Convergence of boundary element methods on fractals

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This talk is concerned with BEM for time-harmonic acoustic scattering by infinitely thin, bounded planar screens. This is a classical subject [8]: our substantial and novel twist is that we consider the case where the screen is fractal or has fractal boundary. One motivation is the application of fractal antennae in electromagnetics [9]. For details, including numerical examples, see the preprint [5].

We work in n dimensions ($n = 2$ or 3), and suppose that the screen Γ is a bounded subset of $\Gamma_\infty := \{x = (x_1, ..., x_n) \in \mathbb{R}^n : x_n = 0\}$, which we identify with $\mathbb{R}^{n-1}$. We restrict attention to the case that Γ is either a closed or an open[1] subset of Γ_∞, and set $D := \mathbb{R}^n \setminus \overline{\Gamma}$. We suppose that an incident wave u^i, to be concrete the plane wave $u^i(x) = \exp(\mathrm{i}kx \cdot d)$, where d is a unit vector, is incident on the screen Γ. This incident wave is a solution of the Helmholtz equation

$$\Delta u + k^2 u = 0 \tag{1}$$

for wavenumber $k > 0$. We focus on the case of a sound soft screen (see [3] for the sound hard case). The scattering problem that we consider is the following:

Given the incident field u^i, find the total field $u \in C^2(D) \cap W_0^{1,\mathrm{loc}}(D)$ *such that* (1) *holds in D, and $u^s := u - u^i$ satisfies the standard Sommerfeld radiation condition.*

Here $\phi \in W_0^{1,\mathrm{loc}}(D)$ if $\chi\phi \in W_0^1(D)$, for every $\chi \in C_0^\infty(\mathbb{R}^n)$; and $W_0^1(D)$ is the closure of $C_0^\infty(D)$ in $W^1(D)$, where $W^1(D) := \{\phi \in L^2(D) : \nabla\phi \in L^2(D)\}$.

Our numerical scheme is: approximate the screen Γ by a sequence Γ_j of more regular screens; compute the solution for Γ_j by a conventional BEM with some maximum element diameter h_j. BEM computations for fractals have been carried out previously using this methodology, for example in potential theory [6]. Our main novelty is that we present the first results, conditions on the sequences Γ_j and h_j, that guarantee convergence in the limit $j \to \infty$. Our focus is concrete fractal scattering problems, but our numerical analysis ideas are widely applicable, to general classes of BIEs/pseudo-differential equations on rough sets approximated by sequences of more regular sets.

BIE and Variational Formulations. Our Sobolev space notations are those of [7], and we identify $H^s(\Gamma_\infty)$ with $H^s(\mathbb{R}^{n-1})$ in the obvious way. In particular, for a closed set $F \subset \Gamma_\infty$, H^s_F is the set of those $\phi \in H^s(\Gamma_\infty)$ with support in F and, for an open set $O \subset \Gamma_\infty$, $\widetilde{H}^s(O) \subset H^s(\Gamma_\infty)$ is the closure of $C_0^\infty(O)$ in the $H^s(\Gamma_\infty)$ norm. Both H^s_F and $\widetilde{H}^s(O)$ are closed subspaces of $H^s(\Gamma_\infty)$. Further, $\widetilde{H}^s(O) \subset H^s_{\overline{O}}$, with equality if: O is C^0 [7]; $|s| \le 1/2$ and O is C^0 except at a set of countably many points in ∂O that has only finitely many limit points [4]; ∂O has $(n-1)$-dimensional Lebesgue measure zero, and O is 'thick' in the sense of Triebel, the case for many open O with fractal boundaries, for example the interior of the Koch snowflake [1]. See [4] for examples where equality does not hold.

[1] Strictly speaking 'relatively open', which we abbreviate as 'open' throughout.

In the case when the screen Γ is some C^∞ open subset of Γ_∞, it is well-known [8, 2] that u satisfies the above scattering problem iff

$$u(x) = u^i(x) - \int_\Gamma \Phi(x,y) \left[\frac{\partial u}{\partial n}\right](y)\mathrm{d}s(y), \quad x \in D, \tag{2}$$

and

$$S[\partial u/\partial n] = g := u^i|_\Gamma. \tag{3}$$

Here $[\partial u/\partial n] = [\partial u/\partial x_n] \in H_{\overline{\Gamma}}^{-1/2} = \widetilde{H}^{-1/2}(\Gamma)$ is the jump in the normal derivative across Γ_∞ and S is the standard acoustic single-layer potential operator on Γ. S is an isomorphism from $\widetilde{H}^{-1/2}(\Gamma)$ to its dual space $H^{1/2}(\Gamma)$, indeed is coercive [2]. In particular, in the case that $\Gamma = \Gamma_R := \{x \in \Gamma_\infty : |x| < R\}$, it holds that

$$|\langle S\phi, \phi\rangle| \;\geq\; C_R \|\phi\|^2_{\widetilde{H}^{-1/2}(\Gamma_R)}, \tag{4}$$

for $\phi \in \widetilde{H}^{-1/2}(\Gamma_R)$, where $\langle\cdot,\cdot\rangle$ is the usual extension of the inner product on $L^2(\Gamma_\infty)$ to a sesquilinear form on $H^s(\Gamma_\infty) \times H^{-s}(\Gamma_\infty)$ and $C_R > 0$ depends only on k and R. This implies, by Lax-Milgram, that the variational form of (3) has exactly one solution. Where $a(\phi,\psi) := \langle S\phi, \psi\rangle$, for $\phi, \psi \in \widetilde{H}^{-1/2}(\Gamma_R)$, this variational form is to find $[\partial u/\partial n] \in \widetilde{H}^{-1/2}(\Gamma_R)$ such that

$$a\left([\partial u/\partial n], \psi\right) = \langle g, \psi\rangle, \quad \forall \psi \in \widetilde{H}^{-1/2}(\Gamma_R). \tag{5}$$

These observations immediately give us well-posedness of variational formulations of integral equations on *arbitrary* bounded open or closed subsets of Γ_∞. For any such subset Γ is contained in Γ_R for some $R > 0$. These variational formulations are (5) with $\widetilde{H}^{-1/2}(\Gamma_R)$ replaced by the closed subspace $V \subset \widetilde{H}^{-1/2}(\Gamma_R)$, where $V := H_\Gamma^{-1/2}$ if Γ is closed, $V := \widetilde{H}^{-1/2}(\Gamma)$ if Γ is open. It is immediate from (4) and the Lax-Milgram lemma that these variational formulations are well-posed. This is part of the proof of the following theorem.

Theorem 1. [3, 5] *If $\Gamma \subset \Gamma_R$ is closed,* (5)*, with $\widetilde{H}^{-1/2}(\Gamma_R)$ replaced by $V = H_\Gamma^{-1/2}$, has exactly one solution, and u given by* (2) *is the unique solution of the above scattering problem. If $\Gamma \subset \Gamma_R$ is open,* (5)*, with $\widetilde{H}^{-1/2}(\Gamma_R)$ replaced by $V = \widetilde{H}^{-1/2}(\Gamma)$, has exactly one solution. Further, u given by* (2) *is the unique solution of the above scattering problem, provided $\widetilde{H}^{-1/2}(\Gamma) = H_{\overline{\Gamma}}^{-1/2}$.*

Generally the integral in (2) has to be interpreted as a duality pairing, in particular if Γ is closed with empty interior, when the solution of (5) is zero iff $H_\Gamma^{-1/2} = \{0\}$, but $H_\Gamma^{-1/2} \neq \{0\}$ if the Hausdorff dimension of Γ exceeds $n-2$ [4].

BEM and Mosco convergence. In our BEM we approximate Γ by a sequence of (more regular) open sets $\Gamma_j \subset \Gamma_R$, and we mesh Γ_j with what we call a *pre-convex* mesh $M_j = \{T_{j,1}, ..., T_{j,N_j}\}$, meaning that: each element $T_{j,\ell} \subset \Gamma_j$ is open; Γ_j is the interior of the union of the closures of the elements $T_{j,\ell}$; the convex hulls of the elements are pairwise disjoint; and each $\partial T_{j,\ell}$ has $(n-1)$-dimensional Lebesgue measure zero. Let $h_j := \max_\ell \mathrm{diam}(T_{j,\ell})$ and let $V_j^h \subset L^2(\Gamma_j) \subset \widetilde{H}^{-1/2}(\Gamma_j)$

denote the piecewise constant BEM approximation space, the set of functions that are constant on each element $T_{j,\ell}$. Then the solution $\phi := [\partial u/\partial n] \in V$ and its BEM approximation $\phi_j^h \in V_j^h$ are defined by

$$a(\phi,\psi) = \langle g,\psi\rangle, \ \ \forall \psi \in V, \quad a(\phi_j^h,\psi) = \langle g,\psi\rangle, \ \ \forall \psi \in V_j^h. \tag{6}$$

In contrast to usual BEM analysis, it need not be the case that $V_j^h \subset V$, in particular this cannot be the case if Γ is closed with empty interior. However, V and V_j^h are both subsets of the larger Hilbert space $H := \widetilde{H}^{-1/2}(\Gamma_R)$.

The following results are a partial extension to this case of the standard Céa's lemma. In these results we suppose temporarily that: H is any Hilbert space; $\langle\cdot,\cdot\rangle$ is the duality pairing on $H^* \times H$; $a(\cdot,\cdot)$ is any continuous sesquilinear form on H; $V \subset H$ and $V_j^h \subset H$, for $j \in \mathbb{N}$, are closed subspaces; and $\phi \in V$ and $\phi_j^h \in V_j^h$ are defined by (6), with $g \in H^*$. Further, $\to$ is norm and $\rightharpoonup$ weak convergence in H.

Theorem 2. *Suppose that $a(\cdot,\cdot)$ is invertible on V and, for some $J \in \mathbb{N}$, on V_j^h for $j \geq J$ (meaning that, for every $g \in H^*$, the problems* (6) *have exactly one solution $\phi \in V$ and $\phi_j^h \in V_j^h$, for $j \geq J$). Suppose also, for every $g \in H^*$, that $\phi_j^h \to \phi$ as $j \to \infty$. Then V_j^h* Mosco-converges *to V ($V_j^h \xrightarrow{\mathcal{M}} V$), meaning that*

(i) *for every $v \in V$ and $j \in \mathbb{N}$ there exists $v_j \in V_j^h$ such that $v_j \to v$;*
(ii) *if j_m is a subsequence of $\mathbb{N}$, $w_{j_m} \in V_{j_m}^h$, and $w_{j_m} \rightharpoonup w \in H$, then $w \in V$.*

Proof. Suppose $v \in V$ and define $g \in H^*$ on V by $\langle g,\psi\rangle = a(v,\psi)$, for all $\psi \in V$, and then extend g to a linear functional on H by Hahn-Banach. Then, where ϕ, ϕ_j^h are the solutions of (6), $\phi_j^h \to \phi$, but also $\phi = v$ by construction, so that (i) holds. To see that (ii) holds suppose that a weakly convergent sequence w_{j_m} exists as in (ii), but that its limit $w \notin V$. Define $g \in H^*$ on $\mathbb{C}w + V$ by $\langle g, cw+v\rangle = c$, for $c \in \mathbb{C}$ and $v \in V$, and extend g to H by Hahn-Banach. Then, where ϕ, ϕ_j^h are the solutions of (6), $\phi_j^h \to \phi$ as $j \to \infty$ and $\phi = 0$ as $\langle g,\psi\rangle = 0$, $\psi \in V$. Thus $a(\phi_{j_m}^h, w_{j_m}) \to 0$ as $m \to \infty$, but also $a(\phi_{j_m}^h, w_{j_m}) = \langle g, w_{j_m}\rangle \to \langle g, w\rangle = 1$. ☐

Theorem 3. [5] *Suppose that $a(\cdot,\cdot)$ is invertible on V and is a compact perturbation of a coercive form on H. Then, for every sequence of closed subspaces $V_j^h \subset H$ such that $V_j^h \xrightarrow{\mathcal{M}} V$, there exists $J \in \mathbb{N}$ such that $a(\cdot,\cdot)$ is invertible on V_j^h for $j \geq J$, and, for every $g \in H^*$, $\phi_j^h \to \phi$, where ϕ and ϕ_j^h are the solutions of* (6).

In the case that $H = V$ (so $V_j^h \subset V$): (ii) holds automatically in Theorem 2; $V_j^h \xrightarrow{\mathcal{M}} V$ iff $m_j(v) := \inf_{\psi \in V_j^h} \|v-\psi\| \to 0$ as $j \to \infty$ for every $v \in V$; Theorem 3 reduces to (part of) a generalised Céa's lemma. An **open problem** is what should replace the standard Céa's lemma quasi-optimality bound that $\|\phi - \phi_j^h\| \leq Cm_j(\phi)$ in cases where $V_j^h \not\subset V$?

Returning to the case in which (6) is the BIE variational problem and its BEM approximation, with V_j^h the piecewise constant BEM approximation space on a pre-convex mesh M_j, it follows from the above theorems that the BEM approximation $\phi_j^h \to \phi = [\partial u/\partial n]$ in $\widetilde{H}^{-1/2}(\Gamma_R)$, for every incident wave direction d, iff

$V_j^h \xrightarrow{\mathcal{M}} V$. The following theorem gives conditions in the case when Γ is compact (see [5] for the case Γ open) that guarantee that $V_j^h \xrightarrow{\mathcal{M}} V$. For $\epsilon > 0$, $\Gamma(\epsilon) := \{x \in \Gamma_\infty : \mathrm{dist}(x, \Gamma) < \epsilon\}$.

Theorem 4. *Let Γ_j be a sequence of open subsets of Γ_R such that $\Gamma \subset \Gamma(\epsilon_j) \subset \Gamma_j \subset \Gamma(\eta_j)$, for some $0 < \epsilon_j < \eta_j$, with $\eta_j \to 0$ as $j \to \infty$. If H^t_Γ is dense in $H_\Gamma^{-1/2}$ for some $t \in [-1/2, 0]$ (always true for $t = -1/2$), and $h_j = o((\epsilon_j)^{-2t})$, then $V_j^h \xrightarrow{\mathcal{M}} V$ so that $\phi_j^h \to \phi$ as $j \to \infty$.*

As an example, consider the $n = 2$ case when $\Gamma = C \times \{0\} \subset \mathbb{R}^2$, where $C \subset [0, 1]$ is the Cantor set, with Hausdorff dimension $d = \log(2)/\log(1/\alpha)$, that is the attractor of the iterated function system $\{s_1, s_2\}$, where, for some $\alpha \in (0, 1/2)$, $s_1(t) := \alpha t$, $s_2(t) := \alpha t + 1 - \alpha$, for $t \in \mathbb{R}$. Choose $\delta \in (0, -1 + 1/(2\alpha))$, set $C_0 := (-\delta, 1 + \delta)$, $C_j := s_1(C_{j-1}) \cup s_2(C_{j-1})$, for $j \in \mathbb{N}$. Then $C_0 \supset C_1 \supset ... \supset C$, $C = \cap_j C_j$, and C_j is the disjoint union of 2^j intervals of length $H_j := \alpha^j(1 + 2\delta)$. Define $\Gamma_j := C_j \times \{0\}$, for $j \in \mathbb{N}$, choose $i = i(j) \in \{1, ..., j\}$ and mesh Γ_j with a pre-convex mesh M_j of $N_j = 2^i$ elements, each element comprising 2^{j-i} intervals of length H_j, and each element having diameter $h_j = \alpha^i(1 + 2\delta\alpha^{j-i})$. It is shown in [1] that H^t_Γ is dense in $H_\Gamma^{-1/2}$ for $-1/2 \le t < (d - 1)/2$, so that it follows from the above theorem [5] that $\phi_j^h \to \phi$ in $\widetilde{H}^{-1/2}(\Gamma_R)$ provided $i(j) > \mu j$ for some $\mu > 1 - \log(2)/\log(1/\alpha)$. An **open problem** is to prove convergence, even in the case when $i(j) = j$, when $\delta = 0$ and C_j is the standard prefractal sequence for C.

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