge $st$ is $k$-splittable with all other edges $sx$) we call $t$ type-$0$. If there exists a unique maximal dangerous set containing $t$, we say $t$ is type-$1$. If there are precisely two maximal dangerous sets containing $t$, then $t$ is type-$2$ and if there are three or more maximal dangerous sets containing $t$, we call $t$ type-$3$.

If $t$ is a type-$0$ $s$-neighbour it is clearly isolated in $B(s)$. If $t \in N(s)$ is type-$1$, then it and the other $s$-neighbours in the maximal dangerous set containing it induce a complete subgraph of $B(s)$. The following proposition is straightforward.

**Proposition 6.3.3** If every $s$-neighbour is type-$0$ or type-$1$, $B(s)$ is a disjoint union of cliques and isolated vertices. □

We now consider type-$2$ and type-$3$ neighbours and the following result is useful for both.

**Lemma 6.3.4** Suppose $t \in N(s)$ and let $X,Y \subseteq V$ be two maximal dangerous sets that both contain $t$. If both $X-Y$ and $Y-X$ are both non-empty, $d(X) = d(Y) = k + 1$, $d(X-Y) = d(Y-X)$, $d_3(X,Y) = 1$, $d_4(X,Y) = 0$, and thus any hyperedge intersecting both $X \cap Y$ and $V - (X \cup Y)$ intersects both $X-Y$ and $Y-X$. Furthermore, if $X \cup Y \neq V$ we also have $d(X \cap Y) = k$, $d(X \cup Y) = k + 2$ and $d_2(X,Y) = d_1(X,Y) = 0$.

Proof: We use Proposition 2.1.1, the maximality of $X,Y$ and the fact that $V$ is $k$-edge-connected, to get the following.

$$2k + 2 \geq d(X) + d(Y) \geq d(X-Y) + d(Y-X) + 2d_3(X,Y) + d_4(X,Y) \geq k + k + 2 + 0.$$  
$$2k + 2 \geq d(X) + d(Y) \geq d(X \cap Y) + d(X \cup Y) + 2d_4(X,Y) + d_2(X,Y) \geq k + (k + 2) + 0 + 0.$$  

Therefore, equality holds throughout in both, and the lemma follows. □

We now look at type-$3$ vertices.
Lemma 6.3.5 Suppose $t$ is a type-3 neighbour of $s$. Let $Y_1, Y_2, \ldots Y_r$ be the largest family of maximal dangerous sets containing $t$. Then there exists a set $A$ of $k - 1$ hyperedges such that $G - s - A$ has at least $r + 1$ components with exactly one $s$-neighbour in each of $r + 1$ of them, and each edge in $A$ intersects every component of $G - s - A$.

Proof: This is very similar to the “fan” arguments used in Chapter 3, and we begin by finding the “heart” of the fan.

Claim 6.3.5.1 There exists $M \subset V$ such that $M = Y_i \cap Y_j$ for all $i, j \in \{1, 2, \ldots, r\}$.

Proof: Because $t$ is type-3, $r \geq 3$. Let $M = Y_1 \cap Y_2$. Because we have $r \geq 3$, $Y_1 \cup Y_2 \neq V$, so by Lemma 6.3.4, $M$ is tight. Now consider $Y_i, Y_j$. They are both maximal dangerous sets containing $t$, so they both intersect $M$ and so by Lemma 6.3.2 they both contain $M$. Therefore, $Y_i \cap Y_j$ contains $M$. Also, $Y_i \cup Y_j \neq V$, so Lemma 6.3.4 implies $Y_i \cap Y_j$ is tight. Suppose there is a vertex $v$, such that $v \in Y_i \cap Y_j$ and $v \notin M = Y_1 \cap Y_2$. So we can assume that $v \notin Y_1$. Then $Y_1$ crosses $Y_i \cap Y_j$, which contradicts Lemma 6.3.2. Thus, there is no such vertex $v$ and $M = Y_i \cap Y_j$ for all $i, j \in \{1, 2, \ldots, r\}$. □

This claim means that each $Y_i = M \cup X_i$, where $X_1 = Y_1 - Y_2$ and $X_i = Y_i - Y_1$ for $i \neq 1$. Then by Lemma 6.3.4, each $X_i$ is tight. Now, let $e$ be an edge intersecting $M$ and $V - M$. Then, by Lemma 6.3.4, for every pair of sets $Y_i, Y_j$, $e$ intersects $Y_i - Y_j = X_i$ and $Y_j - Y_i = X_j$. That is, $e$ intersects every $X_i$. Furthermore, $M$ is tight, and has $d_1(s, M) = 1$, so there are $k - 1$ edges intersecting $M$ and $V - M$. By the above, they all intersect every $X_i$, and because each $X_i$ is tight, we have $d_1(s, X_i) \geq 1$, and the Lemma holds. □

From this, we can make the following deduction about the structure of $B(s)$.

Lemma 6.3.6 Suppose $t$ is a type-3 neighbour of $s$, with $Y_1, Y_2, \ldots, Y_r$ as before. Then each $Y_i$ contains exactly one $s$-neighbour distinct from $t$. Further, the $r + 1$ $s$-neighbours covered by $\bigcup Y_i : i = 1$ to $r$, form a complete component of $B(s)$.

Proof: There exists a set of $k - 1$ hyperedges with the properties of the last lemma. The components of $G - s - A$ can be labelled $X_0, X_1, \ldots X_r, X_{r+1}, \ldots$, with each $Y_i = X_0 \cup X_i$ for $i = 1$ to $r$. Each $X_i$ contains exactly one neighbour of $s$ (with just one edge to $s$) because the degree of each $X_i$ is $k$. (Call these neighbours $t = x_0, x_1, \ldots x_r$.) Also, for $i \neq j \leq r$, $X_i \cup X_j$ is dangerous, and so the pair of edges from $x_i$ and $x_j$ are not a $k$-split. This is true for all such $i, j$ so the specified $s$-neighbours certainly induce a complete subgraph of $B(s)$. 77
Now suppose there is some $s$-neighbour $v \neq x_i$ for any $i = 0$ to $r$. Each $x_i$ is certainly type-3 and if (in addition to all the dangerous sets $X_i \cup X_j$) there was a dangerous set $Y_v$, containing $x_i$ and $v$, it would give rise to a component $X_v \subset Y_v - M$, of $G - s - A$ with $X_v \cap N(s) = v$. Therefore the sets $X_j \cup X_v$ with $j \neq i$ would also be dangerous, contradicting the maximality of $r$. Therefore, in $B(s)$ there is no edge from any $x_i$ to $N(s) - \bigcup Y_i$. \hfill $\Box$

**Corollary 6.3.7** If $t$ is type-1 or type-2 and $X$ is a maximal dangerous set containing $t$, then $X$ contains no type-3 $s$-neighbours. \hfill $\Box$

We now consider the type-2 vertices in $N(s)$. The next lemma shows that if there are two type-2 $s$-neighbours in a maximal dangerous set, there are no other members of $N(s)$ in that set. If $x \in N(s)$ is type-2, and $X, Y$ are the two maximal dangerous sets containing $x$, then by Lemma 6.3.4 there are no other $s$-neighbours in $X \cap Y$. Therefore, any other $s$-neighbour in $X$, is in $X - Y$.

**Lemma 6.3.8** Suppose that $t$ is a type-2 $s$-neighbour and $X$ and $Y$ are the two maximal dangerous sets containing $t$. Let $u \in Y - X$ be type-2 and $Y$ and $Z$ be the two maximal dangerous sets containing $u$. Then $(X \cap Y) \cap (Z \cap Y) = \emptyset$, $Y = (X \cap Y) \cup (Y \cap Z)$ and $d_1(s,Y) = 2$. Further, if there is no $s$-neighbour that lies in both $Z$ and $X$, then $X \cap Z$ is empty.

**Proof:** Lemma 6.3.4 implies that $d_1(s,Z \cap Y) = 1$, so there is at least one $s$-neighbour in $Z - Y$. There are two cases.

**Case 1:** $Z$ contains an $s$-neighbour, $v \in X - Y$.

Because $t$ is type-2, and lies in $X$ and $Y$, we know $t \notin Z$ and hence $Z \neq X$. Also, by the Corollary 6.3.7, $v$ cannot be type-3 and so $v$ is type-2 and its two maximal dangerous sets are $X$ and $Z$. Both $X - Z$ and $Z - X$ are non-empty, so Lemma 6.3.4 implies that $d(Z) = d(X) = d(Y) = k + 1$ and $d(X - Y) = d(Y - X) = k$. Also, because $u \notin X$, we have $X \neq Z$ and so $X \cup Y \neq V$ and Lemma 6.3.4 implies $d(X \cap Y) = k$. So by Lemma 6.3.2 $Z$ does not cross $X - Y$. Because $x$ is in both $X - Y$ and $Z$, $t$ is in neither $X - Y$ nor $Z$ and $u$ is in $Z$ but not in $X - Y$, we have that $X - Y \subset Z$. Similarly $Y - X \subset Z$. Also by Lemma 6.3.2 $Z$ does not cross $X \cap Y$. Neither set difference is empty, so either $Z \cap X \cap Y$ is empty, or $Z \cup (X \cap Y) = V$. The latter would imply $d_1(s,Z) \geq d(s) - 1 > d_1(s,V - Z)$ because $d(s) \geq 4$ which contradicts Lemma 6.3.1. Thus $Z \cap X \cap Y = \emptyset$, and so $(X \cap Y) \cap (Z \cap Y) = \emptyset$ as required. Also $Y = (X \cap Y) \cup (Z \cap Y)$. Lemma 6.3.4 implies that $d_1(s,X \cap Y) = d_1(Y \cap Z) = 1$ and thus $d_1(s,Y) = 2$ as required. This completes the proof for case 1.

**Case 2:** There is no $s$-neighbour that lies in both $Z$ and $X$.

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Let \( v \in Z - Y \) be an \( s \)-neighbour. (Then \( v \in V - (X \cup Y) \).) We know that \( t \notin Z \) and by the definition of a dangerous set, we have some \( s \)-neighbour, \( w \in X - Y \), and so \( w \notin Z \). Again we have that both \( X - Y \) and \( X \cap Y \) are tight. Thus, by Lemma 6.3.2 neither cross \( Z \), and so \( Z \cap (X - Y) = Z \cap (X \cap Y) = \emptyset \). So \( Z \cap X \) is empty and thus, so is \( (X \cap Y) \cap (Z \cap Y) \), as required. Using Lemma 6.3.2 again we see that \( Y - X \subset Z \), because otherwise they would cross. Therefore \( Y = (X \cap Y) \cup (Y \cap Z) \) and \( d_1(s, Y) = 2 \) as required. This completes the proof of the lemma.

Before we can state our results about the structure of \( B(s) \) we need a couple more definitions. Suppose \( X = \{X_0, X_1, \ldots, X_{d(s)-1}\} \) is a partition of \( V \) in \( G = (V + s, E) \). The edge-plate of \( X \) is the set \( A = \{e \in E : e \text{ intersects every } X_i \in \mathcal{X}\} \). We say \( X \) is uniform when it satisfies the following properties.

\[
\begin{align*}
& (a) \; d_1(s, X_i) = 1 \text{ for each } X_i \in \mathcal{X}, \\
& (b) \; d(X_i) = k \text{ for all } X_i \in \mathcal{X}, \\
& (c) \; d(X_i, X_{i+1}) = \frac{k-1-a}{2} \text{ for } 0 \leq i \leq d(s) - 1, \text{ where } a = |A| \text{ and } X_{d(s)} := X_0.
\end{align*}
\]

A hypergraph \( G = (V + s, E) \) that is \( k \)-edge-connected in \( V \) with \( s \) special, is semi-brittle at \( s \) when there exists a uniform partition of \( V \). We say that \( G \) is brittle at \( s \) when it is semi-brittle at \( s \) with \( a = k - 1 \). When \( G \) is semi-brittle but not brittle it is properly semi-brittle.

\textbf{(Remark :} When \( G \) is semi-brittle at \( s \), all its edges fit into one of four categories; those incident with \( s \), those in \( A \), those wholly contained in some \( X_i \) and those intersecting both of, and wholly contained in the union of, some consecutive pair \( X_i, X_{i+1} \). This means that if an edge intersects three or more members of \( \mathcal{X} \) it must intersect all of them. Also, note that property \( (c) \) implies that in a uniform partition, \( k - a \) must be odd. This fact appears again later.\textbf{)}

In [42] Jordán shows that for graphs, when \( k \) is odd, \( B(s) \) is 2-edge-connected if and only if \( G \) is semi-brittle at \( s \) with \( a = 0 \). In the hypergraph case the parity of \( k \) does not matter and we must be aware that \( B(s) \) can be complete on \( N(s) \), which cannot happen for graphs. In [1], Bang-Jensen and Jackson characterise the hypergraphs for which \( B(s) \) is complete, that is, for which there is no \( k \)-split at \( s \). Their result can be stated as follows.

\textbf{Theorem 6.3.9 (Bang-Jensen and Jackson)} \textit{For a hypergraph} \( G = (V + s, E) \) \textit{with} \( s \) \textit{special and} \( V \) \textit{k-edge-connected,} \( B(s) \) \textit{is the complete graph on} \( N(s) \) \textit{if and only if} \( G \) \textit{is brittle at} \( s \).

Following Jordán’s line, we now characterise those hypergraphs for which \( B(s) \) is 2-edge-connected.
**Theorem 6.3.10** Let \( G = (V + s, E) \) be \( k \)-edge-connected in \( V \) for some \( k \geq 2 \), with \( s \) special. Then \( B(s) \) is 2-edge-connected if and only if \( G \) is semi-brittle at \( s \).

**Proof:** Firstly suppose that \( G \) is semi-brittle at \( s \) and let the uniform partition of \( V \) be \( X = (X_0, X_1, \ldots, X_{d(s) - 1}) \). Then each set \( X_i \cup X_{i+1} \) is dangerous and so the cycle on \( N(s) \) is a spanning subgraph of \( B(s) \). Therefore \( B(s) \) is 2-edge-connected.

The other direction follows from the following stronger result, that we shall use later when trying to find \( Pk\)-splits. \( \square \)

**Theorem 6.3.11** Let \( G = (V + s, E) \) be \( k \)-edge-connected in \( V \) for some \( k \geq 2 \), with \( s \) special. If \( B(s) \) is 2-edge-connected then either

(a) \( B(s) \) is complete and \( G \) is brittle at \( s \), or

(b) \( B(s) \) is a cycle and \( G \) is properly semi-brittle at \( s \).

**Proof:** Suppose that \( B(s) \) is 2-edge-connected. If \( B(s) \) is the complete graph on \( N(s) \) then there is no \( k \)-split, and so no \( Pk\)-split. Therefore by Theorem 6.3.9, \( G \) is brittle at \( s \).

So from this point we assume that \( B(s) \) is not complete. These means that there are no type-3 \( s \)-neighbours, because if there were, by Lemma 6.3.6, \( B(s) \) would have a complete component, contradicting the 2-edge-connectedness of \( B(s) \). This implies that \( N(s) \) must contain a type-2 vertex - otherwise, by Proposition 6.3.3 \( B(s) \) would be disconnected.

Let \( t \) be a type-2 \( s \)-neighbour and \( X \) and \( Y \) be the two maximal dangerous sets with \( t \in X \cap Y \). Suppose there is no \( r \in N(s) \) such that \( st, sr \) form a \( k \)-split. Then \( N(s) \subseteq X \cup Y \). By Lemma 6.3.4, \( d_3(X, Y) = 1 \) and hence \( d_1(s, X \cap Y) = 1 \). Therefore \( d(s) = 1 + d_1(s, X - Y) + d_1(Y - X) \). By hypothesis, \( d(s) \) is even, and so (wlog) we can assume \( d_1(s, X - Y) = d_1(Y - X) \). But then \( d_1(s, X) = d_1(s, X - Y) + 1 > d_1(s, Y - X) = d_1(s, V - X) \) contradicting Lemma 6.3.1. Therefore there is some \( r \in N(s) \) such that \( st, sr \) form a \( k \)-split. This means that \( r \in V - (X \cup Y) \).

Since \( B(s) \) is 2-edge-connected, there are two paths \( Q_1, Q_2 \) from \( t \) to \( r \) in \( B(s) \). If we let \( uv \in E(B(s)) \) be the first edge of \( Q_1 \) that has one end in \( N(s) \cap (X \cup Y) \) and the other in \( N(s) \cap (V - (X \cup Y)) \), then either \( u \in X - Y \) or \( u \in Y - X \) (in \( G \)). We assume (wlog) that \( u \in Y - X \). Because \( v \in V - (X \cup Y) \), and there is an edge \( uv \) in \( B(s) \), there must be a maximal dangerous set, distinct from \( X \) and \( Y \), containing both \( u \) and \( v \). Therefore \( u \) is type-2 and one of its two
maximal dangerous sets is \( Y \). Therefore by Lemma 6.3.8, \( d_1(s, Y) = 2 \). Hence \( u \) is the only neighbour of \( t \) in \( B(s) \) in \( Y - X \) and so \( Q_2 \) contains an \( s \)-neighbour \( w \in X - Y \), which is adjacent to \( t \) in \( B(s) \). As above, we see that \( w \) is type-2 and \( d_1(s, X) = 2 \). So \( t \) has degree two in \( B(s) \) and both of its neighbours are type-2. We can apply this argument to every type-2 \( s \)-neighbour. So, because \( B(s) \) is 2-edge-connected, every \( s \)-neighbour is type-2 and \( B(s) \) is a cycle.

Further, because each \( s \)-neighbour is type-2, Lemma 6.3.4 implies that there are no parallel edges incident with \( s \). So there are \( d(s) \) vertices in \( B(s) \).

Call these \( x_0, x_1, \ldots, x_{d(s) - 1} \) following the cyclic ordering implied by \( B(s) \). Let \( V_i = X_{i,1} \cap X_{i,2} \) for \( 0 \leq i \leq d(s) - 1 \), where \( X_{i,1} \) and \( X_{i,2} \) are the two maximal dangerous sets containing \( x_i \) in \( G \), with \( X_{i,1} \) containing \( x_i \) and \( x_{i-1} \) and \( X_{i,2} \) containing \( x_i \) and \( x_{i+1} \). We shall show that \( V_0, V_1, \ldots, V_{d(s) - 1} \) is a uniform partition, with \( d(s) \) members. In what follows, subscripts on the sets \( V_i \) will be read modulo \( d(s) \), that is, \( V_0 = V_{d(s)}, V_1 = V_{d(s)+1} \) and so on.

By Lemma 6.3.4, for each \( i \) with \( 0 \leq i \leq d(s) - 1 \), we have \( d_3(X_{i,1}, X_{i,2}) = 1 \) and \( X_{i,1} \cap X_{i,2} \) is tight. That is, each \( V_i \) has \( d_1(s, V_i) = 1 \) and \( d(V_i) = k \). This deals with properties (a) and (b) from the definition of a uniform partition.

Before we consider property (c), we shall show that the \( V_i \)'s are disjoint. Lemma 6.3.8 implies that \( V_i \) and \( V_{i+1} \) are disjoint for any \( i \). Now suppose \( V_i \cap V_j \neq \emptyset \) for some \( i, j \) with \( j \geq i + 1 \). The fact that \( s \) is special means \( d(s) \geq 4 \) (by definition), which ensures that \( V_j \neq V_{i-1} \). Each \( V_i \) only contains a single \( s \), so \( V_i \cap V_j \neq V \). Also, \( V_i, V_j \) are both tight, so by Proposition 2.1.1, \( 2k \geq d(V_i) + d(V_j) \geq d(V_i \cap V_j) + d(V_i \cup V_j) \geq k + k \). Therefore equality holds throughout and \( d(V_i \cup V_j) = k \). But, \( B(s) \) is a cycle, and so \( x_i, x_j \) is a \( k \)-split, so \( d(X) \geq k + 2 \) for all \( X \) containing both \( x_i \) and \( x_j \). This contradiction shows that the \( V_i \)'s are disjoint, and so form a subpartition.

We now consider property (c). Let \( A = \left\{ e \in E : e \text{ intersects every } V_i \right\} \). Then, because \( d_1(s, V_i) = 1 \) and \( d(V_i) = k \) for all \( i \), we have \( a := |A| \leq k - 1 \). All triples \( V_{i-1}, V_i, V_{i+1} \), are such that \( V_{i-1} \cup V_i \cup V_{i+1} = X_{i,1} \cup X_{i,2}, V_i = X_{i,1} \cap X_{i,2}, V_{i-1} = X_{i,1} - X_{i,2} \) and \( V_{i+1} = X_{i,2} - X_{i,1} \) for a pair of maximal dangerous sets \( X_{i,1}, X_{i,2} \). Now, by Lemma 6.3.4, \( d_3(X_{i,1}, X_{i,2}) = 1 \) and \( d_4(X_{i,1}, X_{i,2}) = 0 \). Therefore any edge intersecting both \( V_i \) and \( V - (V_{i-1} \cup V_i \cup V_{i+1}) \) intersects both \( V_{i-1} \) and \( V_{i+1} \).

Also, if \( i - j \geq 2 \) for some pair \( i, j \) with \( 0 \leq j \leq i \leq d(s) - 1 \), then any edge intersecting both \( V_i \) and \( V_j \) intersects all of \( V_{i-1}, V_{i+1}, V_{j-1}, V_{j+1} \). Furthermore, for all \( i, j \), if \( e \) intersects all of \( V_{i-1}, V_i, V_{i+1} \), then it also intersects \( V - (V_{i-1} \cup V_{i+1}) \). This implies the following.

If \( e \in E \) intersects both \( V_i \) and \( V - V_i \) for some \( i \), then either
(a) $e$ intersects just $V_i$ and $V_{i-1}$, or
(b) $e$ intersects just $V_i$ and $V_{i+1}$, or
(c) $e \in A$ (that is, $e$ intersects every $V_j$).

Now, for every $i$, we have already seen that $d(V_i) = k$, and $d_1(s, V_i) = 1$. This means that for every $i$ we have

\[
k - 1 - a = d(V_{i-2}, V_{i-1}) + d(V_{i-1}, V_i) = d(V_{i-1}, V_i) + d(V_i, V_{i+1}) = d(V_i, V_{i+1}) + d(V_{i+1}, V_{i+2}).\]

(Note that because $d(s) \geq 4$, there are at least four distinct $V_i$'s, and also that $V_{i-2}$ and $V_{i+2}$ may or may not be distinct, but this does not matter.)

So we have $d(V_{i-2}, V_{i-1}) = d(V_i, V_{i+1}) =: b_1$ and $d(V_{i-1}, V_i) = d(V_{i+1}, V_{i+2}) =: b_2$. We assume (wlog) that $X_{i,1} = V_{i-1} \cup V_i$, $X_{i,2} = V_i \cup V_{i+1}$, and that $b_1 \geq b_2$.

But $d(X_{i,1}) = a + 1 + 2b_1 = a + 1 + 2b_2 = d(X_{i,2})$ and so $b_1 = b_2 = (k - 1 - a)/2$.

That is, uniform partition property (c) holds.

All that remains is to show that $V_0, V_1, \ldots, V_{d(s)-1}$ is indeed a partition of $V$.

Let $V^* = V - \bigcup V_i$ and suppose that $V^*$ is non-empty. Let $e$ be an edge intersecting both $V^*$ and $V - V^* = \bigcup V_i$. Then $e$ intersects some $V_j$ and it intersects $V - (V_j \cup V_{j-1} \cup V_{j+1})$. Therefore, $e \in A$ and so $d(V^*) \leq |A| = k - 1$.

This contradicts the $k$-edge-connectivity-in-$V$ of our hypergraph $G$. Hence, $V^*$ is empty, and $V_0, V_1, \ldots, V_{d(s)-1}$ is indeed a uniform partition of $V$. Hence $G$ is semi-brittle at $s$. \hfill \qed \hfill \qed

### 6.4 Complete $k$-splits

Because a complete $Pk$-split at $s$ is merely a constrained complete $k$-split at $s$, we often need to consider whether or not a hypergraph has a complete $k$-split. Bang-Jensen and Jackson describe the structure that prevents complete $k$-splits in [1]. Let $G = (V + s, E)$ be a hypergraph with $s$ special. A shredder is a set of edges $E^*$ such that $|E^*| = k - 1$ and $G - s - E^*$ has at least $\frac{d(s)}{2} + 2$ components.

In the hypergraphs we are considering, every shredder takes on further properties.

**Lemma 6.4.1** Let $G = (V + s, E)$ be a hypergraph that is $k$-edge-connected in $V$ and has $s$ special. If $E^*$ is a shredder in $G$ then
(a) \( d_1(s, X) \geq 1 \) for each component \( X \) of \( G - s - E^* \),

(b) \( d_1(s, X) = 1 \) for at least four components of \( G - s - E^* \), and

(c) \(|e| \geq 4\) for all \( e \in E^* \).

**Proof:** Property (a) follows from the fact that \(|E^*| = k - 1\) and the \(k\)-edge-connectivity of \( V \). To see that (b) holds, suppose that fewer than four of the components of \( G - s - E^* \) have \( d_1(s, X) = 1 \). Then at least \( d(s)/2 - 1 \) components have at least two edges to \( s \) and we have \( d(s) \geq 2(d(s)/2 - 1) + 3 \) - a clear contradiction. The \(k\)-edge-connectivity of \( V \) and (b) imply (c). \( \square \)

We are interested in when there is a complete \(k\)-split at \( s \), and we need the following theorem from [1].

**Theorem 6.4.2 (Bang-Jensen and Jackson)** Let \( G = (V + s, E) \) be a hypergraph that is \(k\)-edge-connected in \( V \) and has \( s \) special. Then there is a complete \(k\)-split at \( s \) if and only if \( G \) has no shredder. \( \square \)

We will often be starting with a hypergraph in which we know there is no shredder, but when we perform a \(k\)-split we find a shredder in the resulting hypergraph. The following lemma will be useful.

**Lemma 6.4.3** Let \( G = (V + s, E) \) be \(k\)-edge-connected in \( V \), have \( s \) special and have no shredder. Let \( x, y \in N(s) \) form a \(k\)-split such that \( G_{xy} \) has a shredder, \( E^* \). Then \( G - s - E^* \) has \( d_G(s)/2 + 1 \) components, with \( x \) and \( y \) in the same one.

**Proof:** Let \( a = d_G(s)/2 \). Then \( d_{G_{xy}}(s) = 2a - 2 \) and \( G_{xy} - s - E^* \) has at least \((a - 1) + 2 = a + 1\) components, with \( x \) and \( y \) contained in exactly one of them. When we “unsplit” \( xy \) to get back to \( G \), the component containing \( x \) and \( y \) can either remain as a component in \( G - s - E^* \) or can fall into two parts. However, if the latter occurs we have that \( G - s - E^* \) has \( a + 2 \) components, so \( E^* \) is a shredder in \( G \), which is contrary to hypothesis. So the former case occurs and the lemma follows. \( \square \)

We will also need to be able to contract \(k\)-tight sets in \( G \). Lemma 2.4.4, (considering the routing function \( r \equiv k \)) implies that \( x, y \) form a \(k\)-split in \( G \) if and only if they do so in \( G/T \), for any \(k\)-tight set \( T \). Repeated application of this idea gives the following result.

**Lemma 6.4.4** Let \( G = (V + s, E) \) be \(k\)-edge-connected in \( V \), have \( s \) special and let \( T \) be a proper subset of \( V \) with \( d(T) = k \). Then there is a shredder in \( G \) if and only if there is a shredder in \( G/T \). \( \square \)
6.5 $P_k$-Splits and $C_4$-odd-obstacles

We now consider our specific problem - that is, finding $P_k$-splits in $G = (V + s, E)$, where $P$ is a bipartition of $V$. We are trying to characterise those graphs for which there is no complete $P_k$-split. Throughout this section (unless otherwise stated) we suppose that $G = (V + s, E)$ is $k$-edge-connected in $V$, has $s$ only incident with graph edges and of even degree, and that $P$ is a bipartition of $V$. If we recall that, by Theorem 6.4.2, there is a complete $k$-split in $G$ if and only if $G$ has no shredder, the following Proposition is straightforward.

**Proposition 6.5.1** If $G = (V + s, E)$ does not satisfy both of the following, then it has no complete $P_k$-split.

(a) $G$ has no shredder, and

(b) $d(s, P_1) = d(s, P_2)$ for both $P_1$ and $P_2$.

In the following results, we assume that our starting hypergraph satisfies (b) and deal with any shredders as necessary. Before trying to find a complete $P_k$-split, we must determine the structures that prevent us performing a single $P_k$-split at any time.

**Lemma 6.5.2** Let $G = (V + s, E)$, and $P$ be a bipartition of $V$ such that $\frac{d(s)}{2} = d_1(s, P_1)$. Then, if there is no $P_k$-split at $s$, either,

(a) $G$ is brittle at $s$, or

(b) $G$ is semi-brittle at $s$ with $d(s) = 4$.

**Proof:** Suppose that $G$ has no $P_k$-split. We consider the graphs $B(s)$ and $D(s)$. In our case, $D(s)$ is the complete bipartite graph on $N(s) \cap P_1$ and $N(s) \cap P_2$. Because $d(s) \geq 4$ either $D(s)$ is a star, or it is 2-edge-connected. However, if $D(s)$ is a star, say $N(s) \cap P_1 = \{x\}$, $d_1(s, x) \geq 2$ and so every edge $sx$ can be split with an edge $sy$ where $y \in P_2$. This is contrary to our initial supposition and so $D(s)$ is 2-edge-connected.

Because $G$ has no $P_k$-split, by Proposition 6.2.1, $D(s)$ is a subgraph of $B(s)$. Therefore $B(s)$ is 2-edge-connected and by Theorem 6.3.11 either $B(s)$ is complete and $G$ is brittle at $s$, or $B(s)$ is a cycle and $G$ is properly semi-brittle at $s$. Further, the only complete bipartite graph that is a subgraph of a cycle is $K_{2,2}$, so in the latter case, $D(s) = K_{2,2}$. This implies that $s$ has four neighbours and the definition of semi-brittle at $s$ means that $d(s) = 4$. \qed
Corollary 6.5.3 If \( G \) has \( d(s) \geq 6 \) and there is a \( k \)-split at \( s \), then there is a \( Pk \)-split. \( \square \)

This is very useful. It seems to say that when faced with trying to perform a complete \( Pk \)-split, we are only going to “get stuck” if we get down to a hypergraph with \( d(s) = 4 \) that is semi-brittle at \( s \). However, as we show later on, unfortunately it isn’t quite as simple as that! But for now, we carry on with this idea, and consider the structure of hypergraph that may force us towards a semi-brittle hypergraph. The terminology for what follows originates in Bang-Jensen et al. [3], but the definition takes into account the possible existence of a set of hyperedges “in the middle” of the obstacle.

Let \( G = (V + s, E) \) be a hypergraph with \( s \) special and \( V \) \( k \)-edge-connected. Let \( C = \{A_1, A_2, B_1, B_2\} \) be a partition of \( V \), and let \( A = \{e \in E : e \text{ intersects every } X \in C\} \) be its edge-plate. Then \( C \) is a \( C_4 \)-odd-obstacle (or \( C_4-o-o \) for short), when it has the following properties.

(a) \( s \) has a neighbour in each element of \( C \) with \( N(s) \cap (A_1 \cup A_2) = N(s) \cap P_1 \),

(b) \( d(X) = k \) for all \( X \in C \),

(c) if an edge \( e \) intersects both \( A_1 \) and \( A_2 \), or both \( B_1 \) and \( B_2 \), then \( e \in A \), and

(d) \( k - |A| \) is odd.

Notice that if \( d(s) = 4 \) and \( G \) has a \( C_4-o-o \), then \( G \) is semi-brittle at \( s \) (and is brittle at \( s \) if \( |A| = k - 1 \)). This implies the following.

Lemma 6.5.4 Let \( G = (V + s, E) \) have \( s \) special, and let bipartition \( P = \{P_1, P_2\} \) with \( d_{1,G}(s, P_i) = \frac{d(s)}{2} \). If \( d_{C}(s) = 4 \), then \( G \) has a complete \( Pk \)-split if and only if there is no \( C_4-o-o \) in \( G \). \( \square \)

Also, if \( d(s) > 4 \) and \( G \) has a \( C_4-o-o \) and no shredder, any sequence of \( Pk \)-splits must end at \( d(s) = 4 \), because we will have a semi-brittle at \( s \). We will show eventually that \( G \) has a complete \( Pk \)-split if and only if \( G \) has no shredder and no \( C_4-o-o \). However, in order to do this we need to analyse how sets cross, and to be able to handle tight sets.

Lemma 6.5.5 If \( G = (V + s) \) is \( k \)-edge-connected in \( V \), and \( X, Y \subset V \) cross, then \( d(X) + d(Y) \geq 2k + 2d(s, X \cup Y) - d(s) \).
Proof: Firstly note that $d(X \cup Y) = d(V - (X \cup Y)) - d(s, V - (X \cup Y)) + d(s, X \cup Y)$ and $d(s) = d(s, X \cup Y) + d(s, V - (X \cup Y))$. Using our old friend, we see that $d(X) + d(Y) \geq d(X \cap Y) + d(X \cup Y) \geq d(X \cap Y) + d(V - (X \cup Y)) + d(s, X \cup Y) - d(s, V - (X \cup Y)) \geq k + k + 2d(s, X \cup Y) - d(s)$. \hfill \qed

We know more when the two sets are tight.

Lemma 6.5.6 Let $G = (V + s, E)$ be $k$-edge-connected in $V$, and have $s$ special. If $X, Y$ are intersecting tight subsets of $V$ and both $X - Y$ and $Y - X$ are non-empty then $d_i(X, Y) = 0$ for $i = 3, 4$ and $d(X - Y) = d(Y - X) = k$. Furthermore, if they cross, then $d_i(X, Y) = 0$ for $i = 1, 2$ and $d(X \cap Y) = d(X \cup Y) = k$.

Proof: We use Proposition 2.1.1 and the fact that $V$ is $k$-edge-connected, and get the following.
\[2k = d(X) + d(Y) \geq d(X - Y) + d(Y - X) + 2d_3(X, Y) + d_4(X, Y) \geq k + k + 0 + 0.
2k = d(X) + d(Y) \geq d(X \cap Y) + d(X \cup Y) + 2d_1(X, Y) + d_2(X, Y) \geq k + k + 0 + 0.
\]
Therefore, equality holds throughout in both, and the lemma follows. \hfill \qed

We are interested in how tight sets interact with $C_4$-odd-obstacles. In the next five lemmas, we suppose that $G = (V + s, E)$ has $s$ special, and $P = \{P_1, P_2\}$ is a bipartition with $d_i(G(s, P)) = \frac{d(s)}{2}$. Further, we suppose $C = \{A_1, A_2, B_1, B_2\}$ is a $C_4$-o-o in $G$ and $Z \subset V$ is tight, and is such that $Z \cap N(s)$ is non-empty, and contained in $P_i$ for $i = 1$ or $2$. We call such $Z$ monochrome with respect to $N(s)$.

Lemma 6.5.7 $Z$ contains at most one member of $C$.

Proof: Suppose the contrary. Then $Z$ must contain exactly two members of $C$, for if it contained three or more, it would contradict Lemma 6.3.1. By the monochromicity of $Z \cap N(s)$, it must contain a non-consecutive pair from $C$, say (wlog) $A_1$ and $A_2$. Also, since $Z$ only has $s$-neighbours of one colour, both $B_1 - Z$ and $B_2 - Z$ contain $s$-neighbours.

Let $X = V - Z = (B_1 - Z) \cup (B_2 - Z)$. Then because $Z$ contains no $s$-neighbours from $B_1 \cup B_2$ and $C$ is a $C_4$-o-o, $d(X) = d(Z) = k$. Any edge intersecting both $B_1 - Z$ and $B_2 - Z$ intersects $Z$, by the definition of a $C_4$-o-o. Suppose there are $t$ such edges. Suppose that there are $b_i$ further edges in $\Delta(B_i - Z)$ for $i = 1, 2$. The $s$-neighbours in both $B_i - Z$ ensure that $b_1 > 0$ for $i = 1, 2$. Then $k = d(X) = b_1 + b_2 + t$ and, by Lemma 6.5.6, $k = d(B_1 - Z) = b_1 + t$, contradicting the fact that $b_2 > 0$. \hfill \qed
In the next result, we use Lemma 6.5.7 on both \( Z \) and other monochrome tight sets we form as we go along.

**Lemma 6.5.8** If \( Z \) intersects \( X \in C \) it does not intersect \( Y \in C \), where \((X,Y)\) is a non-consecutive pair.

**Proof:** Suppose the contrary, and that (wlog) \( Z \) intersects both \( A_1 \) and \( A_2 \). By Lemma 6.5.7, \( Z \) at least one of \( B_1 - Z \) and \( B_2 - Z \) is non-empty, so neither \( A_1 \cup Z \) nor \( A_2 \cup Z \) equal \( V \). For \( i = 1, 2 \), if \( Z \) contains \( A_i \) then \( A_i \cup Z = Z \) and so \( d(A_i \cup Z) = k \). Otherwise \( Z \) crosses \( A_i \) and Lemma 6.5.6 implies that \( d(A_i \cup Z) = k \). That is both \( A_1 \cup Z \) and \( A_2 \cup Z \) have degree \( k \). Now, because at least one of \( B_1 - Z \) and \( B_2 - Z \) is non-empty, we have \( (A_1 \cup Z) \cup (A_2 \cup Z) = A_1 \cup A_2 \cup Z \neq V \). Also, Lemma 6.5.7 implies that at least one \( A_i - Z \) is non-empty, and hence \( A_1 \cup Z \) and \( A_2 \cup Z \) cross. Then using Lemma 6.5.6, we have \( d(A_1 \cup A_2 \cup Z) = k \) which contradicts Lemma 6.5.7. \( \square \)

When we contract a tight set, clearly, we change \( V \). We need to be able to deal with how this affects the bipartition. For this reason, here we are only considering tight sets that contain \( s \)-neighbours, and have all those \( s \)-neighbours in the same member of the bipartition. When we contract \( Z \) we replace its vertices with a new vertex \( z' \). We say that \( z' \in P_i \), where \( i \) is such that \( Z \cap N(s) \subseteq P_i \). Usually, we contract tight sets that contain \( s \)-neighbours. Later on, though, it is sometimes necessary to contract tight sets with no \( s \)-neighbours, when we are looking for \( k \)-splits. In these cases the bipartition becomes irrelevant.

**Lemma 6.5.9** If \( Z \subseteq X \in C \), then \( C' = C - X + X/Z \) is a \( C_4 \)-o-o in \( G/Z \).

**Proof:** To see this, suppose (wlog) that \( Z \subseteq A_1 \). We consider each of the properties of a \( C_4 \)-o-o. We do not change the location of \( s \)-neighbours by contracting \( Z \), so (a) holds for \( C' \) and (b) holds by Lemma 2.4.1. Also \( e \in A \) if and only if \( e' \in A' \), where \( A \) and \( A' \) are the respective edge-plates for \( C \) and \( C' \). So \( A' = A \) and (c) and (d) hold. \( \square \)

We now have the main contraction lemma.

**Lemma 6.5.10** \( G \) has a \( C_4 \)-o-o if and only if \( G/Z \) has a \( C_4 \)-o-o.

**Proof:** Throughout we deal with properties (a), (b), (c) and (d) from the definition of a \( C_4 \)-o-o. One direction is not difficult. Suppose that \( C' = \{A'_1, A'_2, B'_1, B'_2\} \) is a \( C_4 \)-o-o in \( G/Z \). Then we claim that \( C = \{A_1, A_2, B_1, B_2\} \),
where $X_i$ is the inflation of $X'_i$ is a $C_4$-o-o in $G$. Because $Z$ is monochrome with respect to $N(s)$, (a) holds for $C$, and (b) holds by Lemma 2.4.1. Suppose an edge $e$ violates (c) in $G$. Then its contraction would violate (c) for $C'$ in $G'$ so (c) holds. Finally, if $e$ is in $A$ then $e'$ is in $A'$ and vice-versa, so $a' = a$ and (d) holds for $C$ in $G$.

To see the other direction, we suppose that $C = \{A_1, A_2, B_1, B_2\}$ is a $C_4$-o-o in $G$. If $Z$ is contained in some $X \in C$, by Lemma 6.5.9, $G/Z$ has a $C_4$-o-o and we are done. Hence, from this point on, we assume that $Z$ intersects at least two elements of $C$.

By Lemma 6.5.8, $Z$ intersects both of a consecutive pair in $C$, and by monochromaticity (and indeed by Lemma 6.5.7) contains at most one of them. Without losing generality, we can say that we either have $Z$ contains $A_1$ and crosses $B_1$, or $Z$ crosses both $A_1$ and $B_1$. In both cases, we show that $C^* = \{A_1 \cup Z, A_2, B_1 - Z, B_2\}$ is a $C_4$-o-o in $G$.

Now, $Z$ and $B_1$ cross and are both tight. Therefore Lemma 6.5.6 implies $0 = d_3(Z, B_1)$ and so $Z \cap B_1$ contains no $s$-neighbours. Hence, $B_1 - Z$ contains all the $s$-neighbours from $B_1$ and so (a) holds for $C^*$. We also have $d(B_1 - Z) = k$ and either by Lemma 6.5.6 or because $Z$ contains $A_1$, we have $d(A_1 \cup Z) = k$. So (b) holds.

Next we show that $A = A^*$ where $A$ and $A^*$ are the edge-plates of $C$ and $C^*$ respectively. Suppose $e \in A^*$. Then $e$ intersects each member of $C^*$, and hence every member of $C$. So $e \in A$ and $A^* \subseteq A$. Now suppose $e \in A$. Then $e$ certainly intersects $A_1 \cup Z, A_2$ and $B_2$. If we suppose that $e$ does not intersect $B_1 - Z$, it intersects $B_1 \cap Z$. But $B_1$ and $Z$ are tight and crossing, so if $e$ intersects both their intersection and $V - (B_1 \cup Z)$, by Lemma 6.5.6 it also intersects $B_1 - Z$. Therefore $e$ intersects all members of $C$, so $e \in A^*$ and $A \subseteq A^*$. So we have both directions and $A = A^*$. Therefore (d) holds for $C^*$ immediately, because $k - a^* = k - a$ which is odd.

All that remains is to show (c). If $e$ intersects both $B_1 - Z$ and $B_2$, it intersects both $B_1$ and $B_2$. So because $C$ is a $C_4$-o-o, $e \in A = A^*$. If $e$ intersects both $A_1 \cup Z$ and $A_2$ a similar argument works, unless $e \cap (A_1 \cup Z) \subseteq (Z - A_1)$. But then $e$ intersects $B_1 \cap Z = B_1 \cap (A_1 \cup Z)$ and because $e$ intersects $A_2$, it intersects $V - (B_1 \cup (A_1 \cup Z))$. Also, $B_1$ and $A_1 \cup Z$ cross, so we can apply Lemma 6.5.6 to see that $e$ must intersect $(A_1 \cup Z) - B_1 = A_1$. Therefore, if $e$ intersects both $A_1 \cup Z$ and $A_2$, it must intersect $A_1$ and hence is in $A = A^*$. Thus, (c) holds, and $C^*$ is a $C_4$-o-o in $G$.

Moreover, $Z$ is contained in $(A_1 \cup Z) \in C^*$, and so by Lemma 6.5.9 there is a $C_4$-o-o in $G/Z$. $\square$
The last Lemma shows that $C_4-o-o$'s survive contraction. We now show that $Pk$-splits do as well.

**Lemma 6.5.11** Vertices $x, y \in N(s)$ form a $Pk$-split in $G$ if and only if they form a $Pk$-split in $G/Z$.

**Proof:** By Lemma 2.4.4, (using routing function $r \equiv k$), $x, y$ form a $k$-split in $G$ if and only if they do in $G/Z$. Therefore, because $Z$ may only contain $s$-neighbours of one colour, the lemma holds. □

### 6.6 Complete $Pk$-Splits

We have seen that in order for a hypergraph to have a complete $Pk$-split it must have a complete $k$-split - that is, it must have no shredder. Also, if a hypergraph has a $C_4-o-o$ then any sequence of $Pk$-splits must halt at a semi-brittle hypergraph with $d(s) = 4$. The main Theorem combines these two properties into an if and only if result. First though, we show that we can find the initial split. The next Lemma finds this split and is used in our algorithm.

**Lemma 6.6.1** Let $G = (V + s, E)$ be $k$-edge-connected in $V$, have $d(s) = 0$ and a bipartition $P_1, P_2$ of $V$. Let $ab, pq, uv \in E$ be such that $a, b, p, q \in P_1$ and $u, v \in P_2$. Then $G' = G - \{ab, pq, uv\} + \{sa, sb, sp, sq, su, sv\}$ has a complete $k$-split using two $Pk$-splits.

**Proof:** Consider $D_{G'}(s)$. Because $ab, pq$ and $uv$ are all edges in $G$, $D_{G'}(s) = K_{2, r}$ where $2 \leq r \leq 4$ and hence is 2-edge-connected.

**Claim 6.6.1.1** There is a $Pk$-split in $G'$.

**Proof:** Suppose not. Then by Proposition 6.2.1 $D_{G'}(s)$ is a subgraph of $B_{G'}(s)$. Thus, $B_{G'}(s)$ is 2-edge-connected and using Theorem 6.3.11 we see that, either $B_{G'}(s)$ is complete or $B_{G'}(s)$ is a cycle and $G'$ is semi-brittle at $s$. However, if $B_{G'}(s)$ was complete, there would be no $k$-split at $s$, which is a contradiction. So $B_{G'}(s)$ is a cycle. This implies that $D_{G'}(s)$ is a cycle, and so $r = 2$. Therefore $d_{1,G'}(s, a) = 2$, which contradicts the fact that $G'$ is semi-brittle at $s$. □

Suppose (wlog) that $a, u$ form the good split. Then, $d_{1,G'_{au}}(s, P_2) = 1$ and if there is a complete $k$-split from $s$ in $G'_{au}$ we are done. So we suppose there is not. Then by Theorem 6.4.2, there is a shredder $E^*$ in $G'_{au}$, such that $G'_{au} - s - E^*$
has four components, $X_1, X_2, X_3, X_4$. By Lemma 6.4.3, $a$ and $u$ are in the same component, say (wlog) $X_1$. Each of $X_2, X_3, X_4$ has just one $s$-neighbour that we will denote by $y_i$ for $i = 2, 3, 4$. Then the each split in the complete $k$-split in $G'$ consists of one vertex in $X_1$ and one $y_i$.

Suppose $x \in X_1 \cap N_{G'}(s)$ then there is $i \in \{2, 3, 4\}$ such that $x, y_i$ is a $k$-split. We claim that $x, y_j$ also form $k$-splits for $j \in \{2, 3, 4\} - \{i\}$. Suppose not, and let $Y$ be a maximal dangerous set containing $x$ and $y_j$. Then by Lemma 6.3.2, $Y$ contains $X_j$, and does not intersect $X_i$. Also, every edge in $E^*$ intersects each of $X_1, X_2, X_3, X_4$, so $\Delta_{G'}(X_i) = \Delta_{G'}(X_j) - sy_j + sy_i$. Therefore $d_{G'}(Y - X_j + X_i) = d_{G'}(Y) \leq k + 1$. That is $Y - X_j + X_i$ is a dangerous set containing $x$ and $y_i$, which is contrary to $xy_i$ being a $k$-split.

Therefore, for each $x \in X_1 \cap N_{G'}(s)$ we can choose any $s$-neighbour outside $X_1$ to form a $k$-split. Thus we can find a complete split using two $P_k$-splits. □□

We extend this idea to give the following result.

**Lemma 6.6.2** Let $G = (V + s, E)$ be a hypergraph that is $k$-edge-connected in $V$, has $s$ special, and has no shredder. Let $P = \{P_1, P_2\}$ be a bipartition of $V$ such that $d_{1,G}(s, P_1) = d_{G}(s)/2$. Then there exist $y, z \in N(s)$ such that $yz$ is a $P_k$-split and $G_{yz}$ has no shredder.

**Proof:** By Theorem 6.4.2 there is a complete $k$-split in $G$. Let $F$ be a set of size two edges resulting from such a complete split. If there is an edge $yz \in F$ where $y \in P_1$ and $z \in P_2$ (or vice versa) then the edges $sy, sz$ form a $P_k$-split in $G$ as required. So we suppose that every edge $ab \in F$ has both $a, b \in P_1$ or both $a, b \in P_2$. (This further implies that $d_{G}(s)/2$ is even and hence $d_{G}(s) \geq 8$.)

Let $G'$ be the hypergraph obtained by performing the complete $k$-split producing $F$ and let $G''$ be the hypergraph formed by “unsplitting” three edges from $F$ such that two of them were in $P_1$ and the other was in $P_2$. Then we have $d_{G''}(s) = 6$, $d_{1,G''}(s, P_1) = 4$ and $d_{1,G''}(s, P_2) = 2$ and $G''$ does have a complete $k$-split.

Then, Lemma 6.6.1 implies that there is a complete $k$-split from $s$ in $G''$ that includes two $P_k$-splits. This then generates $F'$, a new complete split from $s$ in $G$, including two $P_k$-splits. Because the order of performance of individual splits in a complete split does not matter we have that there is a $P_k$-split, $yz$, in $G$ such that $G_{yz}$ has a complete $k$-split, as required. □□

We can now prove our main splitting result.

**Theorem 6.6.3** Let $G = (V + s, E)$ have $s$ special, be $k$-edge-connected in $V$,
and let \( P \) be a bipartition of \( V \) with \( d(s, P_1) = \frac{d(s)}{2} \). Then \( G \) has a complete \( P_k \)-split if and only if it has no shredder and no \( C_4 \)-o-o.

**Proof:** One direction is not difficult. If \( G \) has a shredder, it has no complete \( k \)-split and there can be no complete \( P_k \)-split. If \( G \) has a \( C_4 \)-o-o, any sequence of \( d_G(s)/2 - 2 \) \( P_k \)-splits leaves a semi-brittle hypergraph with \( d(s) = 4 \), and so \( G \) has no complete \( P_k \)-split.

To see the other direction we suppose that \( G \) is a counterexample for which \( |V| + |E| \) is as small as possible.

**Claim 6.6.3.1** If \( X \subset V \) is tight with \( X \cap N(s) \) non-empty and contained in one \( P_i \), then \( X \) is a singleton set.

**Proof:** Suppose that \( X \) is tight, monochrome with respect to its \( s \)-neighbours, with \( |X| \geq 2 \), and let \( G' = G/X \). Then Lemma 6.4.4 implies that \( G' \) has no shredder, Lemma 6.5.10 implies that \( G' \) has no \( C_4 \)-o-o and repeated application of Lemma 6.5.11 implies that \( G' \) has no complete \( P_k \)-split. That is, \( G' \) is a smaller counterexample to the Theorem, which cannot happen.

We have already seen in Lemma 6.5.4, that the statement is true when \( d_G(s) = 4 \). Therefore, in our counterexample, \( d_G(s) \geq 6 \). Our aim is to find a \( P_k \)-split such that the resulting hypergraph (for which \( |V| + |E| \) is smaller than in \( G \)) has no \( C_4 \)-o-o and no shredder. Lemma 6.6.2 implies that there exist \( y, z \in N(s) \) that form a \( P_k \)-split in \( G \) such that \( G_{yz} \) has no shredder. If \( G_{yz} \) has no \( C_4 \)-o-o, we are done.

So we assume that \( C = \{A_1, A_2, B_1, B_2\} \) is a \( C_4 \)-o-o in \( G_{yz} \), with all the blue \( s \)-neighbours in \( A_1 \cup A_2 \) and all the red \( s \)-neighbours in \( B_1 \cup B_2 \). If we consider the degrees of the members of \( C \) there are two cases.

**CASE 1:** \( d_G(A_1) = k + 2 \) and \( d_G(A_2) = d_G(B_1) = d_G(B_2) = k \).

Then \( y \) and \( z \) are both in \( A_1 \). We assume (wlog) that \( y \) is blue and \( z \) is red. Also, Claim 6.6.3.1 implies that each of \( A_2, B_2, B_1 \) is a singleton. We shall say that \( A_2 = \{w\}, B_1 = \{u\} \) and \( B_2 = \{v\} \).

**Claim 6.6.3.2** \( zw \) is a \( P_k \)-split in \( G \).

**Proof:** The colour condition is satisfied, so we must show that \( zw \) is a \( k \)-split. We suppose the contrary and that \( Y \) is a maximal dangerous set containing \( z \) and \( w \).
We first show that $Y$ contains neither $u$ nor $v$. To see this, we suppose the contrary, and assume (wlog) that $Y$ contains $u$. Then $d(s, A_1 \cup Y) \geq d(s)/2 + 2$. Also $Y$ does not contain $v$, for if it did, it would contradict Lemma 6.3.1, and further, $y \notin Y$ because $yz$ is a $P_k$-split. Thus $A_1$ and $Y$ cross, and using Lemma 6.5.5 we get the following contradiction. $2k + 3 \geq d(A_1) + d(Y) \geq 2k + 2d(s, A_1 \cup Y) - d(s) \geq 2k + 4.$

So $Y$ does not intersect $B_1 \cup B_2$. Then every edge in $\Delta_G(w)$ is in $\Delta_G(Y)$. (Recall that $\Delta(X)$ is the set of edges intersecting both $X$ and $(V + s) - X$, and that $\delta(X)$ is the set of edges intersecting both $X$ and $V - X$. That is, $\delta(X)$ does not include edges to $s$.) Because $d_G(w) = k$, and $d_G(Y) = k + 1$, we have $\Delta_G(Y) = \Delta_G(w) \cup \{sz\}$. Therefore, every edge that intersects both $A_1 \cap Y$ and $A_1 - Y$ must be in $\delta_G(w)$. Also, because $C$ is a $C_4$-o-o and $d_G(A_1 \cap Y) \geq k$ we have $\delta_G(w) = A$ and $|A| = k - 1$. But then $G - s - A$ has at least $5$ components - because $A_1$ must fall into at least two parts. Also, because $|A| = k - 1$, $d_{1,G}(s, w) = d_{1,G}(s, u) = d_{1,G}(s, v) = 1$ and hence $d_G(s) = 6$. That is, $A$ is a shredder in $G$, which is contrary to our hypothesis.

Therefore our original assumption was incorrect and $zw$ is a $P_k$-split in $G$. □

We must now show that,

(a) there is no $C_4$-o-o in $G_{zw}$, and

(b) $G_{zw}$ has no shredder.

So, firstly, we suppose that $G_{zw}$ does have a $C_4$-o-o, $C^* = \{A_1^*, A_2^*, B_1^*, B_2^*\}$. Note that the monochromicity of the $s$-neighbours of $B_1 \cup B_2 = \{u, v\}$ imply that $u$ and $v$ are the only red $s$-neighbours in $G_{zw}$ and so we can assume (wlog), that $u$ and $v$ are in $B_1^*$ and $B_2^*$ respectively. The edge-plates of $C$ and $C^*$ are labelled $A$ and $A^*$ respectively.

Claim 6.6.3.3 $|A| < k - 1$.

Proof: We argue by contradiction, assuming that $|A| = k - 1$. Each edge in $A$ is incident with both $u \in B_1^*$ and $v \in B_2^*$ and so must also intersect both $A_1^*$ and $A_2^*$, because $C^*$ is a $C_4$-o-o. Thus every edge in $A$ is in $A^*$ and because $|A^*| \leq k - 1$ we have $A = A^*$.

Then, because $d_G(w) = k$, we have $d_{1,G}(s, w) = 1$ and $d_{1,G_{zw}}(s, w) = 0$. Then there are at least two blue $s$-neighbours in $A_1$ in $G_{zw}$, for otherwise $C^*$ could not be a $C_4$-o-o. We already have $y \in A_1$, let $x$ be another $s$-neighbour in $A_1$, in $G$. Because $|A| = k - 1$ and $x$ and $y$ are in non-consecutive elements of $C^*$, then $x$ and $y$ are in different components of $G - s - A$. Also, because
\(d_G(u) = d_G(v) = k\), we have \(d_{1,G}(s, u) = d_{1,G}(s, v) = 1\) and hence \(d_G(s) = 6\). Further, \(u, v, w\) are all in distinct components of \(G - s - A\), none of which contain \(x\) and \(y\). Thus, \(A\) is a set of \(k-1\) edges such that \(G - s - A\) has at least 5 components. That is \(A\) is a shredder in \(G\). This is contrary to our hypothesis and so \(|A| < k - 1\) as required. \(\square\)

We now look at the position of \(w\).

**Claim 6.6.3.4** \(w \notin B_1^t \cup B_2^s\).

**Proof:** We look at the edges incident with \(w\). Defining \(W_u := \{e \in E(G_{zw}) : e = uw\}\) and \(W_v, W_s\) in similar fashion, we have that \(\Delta_{G_{zw}}(w) = A + zw + W_u + W_v + W_s\). If \(W_s \neq \emptyset\), then \(w \notin B_1^t \cup B_2^s\) because it is the wrong colour. So we can assume that \(W_s\) is empty.

We now show that \(W_v\) is non-empty. To see this, we suppose the contrary. Then \(\Delta_{G_{zw}}(w) = A + zw + W_u\) and because \(d_{G_{zw}}(w) = k\) we have \(|A| + |W_u| = k - 1\). Now defining \(U_s\) (respectively \(U_A\)) as the set of edges in \(G_{zw}\) intersecting only \(u\) and \(s\) (respectively \(A\)) we have, \(k = d_{G_{zw}}(u) = |\Delta_{G_{zw}}(u)| = |A| + |W_u| + |U_s| + |U_A| \geq k - 1 + 1 + 0\). So we have equality throughout and \(|U_s| = d_{1,G_{zw}}(s, u) = 1\), \(|U_A| = 0\). Thus, \(\Delta_{G_{zw}}(\{u, w\}) = A + zw + us\) and so \(|A| + 2 \geq k\) (by the \(k\)-edge-connectivity of \(V\)). That is, \(|A| \geq k - 2\), and hence \(|A| = k - 1\), because \(k - |A|\) must be odd. This is contrary to Claim 6.6.3.3. So \(W_v\) is non-empty.

Therefore \(w \notin B_1^t\) because the edges in \(W_v\) would contradict the definition of a \(C_4\)-o-o. By a similar argument \(W_u\) is non-empty and \(w \notin B_2^s\). So \(w \notin B_1^t \cup B_2^s\) as required. \(\square\)

So \(w\) is in one of \(A_1^t, A_2^s\). We’ll assume (wlog) that \(w \in A_1^t\). Then \(z\) is also in \(A_1^t\). It cannot be in \(A_2^s\) because the edge \(zw\) would imply that \(C^*\) was not a \(C_4\)-o-o in \(G_{zw}\), and if it were in \(B_1^t \cup B_2^s\), then \(C^*\) would be a \(C_4\)-o-o in \(G\) (which has no \(C_4\)-o-o).

Because \(C^*\) is a \(C_4\)-o-o in \(G_{zw}\) we have \(d_{G_{zw}}(A_1^t) = d_{G_{zw}}(w) = k\) and \(\Delta_{G_{zw}}(w) = A + zw + W_u + W_v\) with \(W_u\) and \(W_v\) defined as above. Then \(\Delta_{G_{zw}}(w) - zw \subseteq \Delta_{G_{zw}}(A_1^t)\). Therefore, there is at most one edge intersecting only \(A_1^t - w\) and \((V + s) - A_1^t\) (for otherwise \(d(A_1^t) > k\)). Note also, that the edge \(zw\) intersects just \(w\) and \(A_1^t - w\). Therefore, there are at least \(k - 2\) edges intersecting \(A_1^t - w, w\) and \(V - A_1^t\) by the \(k\)-edge-connectivity of \(V\). That is \(|A| \geq k - 2\), and because \(k - |A|\) is odd, we have \(|A| = k - 1\), which is contrary to Claim 6.6.3.3. This contradiction means that our original assumption is false and \(G_{zw}\) has no \(C_4\)-o-o. That is, \((a)\) is true.

We must now show that \((b)\) holds, and we again argue by contradiction. Suppose that \(G_{zw}\) has a shredder, \(E^*\). Firstly we show that \(u\) and \(v\) must be in different
components of \( G_{zw} - s - E^* \). To see this, suppose that \( u, v \in X \) (where \( X \) is a component of \( G_{zw} - s - E^* \)). Let \( d = d_{G_{zw}}(s) \). Then \( d_{1, G_{zw}}(s, X) = d/2 \), and because (by Lemma 6.4.1) the other \( d/2 + 1 \) components each have an \( s \)-neighbour we have the contradiction \( d_{G_{zw}}(s) \geq d/2 + (d/2 + 1) \). So we suppose that \( u \in X_u \) and \( v \in X_v \) where \( X_u, X_v \) are components of \( G_{zw} - s - E^* \).

Recall that \( G_{yz} \) has a \( C_4 \)-o-o, and its edge plate is \( A \). Then because every element of \( A \) is incident with both \( u \) and \( v \), we have \( A \subseteq E^* \). We can go further than this.

**Claim 6.6.3.5** \(|A| = k - 1, and hence \( E^* = A \).**

(NB. Before proving this claim we point out that this might appear to be contrary to Claim 6.6.3.3. However, 6.6.3.3 was working under the assumption that \( G_{zw} \) has a \( C_4 \)-o-o, which now is not the case.)

**Proof:** There are two cases here. Firstly, that \( w \in X_u \cup X_v \). We assume (wlog) that \( w \in X_u \). Then because every edge in \( E^* \) has size at least four (by Lemma 6.4.1) there are no copies of the edge \( uv \) in \( G_{zw} \). Also, if we suppose that \( d_{1, G_{zw}}(s, w) \geq 1 \), we have \( d_{1, G_{zw}}(s, X_u \cup X_v) \geq d(s)/2 + 1 \). This and the fact that the remaining \( d(s)/2 \) components all have an \( s \)-neighbour give the contradiction that \( d_{G_{zw}}(s) \geq (d(s)/2 + 1) + d(s)/2 \). So \( d_{1, G_{zw}}(s, w) = 0 \).

Because \( d_{G_{zw}}(w) = k \) we have that \( |\Delta_{G_{zw}}(w) - zw| = k - 1 \). Also, \( \Delta_{G_{zw}}(w) - zw \) is made up of all the edges in \( A \) and copies of \( uv \). That is, \( \Delta_{G_{zw}}(w) - zw \subseteq \Delta_{G_{zw}}(u) \). So, because \( d_{G_{zw}}(u) = k \), \( d_{G_{zw}}(s, w) = 1 \) and there are no edges from \( u \) to \( A_1 \). Therefore, because \( d_{G_{zw}}(\{w, u\}) \geq k \) we have \( |A| \geq k - 2 \). Hence \(|A| = k - 1 \) because \( k - |A| \) is odd.

The second possibility is that \( w \notin X_u \cup X_v \), say \( w \in X_w \). In this case, Lemma 6.4.1 implies that there are no edges \( uw \) or \( vw \). Hence, we have \( \delta_{G_{zw}}(w) = A + zw \). Also, in \( G_{zw} \) there is at most one edge \( sw \), for otherwise \( d_{1, G_{zw}}(s, X_u \cup X_v) \geq d(s)/2 + 2 \) and the edges from \( s \) to the remaining \( d(s)/2 - 1 \) components of \( G_{zw} - s - E^* \) would again imply that \( d(s) \geq d(s) + 1 \). Hence \(|A| \geq k - 2 \) and so \(|A| = k - 1 \) as required.

Now, because \( A \subseteq E^* \) and \( |E^*| = |A| \), \( E^* = A \) as required.

Because \(|A| = k - 1 \) and \( A \) is the edge-plate of a \( C_4 \)-o-o, we have that \( w \) and \( z \) are in different components of \( G - s - A \). Also, because \( E^*(= A) \) is a shredder in \( G_{zw} \) and \( G \) has no shredder, by Lemma 6.4.3, \( z \) and \( w \) are in the same component of \( G - s - E^* = G - s - A \). This contradiction implies our supposition was incorrect, and \( G_{zw} \) has no shredder.

This completes Case 1.  

\( \square \square \)

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CASE 2: \( d_G(A_1) = d_G(A_2) = d_G(B_1) = d_G(B_2) = k. \)

Then \( y \) and \( z \) are in different elements of \( C \). Because \( C \) is not a \( C_4\)-o-\( o \) in \( G \), they must be in members with \( s \)-neighbours of the opposite to their colour. So, without losing generality we can assume that \( y \) is blue and in \( B_1 \) and that \( z \) is red and in \( A_1 \). Because \( C \) is a \( C_4\)-o-\( o \) in \( G_{yz} \), we have \( d_G(s, y) = d_G(s, z) = 1 \). By Claim 6.6.1 \( A_2 = \{w\} \) and \( B_2 = \{v\} \) are singletons.

**Claim 6.6.3.6** \( zw \) is a \( Pk \)-split in \( G \).

**Proof:** Suppose the contrary and that \( Y \) is a dangerous set containing \( z \) and \( w \). Then by Lemma 6.3.2, \( Y \) contains \( A_1 \). If \( Y \) contains neither \( u \) nor \( v \), then \( \Delta_G(w) \subseteq \Delta_G(Y) \). But, because \( y \in A_1 \) and there is also a red \( s \)-neighbour in \( A_1 \), we have \( d_G(s, A_1) \geq 2 \), implying that \( d(Y) \geq k + 2 \), contradicting the dangerousness of \( Y \). So \( Y \) does intersect \( B_1 \cup B_2 \). But then it must contain one of them, implying that \( d(s, Y) \geq d(s)/2 + 1 \), contradicting Lemma 6.3.1. \( \square \)

As before, we have to to show that

(a) there is no \( C_4\)-o-\( o \) in \( G_{zw} \), and

(b) there no shredder in \( G_{zw} \).

To see (a) we suppose the contrary, and let \( C^* \) be a \( C_4\)-o-\( o \) as before, and we assume (wlog) that \( B_1^* \cup B_2^* \) contains the red \( s \)-neighbours in \( G_{zw} \).

**Claim 6.6.3.7** \( B_1 = A_1^* \cup B_1^* \).

**Proof:** In \( G_{zw} \), \( B_1 \) is still tight, with one blue \( s \)-neighbour, \( y \), and, at least one red \( s \)-neighbour. Let \( v \) be a red \( s \)-neighbour in \( B_1 \). We can assume (wlog) that \( y \in A_1^* \) and \( v \in B_1^* \) and hence \( B_1 \cap A_1^* \) and \( B_1 \cap B_1^* \) are both non-empty. By Lemma 6.5.6, if two sets are tight and share an \( s \)-neighbour, then they do not cross. So, because \( y \in B_1 \) and \( y \notin B_1^* \), \( B_1 \) contains \( B_1^* \) and, because \( v \in B_1 \) and \( v \notin A_1^* \), \( B_1 \) contains \( A_1^* \). That is \( A_1^* \cup B_1^* \subseteq B_1 \).

Furthermore, \( B_1 \) does not intersect \( A_2^* \). Otherwise, it would cross \( A_2^* \), for to contain it would contradict Lemma 6.3.1. But then Lemma 6.5.6 would imply that \( B_1 \cup A_2^* \) is tight, and thus dangerous, and thus contrary to Lemma 6.3.1 again. The same argument can be used to show that \( B_1 \) cannot intersect \( B_2^* \). Hence \( B_1 = A_1^* \cup B_1^* \) as required. \( \square \)

Suppose that \( p = d_{G_{zw}}(A_1^*, B_2^*), q = d_{G_{zw}}(A_1^*, B_1^*), r = d_{G_{zw}}(B_1^*, A_2^*) \) and \( t = d_{1, G_{zw}}(s, B_1^*) \). We know that \( d_{1, G_{zw}}(s, A_1^*) = 1 \) because \( B_1 \) contains \( A_1^* \) and the
only blue \( s \)-neighbour in \( B_1 \) is \( y \). Lastly, let \( a = |A^*| \). Then \( k = d_{G_{zw}}(B1) = p + r + t + a + 1 = d_{G_{zw}}(A^*_1) = p + q + a + 1 = d_{G_{zw}}(B^*_1) = q + r + t + a \). From these we see that \( q = r + t \), and hence \( 2q + a = k \). Therefore \( k - a = 2q \) is even, contradicting the fact that \( C^* \) is a \( C_4\)-o-o in \( G_{zw} \). So our original assumption was false, and \( (a) \) holds. That is, there is no \( C_4\)-o-o in \( G_{zw} \).

We must now show that \((b)\) holds. We have \( d_{G_{zw}}(X) = k \) for all \( X \in C \). So consider the hypergraph \( G'_{zw} \) formed by contracting \( A_1 \) and \( B_1 \) to single vertices \( u' \) and \( z' \) respectively. Then \( G'_{zw} \) has four vertices \( u', z', w \) and \( v \), and there is a size-two edge \( z'w \). Therefore there is no shredder in \( G'_{zw} \) because removing any set of edges that all have size at least four, can leave at most three components and \( 3 < d_{G_{zw}}(s)/2 + 2 \). Therefore, by repeated application of Lemma \( 6.4.4 \), \( G_{zw} \) has no shredder.

This completes Case 2.

So, if \( G \) is a counterexample with \( |V| + |E| \) as small as possible, there is a \( Pk\)-split, such that the resulting hypergraph, \( G_1 \) say, has no \( C_4\)-o-o and no shredder. Furthermore, because \( G \) has no complete \( Pk\)-split, \( G_1 \) has no complete \( Pk\)-split. That is, \( G_1 \) contradicts the minimality of \( G \) and the theorem holds.

\[ \square \]

**Remark - What next?**

The obvious extension to the problems dealt with in this chapter, is to consider a larger partition of \( V \). We have \( G = (V + s, E) \) and partition \( P = \{P_1, P_2, \ldots, P_n\} \), and say \( u, v \in N(s) \) form a \( Pk\)-split when \( u \in P_i, v \in P_j \) and \( i \neq j \). The graph version of this problem is considered by Bang-Jensen et al. in [3], wherein the "\( C_4\)-o-o" terminology originates. When dealing with partitions having more than two members, we must change the definition of a \( C_4\)-o-o slightly, replacing \((a)\) with the following.

\[ s \text{ has exactly one neighbour in each } X \in C, \text{ and there is some } i \leq n \text{ for which } N(s) \cap (A_1 \cup A_2) = N(s) \cap P_i, \text{ or } N(s) \cap (B_1 \cup B_2) = N(s) \cap P_i \text{ and } d_1(s, P_i) = d(s)/2. \]

Bang-Jensen et al. also define a \( C_6\)-o-o. We could extend this definition to include hypergraphs as follows.

Let \( G = (V + s, E) \) be a hypergraph, let \( P = \{P_1, P_2, \ldots, P_n\} \), be a partition of \( V \) with \( 2 \leq n \leq |V| \), and suppose that \( G \) is \( k\)-edge-connected in \( V \). Let \( C = \{A_1, A_2, B_1, B_2, C_1, C_2\} \) be a partition of \( V \) and let \( A = \{e \in E : e \text{ intersects every } X \in C\} \). Then \( C \) is a **\( C_6\)-odd-obstacle** (or \( C_6\)-o-o) when it satisfies the following properties.
(a) \( d(X) = k \) for all \( X \in C \).

(b) \( d(X, Y) = (k - 1 - |A|)/2 \) for all \((X, Y) \in \{(A_1, B_1), (B_1, C_1), (C_1, A_2), (A_2, B_2), (B_2, C_2), (C_2, A_1)\}\).

(c) \( d_1(s, X) = 1 \) for all \( X \in C \).

(d) There exist distinct \( a, b, c \leq n \) such that \( N(s) \cap (A_1 \cup A_2) = N(s) \cap P_a \), \( N(s) \cap (B_1 \cup B_2) = N(s) \cap P_b \), and \( N(s) \cap (C_1 \cup C_2) = N(s) \cap P_c \).

(e) \( k - |A| \) is odd.

(The graph definitions of \( C_4 \)- and \( C_6 \)-o-o's are contained Section 1.6, in Chapter 1.)

Also, instead of us requiring that \( d_1(s, P_i) = d(s)/2 \) in the bipartition case, we change our restriction to \( d_1(s, P_i) \leq d(s)/2 \) for all \( i \leq n \). Using this terminology we propose the following conjecture.

**Conjecture 6.6.4** Let \( G + (V + s, E) \) be a hypergraph that is \( k \)-edge-connected in \( V \), and let \( P = \{P_1, \ldots, P_n\} \) be a partition of \( V \) such that \( d_1(s, P_i) \leq d(s)/2 \) for all \( i \leq n \). Then there is a complete \( P_k \)-split from \( s \) if and only if \( G \) contains no shredder, no \( C_4 \)-o-o and no \( C_6 \)-o-o. \( \Box \)

This leads to the conjecture closing the next chapter, that considers augmenting a hypergraph \( H = (V, E) \) with respect to any partition of \( V \), with at least two members.
Chapter 7

Bipartition Constrained $k$-Augmentation

7.1 Introduction and Notation

In this chapter, we consider the following problem. We are given a hypergraph, $H = (V,E)$ with a bipartition $P = \{P_1, P_2\}$ of $V$, and an integer $k \geq 2$. We wish to $P_k$-augment $H$ with size-two edges. That is, add a set $F$ of new size-two edges, forming $H^+ = (V,E \cup F)$ such that

(a) every edge in $F$ has one end in $P_1$ and the other in $P_2$, and

(b) $H^+$ is $k$-edge-connected.

We define $\theta_{P_k}(H)$ to be the smallest number of size-two edges required to $P_k$-augment $H$. We use $\omega(H)$ to denote the number of components in any hypergraph $H$. For $H = (V,E)$ with bipartition $P = \{P_1, P_2\}$ and integer $k \geq 2$, we define the following.

\[
\alpha = \max\{\left\lceil \sum_{X \in S} (k - d(X)) / 2 \right\rceil : S \text{ is a subpartition of } V\}; \\
\beta_i = \max\{\sum_{Y \in S_j} (k - d(Y)) : S_j \text{ is a subpartition of } P_i\} \text{ for } i = 1, 2; \\
\gamma = \max\{\omega(H - A) : A \subseteq E, |A| = k - 1\} - 1; \\
\Phi = \max\{\alpha(H), \beta_1(H), \beta_2(H), \gamma(H)\}; \\
\Phi' = \max\{\alpha(H), \beta_1(H), \beta_2(H)\}.
\]
The following result is straightforward.

**Lemma 7.1.1** Let $H = (V, E)$ be a hypergraph and $P = \{P_1, P_2\}$ be a bipartition of $V$. Then $\theta_{P_k}(H) \geq \Phi$. 

In this chapter, we show that this lower bound is often achievable and characterise the situations when this is not so. The augmentation process, as before, follows Frank’s line (see [22]) - add a vertex $s$, add edges from $s$ into $V$, and then perform $P_k$-splits at $s$. Using the results of the previous chapter, we find $\theta_{P_k}(H)$ for all $H$, and describe a polynomial algorithm to produce a minimal $P_k$-augmentation.

### 7.2 Initialising the Hypergraph

We are given $H = (V, E)$, a bipartition, $P = \{P_1, P_2\}$, of $V$ and an integer $k \geq 2$. We use the following process to add the special vertex $s$. It is the same as that used for graphs by Bang-Jensen *et al.* in [3].

**PROCESS: $P_k$-ADD-$s$**

**Step 1:** Add a vertex $s$ and $k$ size-two edges from $s$ to every vertex in $V$. Then delete these edges one by one until no further edge to $s$ can be removed without destroying the $k$-edge-connectivity of $V$. If at this point $d(s)$ is odd, add one edge from $s$ to an arbitrary vertex in $V$.  

**Step 2:** If $d_1(s, P_1) \neq d(s)/2$, we suppose that the labelling has been chosen to make $d_1(s, P_1) > d_1(s, P_2)$, and repeat the following until we have equality.

(a) For each edge $su$, let $X_u$ be a minimal tight set containing $u$.

(b) If $X_u \nsubseteq P_1$ for some $u \in N(s) \cap P_1$, then let $v$ be a vertex in $X_u - P_1$ and remove the edge $su$ and add an edge $sv$. If we now have $d_1(s, P_1) = d_1(s, P_2) = d(s)/2$ we stop. Otherwise we return to (a).

(c) We now have $X_u \subseteq P_1$ for every edge $su$ with $u \in P_1$, and $d_1(s, P_1)$ is still larger than $d_1(s, P_2)$. Then add $d_1(s, P_1) - d_1(s, P_2)$ parallel edges $sx$ where $x \in P_2$.

Before showing that this process produces a useful hypergraph, we prove the following result, that we use again in our algorithm.
Lemma 7.2.1 Let \( G = (V + s, E) \) be a hypergraph with \( s \) special and \( k \)-edge-connected in \( V \). Let \( Z \subset V \) be a minimal (inclusion) tight set, and \( z \in Z \) be an \( s \)-neighbour. For a vertex \( z' \in Z \), the hypergraph \( G' = G - sz + sz' \) is \( k \)-edge-connected in \( V \).

Proof: Suppose the contrary. Then there is a tight set, \( Z' \), containing \( z' \) but not \( z \), and hence \( Z - Z' \) is non-empty. The minimality of \( Z \) implies that \( Z' - Z \) is non-empty. Then, by Lemma 6.5.6, \( Z - Z' \) is tight, contradicting the minimality of \( Z \). \( \square \)

The next Lemma shows that the hypergraph we form by running \( Pk\text{-ADD-}s \) has the properties we need later on.

Lemma 7.2.2 Let \( H = (V, E) \) be a hypergraph and \( P = \{P_1, P_2\} \) be a partition of \( V \). Then by running \( Pk\text{-ADD-}s \), we can form a hypergraph \( G = (V + s, E \cup E(s)) \) that is \( k \)-edge-connected in \( V \) and has \( d_{G}(s)/2 = d_{1,G}(s, P_i) = \Phi' \).

Proof: Run \( Pk\text{-ADD-}s \) on \( H \). Clearly the hypergraph formed at the completion of Step 1 is \( k \)-edge-connected in \( V \). Lemma 7.2.1 implies that if \( Pk\text{-ADD-}s \) stops before reaching \( 2(c) \), the graph obtained is \( k \)-edge-connected in \( V \). We can go further and say the following.

Claim 7.2.2.1 If we stop before reaching \( 2(c) \), then \( \Phi' = \alpha = d(s)/2 \).

Proof: At the end of Step 1, \( d(s)/2 \geq \alpha \) and Bang-Jensen and Jackson proved in [1] that in fact equality holds. If Step 2 stops before reaching \( 2(c) \), the degree of \( s \) is not different to its value after Step 1, so we still have \( d(s)/2 = \alpha \). We must show that in this case, \( \alpha \geq \beta_i \) for \( i = 1, 2 \).

Let \( S_i \) be the subpartition of \( P_i \) that determines \( \beta_i \), for each \( i = 1, 2 \). Then, in order to ensure that \( G \) is \( k \)-edge-connected in each \( P_i \), we must have at least \( \beta_i \) edges from \( s \) into \( P_i \). Therefore, \( d_1(s, P_i) \geq \beta_i \). Also, we have stopped because \( d_1(s, P_1) = d_1(s, P_2) = d(s)/2 = \alpha \). So we have \( \beta_i \leq d_1(s, P_i) = d(s)/2 = \alpha \), for both \( i = 1, 2 \). Hence, \( \Phi' = \alpha = d(s)/2 \). \( \square \)

Therefore, if \( Pk\text{-ADD-}s \) stops before reaching \( 2(c) \) we have created the required hypergraph. If we have to proceed to \( 2(c) \), we do not damage the \( k \)-edge-connectivity in \( V \), because we add edges.

Claim 7.2.2.2 If we stop after \( 2(c) \), then \( \Phi' = \beta_1 = d(s)/2 \).
Proof: At the beginning of Step 2 we chose the labelling of the partition so that \( d_1(s, P_1) > d_1(s, P_2) \). The \( k \)-edge-connectivity of \( V \) implies that \( d_1(s, P_1) \geq \beta_1 \). Hence, \( \beta_1 > \beta_2 \). So to see that \( \Phi' = \beta_1 \) we must show that \( \beta_1 > \alpha \).

Before commencing 2(c), we have \( d(s)/2 = \alpha, d_1(s, P_1) > d_1(s, P_2) \) and every \( X_u \subseteq P_1 \), where \( X_u \) is a minimal tight set containing \( u \in N(s) \cap P_1 \).

Let \( S_1 = \{ X_u : u \in N(s) \cap P_1 \} \) and \( S'_1 \) be the smallest subset of \( S_1 \) that covers \( N(s) \cap P_1 \). We shall show that \( S'_1 \) is a subpartition of \( P_1 \). Let \( X_u, X_v \in S_1 \) with \( X_u \cap X_v \neq \emptyset \). Then \( X_u \subseteq X_v \) or vice versa, for otherwise both \( X_u - X_v \) and \( X_v - X_u \) are non-empty. Then by Lemma 6.5.6, \( d_3(X_u, X_v) = 0 \) and \( X_u - X_v \) and \( X_v - X_u \) are both tight which contradicts the minimality of \( X_u \) and \( X_v \).

From this we deduce that \( S'_1 \) is a subpartition of \( Y \), for otherwise there is a pair \( X_u, X_v \in S'_1 \) which intersect. Hence \( X_u \subseteq X_v \) (or vice versa) and \( S'_1 - X_u \) (or \( S'_1 - X_v \)) contradict the minimality of \( S'_1 \).

Therefore, \( d_1(s, P_1) \leq \beta_1 \). Also, because \( V \) is \( k \)-edge-connected, \( d_1(s, P_1) \geq \beta_1 \) and so we have equality. Therefore, we have \( \alpha < d_1(s, P_1) = \beta_1 \) and so \( \Phi' = \beta_1 \).

Also, after running 2(c) we have \( d(s) = 2\beta_1 \) and the claim holds. \( \square \)

We have shown that whenever \( Pk-ADD-s \) stops, we have a hypergraph \( G = (V + s, E \cup E(s)) \) such that \( d(s)/2 = \Phi' \). Furthermore, the process only stops when \( d(s)/2 = d_1(s, P_1) \) as required. \( \square \square \)

7.3 \( C_4 \)-odd-configurations

The next stage in our augmentation process is to find \( Pk \)-splits at \( s \). If there is a complete \( Pk \)-split at \( s \), it generates an augmenting set with \( \Phi' \) edges. If there is not, then by Theorem 6.6.3 either \( G \) has a shredder or a \( C_4-o-o \). We firstly consider the structure in \( H \) which might force a \( C_4-o-o \) in \( G \).

Let \( H = (V, E) \) be a hypergraph. Let \( C = \{ A_1, A_2, B_1, B_2 \} \) be a partition of \( V \), and let \( A = \{ e \in E : e \) intersects every \( X \in C \} \) be its edge-plate. Then \( C \) is a \( C_4 \)-odd-configuration (or \( C_4-o-c \) for short), when it has the following properties.

(a) \( d(X) < k \) for all \( X \in C \),

(b) each \( X \in C \) has a subpartition \( S(X) \) such that \( k - d(X) = \sum_{U \in S(X)} (k - d(U)) \) and for at least one \( i = 1, 2 \), \( P_i \) contains all the sets in \( S(A_1) \cup S(A_2) \) or all the sets in \( S(B_1) \cup S(B_2) \),

(c) \( k - d(A_1) + k - d(A_2) = k - d(B_1) + k - d(B_2) = \Phi' \),

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(d) if an edge, e intersects both $A_1$ and $A_2$, or both $B_1$ and $B_2$, then $e \in A$, and

(e) $k - |A|$ is odd.

**Lemma 7.3.1** If $H = (V, E)$ has a $C_4$-o-c, then $\theta_{P_k}(H) \geq \Phi' + 1$.

**Proof:** We argue by contradiction. Lemma 7.1.1 implies that $\theta_{P_k}(H) \geq \Phi'$, so we suppose that we can $P_k$-augment $H$ with a set, $E_{new}$, of $\Phi'$ size-two edges. Let $H' = H + E_{new}$.

We assume (wlog) that $P_1$ contains all the sets in $S(A_1) \cup S(A_2)$. Then, because $k-d(A_1)+k-d(A_2) = \Phi'$, every edge $xy \in E_{new}$ has $x \in A_1 \cup A_2$ and $y \in B_1 \cup B_2$ (or vice versa). This and the definition of a $C_4$-o-c imply that in $H'$ every edge is either in $A$ or is contained in some $A_i \cup B_j$. For each $i, j = 1, 2$, let $t_{ij}$ be the number of edges in $H'$ contained in $A_i \cup B_j$. Then $k = d_{H'}(A_1) = |A_1| + t_{11} + t_{12} = d_{H'}(B_1) = |B_1| + t_{11} + t_{21}$ and $t_{11} = t_{22}$ and $t_{12} = t_{21}$. Also, if $t_{11} > t_{12}$, then $d_{H'}(A_2 \cup B_2) = |A_1| + t_{12} + t_{21} < |A_1| + t_{11} + t_{21} = d_{H'}(B_2) = k$ which is a contradiction. The same argument shows $t_{12} \neq t_{11}$, so equality holds. Therefore $k - |A| = 2t_{11}$, which is contrary to the definition of a $C_4$-o-c. \[\Box\]

Sometimes, running $P_k$-ADD-$s$ results in a hypergraph with a $C_4$-o-o that is not a $C_4$-o-c in $H$. The next Lemma describes the hypergraphs for which this is so.

**Lemma 7.3.2** Let $G = (V + s, E \cup E(s))$, for each $u \in N(s)$, let $X_u$ be the minimal tight set containing $u$, and let $C$ be a $C_4$-o-o in $G$. Then $C$ is not a $C_4$-o-c in $G$ if and only if there exists $u \in N_{G}(s) \cap (A_1 \cup A_2)$ with $X_u \not\subseteq P_1$ and $v \in N_{G}(s) \cap (B_1 \cup B_2)$ with $X_v \not\subseteq P_2$.

**Proof:** All we must check is the subpartition condition (b) from the definition of a $C_4$-o-c, as the other properties hold for $C$, by the definition of a $C_4$-o-o.

Firstly we show that for each $Y \subseteq C$, if $u \in N(s) \cap Y$ then $X_u \subseteq Y$. To see this, suppose the contrary. Then $X_u - Y$ is non-empty, and $d_3(X_u, Y) \geq 1$. So by Lemma 6.5.6, $Y - X_u$ is empty and hence $Y$ contradicts the minimality of $X_u$. Now let $S(Y) = \{X_u : u \in N(s) \cap Y\}$ and let $S_1(Y)$ be the smallest subset of $S(Y)$ that covers $N(s) \cap Y$.

We shall show that $S_1(Y)$ is a subpartition of $Y$. Let $X_u, X_v \in S(Y)$ with $X_u \cap X_v \neq \emptyset$. Then $X_u \subseteq X_v$ or vice versa, for otherwise both $X_u - X_v$ and $X_v - X_u$ are non-empty. Then by Lemma 6.5.6, $d_3(X_u, X_v) = 0$ and $X_u - X_v$ and $X_v - X_u$ are both tight which contradicts the minimality of $X_u$ and $X_v$.  

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From this we deduce that $S_1(Y)$ is a subpartition of $Y$, for otherwise there is a pair $X_u, X_v \in S_1(Y)$ which intersect. Hence $X_u \subseteq X_v$ (or vice versa) and $S_1(Y) - X_u$ (or $S_1(Y) - X_v$) contradict the minimality of $S_1(Y)$.

Now, if $X_u \subseteq P_1$ for all $u \in N_G(s) \cap (A_1 \cup A_2)$, (or if $X_v \subseteq P_2$ for all $v \in N_G(s) \cap (B_1 \cup B_2)$, respectively) then the subpartitions $S_1(A_1)$ and $S_1(A_2)$ (respectively $S_1(B_1)$ and $S_2(B_2)$) satisfy property $(b)$ and $C$ is $C_4$-o-c in $G$.

We now show that if there exist $u \in N_G(s) \cap (A_1 \cup A_2)$ with $X_u \not\subseteq P_1$ and $v \in N_G(s) \cap (B_1 \cup B_2)$ with $X_v \not\subseteq P_2$, then there can be no subpartition satisfying $(b)$ and deduce that $C$ is not a $C_4$-o-c in $G$. We do this by showing the following.

**Claim 7.3.2.1** For each $Y \in C$, and for every subpartition $S_2(Y)$, such that $k - d_H(Y) = \sum_{U \in S_2(Y)} (k - d_H(U))$, each $U \in S_2(Y)$ contains at least one $X_u$.

**Further, every $X_u$ is contained in some $U \in S_2(Y)$.

**Proof:** Let $S_2(Y)$ be a subpartition of $Y \in C$. We first note that for any such subpartition, $k - d_H(Y) \geq \sum_{U \in S_2(Y)} (k - d_H(U))$, because $d_G(Y) = k$ for all $Y \in C$. We now suppose that equality holds. Then each $U \in S_2$ is tight in $G$ and contains at least one $s$-neighbour, because in forming $G$ we must add exactly $k - d_H(U)$ edges to each $U$. This means that no $X_u$ crosses a member of $S_2(Y)$, for otherwise, by Lemma 6.5.6, $X_u \cap U$ would be tight, contradicting the minimality of $X_u$.

Now, suppose that $u \in N(s) \cap Y$ such that $u \not\in \bigcup_{U \in S_2(Y)} U$. Then $u$ lies in a minimal tight set $X_u$ which does not cross any member of $S_2$. But then the subpartition $S_2(Y) = S_2(Y) + X_u$ is contrary to $k - d_H(Y) = \sum_{U \in S_2(Y)} (k - d_H(U))$. Hence, every $U \in S_2(Y)$ contains at least one $X_u$, and every $X_u$ is covered.

So, if there exist $u$ and $v$ as in the statement of the theorem, there can be no subpartitions satisfying $C_4$-o-c $(b)$ and the Lemma holds.

When $G$ has a $C_4$-o-o that is not a $C_4$-o-c in $H$, we can swap edges to $s$, in such a way as to maintain both $k$-edge-connectivity in $V$ and the partition constraint, and produce a hypergraph with no $C_4$-o-o and no shredder.

**Lemma 7.3.3** Let $H = (V,E)$ be a hypergraph and $G$ be the result of running $P_k$-ADD-s on $H$. Suppose $G$ has no shredder and that $C$ is a $C_4$-o-o in $G$, that is not a $C_4$-o-c in $H$. Then there exist $u \in N_G(s) \cap P_1$ and $v \in N_G(s) \cap P_2$ such that $X_u \not\subseteq P_1$ and $X_v \not\subseteq P_2$. Furthermore, if $u' \in X_u \cap P_2$ and $v' \in X_v \cap P_2$, then the hypergraph $G^* = G - \{su, sv\} + \{su', sv'\}$ is $k$-edge-connected in $V$, has no shredder and no $C_4$-o-o.
Proof: Let $C = \{A_1, A_2, B_1, B_2\}$ be a $C_4$-o-o in $G$, and assume that the
labelling is chosen such that every $s$-neighbour in $A_1 \cup A_2$ is in $P_1$, and every
$s$-neighbour in $B_1 \cup B_2$ is in $P_2$.

Lemma 7.3.2 implies the existence of $u$ and $v$. By Lemma 7.2.1, $G^* = G - \{su, sw\} + \{sv, sx\}$ is $k$-edge-connected and because we’ve not changed the
degree of $s$, $G^*$ has no shredder.

We show, by contradiction, that $G^*$ has no $C_4$-o-o. So, suppose $C^* = \{A_1^*, A_2^*, B_1^*, B_2^*\}$ is a $C_4$-o-o in $G^*$, with $N(s) \cap P_1 \subseteq A_1^* \cup A_2^*$. In the following Claim, we
show that the existence of such a $C_4$-o-o implies that $A_1$ and $B_1$ each contain
only one $s$-neighbour and hence that $A_1$ and $B_1$ are monochromatic with respect
to $s$ in both $G$ and $G^*$.

Claim 7.3.3.1 $d_{1,G^*}(s, A_1) = d_{1,G^*}(s, B_1) = 1$.

Proof: Suppose not, and that (wlog) $d_{1,G^*}(s, A_1) \geq 2$. Then $A_1$ contains one
$P_2$-$s$-neighbour, $v$, and at least one $P_1$-$s$-neighbour. So we can assume (wlog)
that both $A_1^*$ and $B_1^*$ intersect $A_1$. Then Lemma 6.5.6 implies that neither
set crosses $A_1$, and hence $A_1^* \cup B_1^* \subseteq A_1$. Furthermore, $A_1$ does not intersect
either of $A_2^*, B_2^*$, for otherwise Lemma 6.5.6 implies that $d_{G^*}(A_1 \cup X) = k,$
$(X \in \{A_2^*, B_2^*\})$ which contradicts Lemma 6.3.1. So $A_1 = A_1^* \cup B_1^*$. Then
$A_1^*$ contains all the $s$-neighbours from $A_2$ and the single $P_1$-$s$-neighbour in $B_1$.
So $A_2^*$ contains both $A_2$ and $B_1$, by Lemma 6.5.6. We have already seen that
$A_2 \cap A_1$ is empty, and furthermore $A_2^*$ cannot intersect $B_2$, for then $A_2^* \cup B_2$
would contradict Lemma 6.3.1. So $A_2^* = A_2 \cup B_1$. Then, recalling that $A$ is the
de-edge-plate of $C$ and if $m$ is the number of edges intersecting both of $B_1$, $A_2$ and
contained in $A_2 \cup B_1$, we have that

$$k = d_{G^*}(A_2^*) = d_{G^*}(A_2 \cup B_1) = d_{G^*}(A_2) + d_{G^*}(B_1) - |A| - 2m = 2k - |A| - 2m.$$ 

Therefore $k - |A| = 2m$, is even, contradicting the definition of a $C_4$-o-o. \qed

This implies that in $G^*$, each $X \in C$ is tight and is monochromatic with
respect to its $s$-neighbours. So we can contract each in turn, forming $G^{**}$, a
hypergraph with four vertices, $a_1, a_2, b_1, b_2$ formed by contracting $A_1, A_2, B_1, B_2$
respectively. In $G^{**}$ we have that $a_1, b_2 \in P_2$ and $a_2, b_1 \in P_1$.

Now, Lemma 6.4.4 implies that $G^{**}$ has no shredder and Lemma 6.5.10 implies
that $G^*$ has a $C_4$-o-o, say $C^{**} = \{A_1^{**}, A_2^{**}, B_1^{**}, B_2^{**}\}$. Then we can choose
the labelling such that $B_1^{**} = \{a_1\}, B_2^{**} = \{b_2\}, A_1^{**} = \{b_1\}$ and $A_2^{**} = \{a_2\}.$

The definition of a $C_4$-o-o implies that there are no size-three edges in $G^{**}$,
and no edges $a_1b_2, b_1a_2$. Then because $d_{1,G^{**}}(s, a_1) = d_{1,G^{**}}(s, b_1) = 1$, and
$d_{G^{**}}(\{a_1, b_1\}) \geq k$ we have $|A| \geq k - 2$. Therefore, because $k - |A|$ is odd, $|A| = k - 1$ and $A$ is a shredder in $G^{**}$, which is a contradiction.

Thus, our original assumption was false and $G^*$ has no $C_4$-$o$-$o$, as required. □□

7.4 The Augmentation Result

We can now prove the main result of this chapter, which solves the bipartition constrained $k$-augmentation problem for hypergraphs.

**Theorem 7.4.1** Let $H = (V, E)$ be a hypergraph, $P = \{P_1, P_2\}$ be a bipartition of $V$, and $k \geq 2$ be an integer. Then $\theta_{P_k}(H) = \Phi$ unless $\Phi = \Phi'$ and $H$ contains a $C_4$-$o$-$c$, in which case $\theta_{P_k}(H) = \Phi + 1$.

**Proof:** We have seen (Lemma 7.1.1) that $\theta_{P_k}(H) \geq \Phi$ and Lemma 7.3.1 implies that if $\Phi = \Phi'$ and $H$ has a $C_4$-$o$-$c$, $\theta_{P_k}(H) \geq \Phi' + 1 = \Phi + 1$.

Lemma 7.2.2 implies that given $H$ we can create $G = (V + s, E \cup E(s))$, that is $k$-edge-connected in $V$ and has $d_G(s) = 2\Phi'$.

**Claim 7.4.1.1** If, $\Phi \neq \Phi'$, then $G$ has a shredder and we can $P_k$-augment $H$ with $\Phi$ edges.

**Proof:** If $\Phi \neq \Phi'$ then $\Phi = \gamma$ and hence there is a set of edges $E^*$ such that $H - E^*$ has $\gamma + 1$ components. Therefore $\omega(G - s - E^*) = \gamma + 1 > \Phi' + 2 = d_G(s)/2 + 2$ and hence, $E^*$ is a shredder in $G$. Now form $G^*$ by adding $\Phi - \Phi'$ edges from $s$ to a vertex in $P_1$ and $\Phi - \Phi'$ edges from $s$ to a vertex in $P_2$. Then $d_{G^*}(s) = 2\Phi = 2\gamma$.

Then for all $A \subset E$ with $|A| = k - 1$, $\omega(H - A) \leq \gamma + 1 = d_{G^*}(s)/2 + 1$. So there is no shredder in $G^*$. Also, there is no $C_4$-$o$-$o$ in $G^*$. (To see this, recall that we have added at least 2 edges from $s$ into $V$. Then, if $C$ were a $C_4$-$o$-$o$ in $G^*$, some $X \in C$ would have $d_{G^*}(X) < k$ which cannot happen.) Therefore, by Theorem 6.6.3, there is a complete $P_k$-split in $G$, which generates a $P_k$-augmentation of $H$, with $\Phi$ edges.

Now suppose that $\Phi = \Phi'$. Then $\gamma \leq \Phi'$ and so for all $A \subset E$ with $|A| = k - 1$ we have $\omega(H - A) \leq \Phi' + 1 = d_G(s)/2 + 1$ and hence $\omega(G - s - A) \leq d_G(s)/2 + 1$. That is, there is no shredder in $G$. Furthermore, if $G$ has no $C_4$-$o$-$o$ then by Theorem 6.6.3, there is a complete $P_k$-split from $s$ in $G$, which generates a $P_k$-augmentation of $H$, with $\Phi$ edges.

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So we now suppose that $C$ is a $C_{4-o-o}$ in $G$. If $C$ is not a $C_{4-o-o}$ in $H$, then by Lemma 7.3.3 we can form $G'$ with $d_G(s) = 2\Phi$, so that $G'$ has no shredder and no $C_{4-o-o}$. Then Theorem 6.6.3, implies that there is a complete $P_k$-split from $s$ in $G'$, generating a $P_k$-augmentation of $H$, with $\Phi$ edges.

(We point out that because, in these cases, we have a good augmentation with $\Phi' = \Phi$ edges, Lemma 7.3.1 implies that there is no $C_{4-o-o}$ in $H$.)

If $C$ is a $C_{4-o-o}$ in $H$, then Lemma 7.3.1 implies that any $P_k$-augmentation has at least $\Phi' + 1 = \Phi + 1$ edges. Let $G''$ be formed by adding two edges $sx, sy$, where $x \in P_1$ and $y \in P_2$. Then $G''$ has no shredder and no $C_{4-o-o}$ (for otherwise some $X''$ in the $C_{4-o-o}$, $C''$, would have $d_G(X) < k$). So there is a complete $P_k$-split in $G''$ which generates a $P_k$-augmentation of $H$ with $\Phi + 1$ edges.

7.5 An Algorithm for $P_k$-Augmentation

To close this chapter, we present an algorithm that runs on a starting hypergraph $H = (V, E)$ that has bipartition $P = \{P_1, P_2\}$, adds size-two edges and outputs a hypergraph that is $k$-edge-connected in $V$ and has no new edge contained in just one $P_i$. The process is based around the polynomial algorithm of Bang-Jensen and Jackson [1], for the unconstrained $k$-augmentation problem. We point out here that provided all basic arithmetic operations are taken as constant time operations, the complexity of their algorithm only depends on the number of edges and vertices in $H$. (In particular, it is independent of $k$.)

The algorithm presented below is essentially an adaptation of Bang-Jensen and Jackson’s process from [1]. Our goal is simply to show the existence of a polynomial algorithm for $P_k$-augmentation. We shall not repeat all of the intricacies provided in [1] but we will try to provide enough detail to see how such an algorithm would run and that it is indeed of polynomial complexity when we take all basic arithmetic operations as constant time operations. We acknowledge

The main tool in running the algorithm is the efficient checking of $\lambda(x, y)$ for all pairs of vertices $x, y \in V$. In order to do this, Bang-Jensen and Jackson store the given hypergraph as a simple hypergraph with a capacity function to indicate the multiplicity of the edges, and to then form a flow network based on this simple capacitated hypergraph. Using this technique, and well known results about calculating the size of a maximum flow in a network, (see [7],[15]), Bang-Jensen and Jackson showed the following in [1].

Lemma 7.5.1 (Bang-Jensen and Jackson) For any hypergraph $H = (V, E)$ and any pair of distinct vertices, $x, y \in V$, we can find $\lambda_H(x, y)$ in time
\(O((|V| + \epsilon(E))^3)\), where \(\epsilon(E)\) is the number of edges in the simple hypergraph, underlying \(H\).

In particular, we can find \(\lambda(x, y)\) across the whole hypergraph in time complexity \(O(|V|^2(|V| + \epsilon(E))^3)\), since there are \(O(|V|^2)\) pairs of vertices. This allows us, for instance, to check whether or not a split is a \(k\)-split in time \(O(|V|^3(|V| + \epsilon(E))^3)\), by simply performing the split, finding \(\lambda\) for all pairs in \(V\) and comparing (in constant time) each \(\lambda\) value with \(k\).

We present the algorithm in stages.

The Initialisation Stage

Given our starting hypergraph \(H = (V, E)\), we start by running \(Pk\)-ADD-s on \(H\). From this we obtain \(G = (V + s, E \cup E(s))\) with \(d_G(s) = 2\Phi'\) by Lemma 7.2.2.

Step 1 of \(Pk\)-ADD-s was shown to be efficient by Bang-Jensen and Jackson. To see that Step 2 is also, we show the following. We point out that when dealing with the capacitated flow network, we have \(|E(s)| \leq |V|\), and so our complexity bounds continue to use \((|V| + \epsilon(E))^3\).

**Lemma 7.5.2** In \(G = (V + s, E \cup E(s))\), for every \(u \in N(s)\), we can find the minimal tight set, \(X_u\), containing \(u\) in time \(O(|V|(|V| + \epsilon(E))^3)\).

**Proof:** For \(u \in N_G(s)\), set \(T := \{u\} \cup \{v \in V - u : \lambda_G(u, v) > k\}\). We can find this set in time \(O(|V|(|V| + \epsilon(E))^3)\). We show that \(T = X_u\). We begin, by pointing out that for every \(v \in X_u - u\), every set separating \(u\) and \(v\), has degree at least \(k + 1\). Therefore, for every tight set, \(X\), containing \(u\), we have \(T \subseteq X\). In particular, \(T \subseteq X_u\).

We suppose that \(T \neq X_u\). Then there is \(x \in X_u\) such that \(x \notin T\). Therefore, \(\lambda(u, x) = k\), and there is a set \(X\) separating \(u\) and \(x\), such that \(d(X) = k\). We can assume (by changing \(X\) for \(V - X\) if necessary) that \(u \in X\) and \(x \notin X\), and hence \(X_u - X\) is non-empty. If \(X - X_u\) is also non-empty, by Lemma 6.5.6, \(X_u - X\) is tight, contradicting the minimality of \(X_u\). If \(X - X_u\) is empty, then \(X \subset X_u\), contradicting the minimality of \(X_u\). Therefore, \(T = X_u\), as required.  

The Splitting and Unshredding Stage

This stage takes the initialised hypergraph \(G = (V + s, E \cup E(s))\), and outputs a hypergraph, \(G_1 = (V + s, E \cup F)\), that has \(d_{G_1}(s) = 0\).

We start by using the **Splitting step**, the **Backtracking Step** and the **Forward Step** from Bang-Jensen and Jackson’s \(k\)-augmentation algorithm on our hypergraph
G = (V + s, E ∪ E(s)). We summarise the details as the process k-SPLIT. This process will either perform a complete k-splitting from s, and G₁ is the hypergraph formed in this way. Otherwise, k-SPLIT outputs the components of a shredder in G. Pk-UNSHRED then adds edges to s, and we re-run k-SPLIT to produce the output (completely split) hypergraph G₁.

**Process: k-SPLIT**

**Step 1:** Form a sequence of hypergraphs \( G = G₀, G₁, \ldots, Gᵣ \) by performing k-splits, until either \( d_{Gᵣ}(s) = 0 \), or there is no further k-split and \( d_{Gᵣ}(s) ≥ 4 \). In the first case we stop, outputting \( G₁ = (V + s, E ∪ F) \) where \( F \) is the set of new edges generated by the splits.

**Step 2:** Because there is no k-split, \( Gᵣ \) is brittle, by Theorem 6.3.9. Find the \( d_{Gᵣ}(s) \) components of \( Gᵣ − s - A \), for some set \( A ⊆ E \) of \( k - 1 \) edges.

**Step 3:** Unsplit the new edges one by one, that is, backtrack through the sequence \( Gᵣ, Gᵣ − 1, \ldots, G₀ \), until either we reach \( G₀ \) or we find the largest \( i \) such that \( ω(Gᵢ − s − A) = ω(Gᵢ₊₁ − s − A) \). In the first case we stop, outputting the partition \( \{X₁, X₂, \ldots Xᵣ\} \) of \( V \) where each \( Xᵢ \) is a component of \( G − s − A \) and \( t ≥ d_G(s)/2 + 2 \).

**Step 4:** Form a new sequence of hypergraphs, \( Gᵢ = Gᵢ', Gᵢ₊₁', \ldots, Gᵣ' \), such that each \( Gᵢ₊₁' \) is obtained by performing a k-split from s in \( Gᵢ' \), using an edge \( sw \), where \( w \) is the unique s-neighbour in some component of \( Gᵢ' − s − A \) and \( q \) is as large as possible. This will either produce a complete k-split, or we reach \( Gᵣ' \) with \( q > r \), \( d_{Gᵣ}(s) ≥ 4 \) and no further k-split. In the first case we stop, outputting \( G₁ = (V + s, E ∪ F) \) where \( F \) is the set of new edges generated by the splits. In the second we return to Step 2.

We point out, here, that we have omitted how to find both the splits and (when necessary) the components of the shredder. The details are contained in [1] along with proof that this process runs in time \( O(|V|^2(|V| + ϵ(E))^2) \).

k-SPLIT outputs either a set of new edges \( F \) formed by a complete k-split from s, or the components of a shredder. If the former, we proceed to the next stage, which alters this set of new edges (whilst staying inside the vertex set induced by \( F \)) to ensure that we have as many Pk-edges as possible. If the latter, we follow the line of Bang-Jensen and Jackson, and add new edges from s to V in G. However, in this case we must consider the bipartition constraint. Although only one step, we state it in a similar manner to our other sub-routines.

**Process: Pk-UNSHRED**

**Step 1:** Add \( (t − 1) − dₙ(s)/2 \) edges from s to an arbitrary s-neighbour in \( P₁ \) and \( (t − 1) − dₙ(s)/2 \) edges from s to an arbitrary s-neighbour in \( P₂ \). Output
\[ G' = (V + s, E \cup E'(s)). \]

We now run \( k \)-SPLIT on \( G' \). The proofs of Bang-Jensen and Jackson show that \( G' \) has a complete \( k \)-split. and so, process \( k \)-SPLIT, will output, \( G_1 = (V + s, E \cup F) \) with \( d_{G_1}(s) = 0 \).

In [1], it is shown that this stage runs in time \( O(|V|^3(|V| + \epsilon(E))^3) \). When not dealing with the bipartition constrained case, we would finish off at this point, by simply deleting \( s \). However, in this case we must make sure that the new edges are all \( Pk \).

The Swapping Stage

This stage looks at the edges in \( F \), in the hypergraph \( G_1 = (V + s, E \cup F) \), generated above. Process \( Pk \)-SWAP replaces edges in \( F \) that are “non-\( Pk \)” with ones that are \( Pk \). We do this by unsplitting groups of edges that are not \( Pk \), and re-splitting with \( Pk \) edges. So, we define unsplitting in a graph \( G_1 = (V + s, E \cup F) \) as removing an edge \( xy \in F \) and adding two edges \( sx, sy \). The trick, though, is finding the “better” split.

\( Pk \)-SWAP outputs either a hypergraph with a minimal \( Pk \)-augmenting set, or sets \( A_1, A_2, B_1, B_2 \) that form a \( C_4 \)-o-o in \( G \).

Process: \( Pk \)-SWAP

**Step 1:** Partition the edges of \( F \) into three sets. \( F_{Pk} \) is the set of \( Pk \)-edges in \( F \), and \( F_1 \) (respectively \( F_2 \)) is the set of all those edges in \( F \) with both ends in \( P_1 \) (respectively \( P_2 \)). Note that because we started \( k \)-SPLIT with \( d_1(s, P_1) = d_1(s, P_2) \) we have \( |F_1| = |F_2| \).

**Step 2:** If \( |F_1| = 0 \), stop and output \( G_1 \). If \( |F_1| = 1 \) go straight to Step 3. Otherwise, find \( a, b, p, q, u, v \in V \) such that \( ab, pq \in F_1 \) and \( uv \in F_2 \). Unsplit these edges to form \( G'_1 = (V + s, E \cup F' \cup E'_1(s)) \). By Lemma 6.6.1, there is a complete split from \( s \) in \( G'_1 \), that includes two \( Pk \)-splits. Find and perform this complete split, re-setting \( F_{Pk}, F_1 \) and \( F_2 \) appropriately. Note that after this re-split, we still have \( |F_1| = |F_2| \). Repeat Step 2 until \( |F_1| = 1 = |F_2| \).

**Step 3:** Suppose that \( F_1 = \{a_1a_2\} \) and \( F_2 = \{b_1b_2\} \). Unsplit these to form \( G''_1 \). Determine (Lemma 7.5.4) whether or not \( G''_1 \) has a complete \( Pk \)-split. If not go to Step 4, otherwise perform this complete split and stop, outputting \( G_2 = (V + s, E \cup F' \cup E_2(s)) \) where \( F' = F_{Pk} \) and \( E_2(s) \) is empty.

**Step 4:** Using Lemma 7.5.5, find \( C = \{A_1, A_2, B_1, B_2\} \), the \( C_4 \)-o-o in \( G''_1 \), choosing the labelling such that \( a_i \in A_i \) and \( b_i \in B_i \) for \( i = 1, 2 \). (That is \( A_1 \cup A_2 \) contains the \( s \)-neighbours that lie in \( P_1 \).) Find the set of edges
Let \( F_{P_k}^* := \{xy \in F_{P_k}, \text{ (where } x \in P_1 \text{ and } y \in P_2): \ xy \text{ does not satisfy the property that } x \in A_1 \cup A_2 \text{ and } y \in B_1 \cup B_2\} \). If \( F_{P_k}^* \) is empty, \( C \) is a \( C_4 \)-o-o in \( G \) so stop and output \( G_2 = (V + s, E \cup F^* \cup E_2(s)) \) where \( F^* = F_{P_k} \) and \( E_2(s) = \{sa_1, sa_2, sb_1, sb_2\} \). (Note that this means the edges \( a_1a_2, b_1b_2 \) are not present in \( G_2 \).)

**Step 5:** Choose an edge \( uv \in F_{P_k} \). Unsplit this edge, forming \( G''_1 \), with \( d_{G''_1}(s) = 6 \). By Lemma 7.5.6 there is a complete \( P_k \)-split in \( G''_1 \). Find and perform this complete split creating three new edges \( f_1, f_2, f_3 \). Stop and output \( G_2 = (V + s, E \cup F^* \cup E_2(s)) \) where \( F^* = F_{P_k} - uv + \{f_1, f_2, f_3\} \) and \( E_2(s) \) is empty. (Again, edges \( a_1a_2, b_1b_2 \) are not present in \( G_2 \).)

To see that this works and runs efficiently we have the following results. The first deals with the process in Step 2 for unsplitting non-\( P_k \) edges, and resplitting them as \( P_k \) edges.

**Lemma 7.5.3** Let \( G = (V + s, E) \) be \( k \)-edge-connected in \( V \), have \( d(s) = 0 \) and a bipartition \( P_1, P_2 \) of \( V \). Let \( ab, pq, uw \in E \) be such that \( a, b, p, q \in P_1 \) and \( u, v \in P_2 \). Then we can find a complete \( k \)-split using two \( P_k \)-splits in \( G' = G - \{ab, pq, uw\} + \{sa, sb, sp, sq, su, sv\} \) in time \( O(|V|^2(|V| + \epsilon(E))^3) \).

**Proof:** The existence of such a complete split is shown in Lemma 6.6.1. Because \( d_1(s, P_1) = 4 \) and \( d_1(s, P_2) = 2 \) there are only 12 possible complete splits using two \( P_k \)-edges. We can check each of these in time \( O(|V|^2(|V| + \epsilon(E))^3) \), so the lemma follows. \( \square \)

The next Lemma deals with testing for a complete \( P_k \)-split in Step 3. Note that as in the previous result, the searching is done simply by trial and error.

**Lemma 7.5.4** If \( G = (V + s, E) \) has \( d(s) = 4 \) with \( d_1(s, P_1) = d_1(s, P_2) = 2 \), we can determine in time \( O(|V|^2(|V| + \epsilon(E))^3) \), whether or not \( G \) has a complete \( P_k \)-split.

**Proof:** There are only 2 possible complete \( P_k \)-splits, and by Lemma 7.5.1 we can test each in time \( O(|V|^2(|V| + \epsilon(E))^3) \). \( \square \)

Before the next Lemma, we note that when starting \( P_k \)-SWAP, we have a hypergraph with \( G = (V + s, E \cup F) \) and \( d(s) = 0 \). Thus, any hypergraph, \( G' \), created by unsplitting edges from \( F \) has no shredder, because resplitting these edges produces a complete \( k \)-split of \( s \) in \( G' \).

The next result deals with when we have \( |F_1| = 1 \) and no complete \( P_k \)-split. Recall that we unsplit the edges from \( F_1 \) and \( F_2 \) giving a hypergraph with
Let \( G = (V + s, E) \) have \( d(s) = 4 \) with \( d_1(s, P_i) = d_1(s, P_j) = 2 \), have no complete \( P_k \)-split and have no shredder. Then, in time \( O(|V|(|V| + \epsilon(E))^3) \), we can find the members \( A_1, A_2, B_1, B_2 \) of a \( C_4\)-o-o in \( G \).

**Proof:** There is a \( C_4\)-o-o in \( G \) by Theorem 6.6.3. This means that \( s \) has four neighbours, say \( a_1, a_2, b_1, b_2 \) in each of \( A_1, A_2, B_1, B_2 \) respectively. Then, because there is no complete \( P_k \)-split and no shredder, \( a_1 a_2 \) and \( b_1 b_2 \) are both \( k \)-splits. We show how to find \( A_1 \). First delete the edge \( sa_1 \). Then find \( W_1 = \{ v \in V : \lambda_{G-sa_1}(v, b_1) = \lambda_{G-sa_1}(v, b_2) = k - 1 \} \). By Lemma 7.5.1 we can do this in time \( O(|V|(|V| + \epsilon(E))^3) \). We show that \( A_1 = W_1 \).

Indeed, \( d_{G-sa_1}(A_1) = k - 1 \) and \( G \) is \( k \)-edge-connected in \( V \), so \( A_1 \subseteq W_1 \). Now, suppose \( x \in W_1 \). Then there is a set \( X \subseteq V \) such that \( d_G(X) = k \) and \( x, a_1 \in X \) (so that \( d_{G-sa_1}(X) = k - 1 \)) and \( b_1 \notin X \). Because both \( X \) and \( A_1 \) contain \( a_1 \) and are tight in \( G \), \( X \) must contain \( A_1 \), for otherwise it would contradict Lemma 6.5.6. Such a set \( X \), does not intersect \( A_2 \). Otherwise, by Lemma 6.5.6, \( d_G(X \cup A_2) = k \), which contradicts the fact that \( a_1 a_2 \) is a \( k \)-split. Thus \( x \notin A_2 \). Also \( X \) cannot intersect both \( B_1 \) and \( B_2 \) for otherwise \( d_G(X \cup B_1 \cup B_2) = k \) by Lemma 6.5.6, which is contrary to Lemma 6.3.1. So we suppose (wlog) that \( x \in B_1 \), and note that Lemma 6.5.6 implies \( X \cup B_1 = A_1 \cup B_1 \) is tight in \( G \).

Recall that, if \( A \) is the set of edges intersecting all elements of \( C \) and \( d(Y, W) \) is the number of edges intersecting both \( W \) and \( Y \) but not \( V - (W \cup Y) \), then \( k = d_G(A_1) = |A| + 1 + d_G(A_1, B_1) + d_G(A_1, B_2) = d_G(B_1) = |A| + 1 + d_G(B_1, A_1) + d_G(B_1, A_2) \). Hence, \( d_G(A_1, B_2) = d_G(B_1, A_2) \). Then from the last paragraph, 
\[
\begin{align*}
k &= d_G(A_1 \cup B_1) = |A| + 2 + d_G(A_1, B_1) + d_G(B_1, A_2) = |A| + 2 + 2d_G(A_1, B_2).
\end{align*}
\]
That is \( k - |A| = 2 + 2d_G(A_1, B_2) \) which is even, and contrary to the definition of a \( C_4\)-o-o. Thus, \( x \notin B_1 \), and a similar argument shows that \( x \notin B_2 \).

Therefore, every \( x \in W_1 \) is in \( A_1 \), hence \( W_1 \subseteq A_1 \) and equality holds. By repeating a similar procedure for each of the other three \( s \)-neighbours, we can find, in the time complexity stated, the four members of \( C \). \( \square \)

To finish step 4, we find \( F^s_{P_k} \) as described above. The following result shows us that if \( F^s_{P_k} \) is non-empty, we can indeed find a complete \( P_k \)-split in \( G''_1 \) as required for Step 5.

**Lemma 7.5.6** Let \( G''_1 \) be as above. There is a complete \( P_k \)-split in \( G''_1 \). Furthermore, we can find this complete split in \( O(|V|^2(|V| + \epsilon(E))^3) \).

**Proof:** Recall that \( G''_1 \) has a \( C_4\)-o-o, \( C = \{A_1, A_2, B_1, B_2\} \), with \( s \)-neighbours
Before we split into the two cases, though, we consider some general properties.

Case 2: $d_{G''}(B)$ contains $v$ and one of $b_1, b_2, v$, and none of $a_1, a_2, u$. Because $b_1, b_2$ are a $k$-split in $G''$, we have that $X$ contains $v$ and one of $b_1, b_2$. Also, because $u \notin X$, we have that $d_{G''}(X) = d_{G''}(X) = k$, and that $X$ has $s$-neighbours of just one colour in $G''$. We now show that in both Case 1 and Case 2, such an $X$ implies a contradiction.

Case 1: We assume (wlog) that $X$ contains $v$ and $b_1$. Then $X$ intersects both $A_1$ and $B_1$. By Lemma 6.5.8, it does not intersect either $A_2$ or $B_2$. Now, $X - B_1$ is non-empty and $d_3(X, B_1) \geq 1$ (because the $s$-neighbour $b_1$ is in the set $X \cap B_1$). Then by Lemma 6.5.6, $B_1 - X$ is empty and $X$ contains $B_1$. Therefore $X \cup A_1 = B_1 \cup A_1$. By Lemma 6.5.6, $d_{G''}(X \cup A_1) = d_{G''}(A_1 \cup B_1) = k$.

Now, $d_{G''}(A_1) = d_{G''}(B_1) = k$. Let $d_{G''}(A_1, B_1) = t_{11}$, $d_{G''}(A_1, B_2) = t_{12}$, $d_{G''}(A_2, B_1) = t_{21}$. Recall that in $G''$, each of the members of $C$ has exactly one $s$-neighbour, with one edge to $s$. Then, $k = d_{G''}(A_1) = t_{11} + t_{12} + 1 + |A| = d_{G''}(B_1) = t_{11} + t_{21} + 1 + |A|$. Thus, $t_{12} = t_{21}$. Also, considering the degree of $A_1 \cup B_1$ (last paragraph) we have that $k = t_{12} + t_{21} + 2 + |A|$ which implies that $k - |A|$ is even, which contradicts the definition of a $C_4$-$o$-$o$. Thus, if unsplitting $u,v$ creates a $G''$, in the form of Case 1, then $G''$ has no $C_4$-$o$-$o$, and hence has a complete $Pk$-split.

Case 2: In this case we suppose that $v \in A_1$ and $u \in B_1$. We know that $X$ contains $v$, and $b_i \in X$ also, for some $i = 1$ or 2. Then $X$ intersects $B_i$, and $d_{G''}(X, B_i) \geq 1$. Therefore, by Lemma 6.5.6, $X$ contains $B_i$. Now, if $i = 1$, we have that $X$ contains $B_i$, contradicting the fact that $uv$ is a $k$-split in $G''$. So $i = 2$. Now, because $X$ intersects both $A_1$ and $B_2$, by Lemma 6.5.8, it does not intersect $A_2$ or $B_1$. So, because $X$ contains $B_2$, we have $X \cup A_1 = B_2 \cup A_1$. Also, $X$
The complexity follows because \( d_{C''_k}(s) = 6 \) with \( d_{1,C''_k}(s, P_1) = d_{1,C''_k}(s, P_2) = 3 \), which means there are only six possible complete \( P_k \)-splits, each of which can be checked in \( O(|V|^2(|V| + \epsilon(E))^3) \) time (Lemma 7.5.1).

\[ \square \]

The Checking and Fixing Stage

At this point, we have either outputted a minimal \( P_k \)-augmenting set, or we have found a \( C_{4-o} \) in the initialised hypergraph \( G \). We must now check whether or not \( C \) is a \( C_{4-o-c} \) in \( H \).

**Process: \( C_4 \)-CHECKFIX**

**Step 1:** For each \( u \in N_G(s) \) find the minimal tight set \( X_u \) containing \( u \). If there exists \( u \in N_G(s) \cap (A_1 \cup A_2) \) with \( X_u \not\subseteq P_1 \) and \( v \in N_G(s) \cap (B_1 \cup B_2) \) with \( X_v \not\subseteq P_2 \) go straight to Step 3. Otherwise go to Step 2.

**Step 2:** Add one edge from \( s \) to \( x \), an arbitrary \( s \)-neighbour in \( A_1 \) and one edge from \( s \) to \( y \), an arbitrary \( s \)-neighbour in \( B_1 \), thus forming \( E'(s) = E(s) + \{sx, sy\} \). Stop and output \( G' = (V + s, E \cup E'(s)) \).

**Step 3:** Replace the edge \( su \) with an edge \( su' \) where \( u' \in X_u \cap P_2 \) and replace \( su \) with an edge \( sv' \) where \( v' \in X_v \cap P_1 \), thus forming \( E'(s) = E(s) - \{su, sv\} + \{su', sv'\} \). Stop and output \( G' = (V + s, E \cup E'(s)) \).

Lemma 7.3.2 implies that in order to determine whether or not \( C \) is a \( C_{4-o-c} \) in \( H \), it is enough to check the minimal tight sets in this way. By Lemma 7.5.2, we can find each minimal tight set \( X_u \) in time \( O(|V|(|V| + \epsilon(E))^3) \), and there are at most \( |V| \) “colour” checks to make. Therefore \( C_4 \)-CHECKFIX runs in time \( O(|V|^2(|V| + \epsilon(E))^3) \).

To finish the augmentation, after running \( C_4 \)-CHECKFIX we run \( k \)-SPLIT and \( P_k \)-SWAP on \( G' \) (the hypergraph outputted by \( C_4 \)-CHECKFIX). There is no shredder in this hypergraph, so \( k \)-SPLIT must output a complete \( k \)-split from \( s \).

If we have used Step 2, we have added edges from \( s \) to vertices \( x, y \), that are \( s \)-neighbours in \( G \). But every set \( X \subset V \) has \( d_G(X) \geq k \). Therefore, in \( G' \), every set containing either \( x \) or \( y \) has \( d_{G_1}(X) \geq k + 1 \) and so there is no \( C_{4-o} \).
in $G_1$. If we have used Step 3, Lemma 7.3.3 implies that $G_1$ has no shredder and no $C_4$-o-o. Therefore, in both cases, $Pk$-SWAP will output a $Pk$-augmentation for $H$.

**Summary**

Whenever the above process stops, a $Pk$-augmenting set for $H$ is provided. That it is minimal follows from the previous sections of this chapter. To finish, we state that the above algorithm has been presented as proof for the following theorem.

**Theorem 7.5.7** Provided all basic arithmetic operations are taken as constant time operations, there is a polynomial algorithm that takes as input a hypergraph $H = (V, E)$, with bipartition $P = \{P_1, P_2\}$ and an integer $k \geq 2$, and outputs an optimal set of size-two edges, none of which is contained in just one $P_i$, whose addition to $H$ forms a $k$-edge-connected hypergraph.

**Remark - What next?**

The natural extension to the augmentation problem considered in this chapter, is the case where the partition may have more than two elements.

Given a hypergraph $H = (V, E)$ with partition $P = \{P_1, \ldots, P_n\}$, of $V$, with $n \geq 2$, what is the smallest number of size-two edges we must add to $H$, such that the resulting hypergraph is $k$-edge-connected, and no new edge is contained in a single $P_i$?

We define $\alpha, \beta, \Phi$ and $\Phi'$ as before - but here we have $\beta_1, \ldots, \beta_n$. Again we use the $C_4$-o-c, and need to construct, (following Bang-Jensen et al.’s line) a $C_6$-o-c, as follows.

Let $H = (V, E)$ be a hypergraph with prescribed partition $P = \{P_1, P_2, \ldots, P_n\}$ with $2 \leq n \leq |V|$, and $k \geq 2$ be an integer and suppose that $\Phi = 3$. Let $C = \{A_1, A_2, B_1, B_2, C_1, C_2\}$ be a partition of $V$ and let $A = \{e \in E : e$ intersects every $X \in C\}$. Then $C$ is a $C_6$-odd-configuration (or $C_6$-o-c) when it satisfies the following properties.

(a) $d(X) = k - 1$ for all $X \in C$.

(b) $d(X, Y) = (k - 1 - |A|)/2$ for all $(X, Y) \in \{(A_1, B_1), (B_1, C_1), (C_1, A_2), (A_2, B_2), (B_2, C_2), (C_2, A_1)\}$.

(c) $d_1(s, X) = 1$ for all $X \in C$. 

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(d) For three distinct partition classes, $P_a, P_b, P_c$, there are six sets $A'_1, A'_2$, $B'_1, B'_2$, $C'_1, C'_2$ contained in $A_1 \cap P_a$, $A_2 \cap P_a$, $B_1 \cap P_b$, $B_2 \cap P_b$, $C_1 \cap P_c$, $C_2 \cap P_c$ respectively.

(e) $k - |A|$ is odd.

Then if $\theta_{P_k}(H)$ is the size of a smallest good-augmentation of $H$, we conjecture the following.

**Conjecture 7.5.8** Let $H = (V, E)$ be a hypergraph, $P = \{P_1, \ldots, P_n\}$ be a partition of $V$, and $k \geq 2$ be an integer. Then $\theta_{P_k}(H) = \Phi$ unless $\Phi = \Phi'$ and $H$ contains a $C_4$-o-c or a $C_6$-o-c, in which case $\theta_{P_k}(H) = \Phi + 1$. \hfill $\square$
Bibliography


